Tau functions, integrable systems random matrices and random processes

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(Joint work with A. Yu. Orlov)

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Three uses of tau functions

1. Classical integrable systems

- Canonical generator for commuting flows

- *Determinant* of a projection operator from linear spaces evolving under an abelian group action

2. Random matrices, quantum integrable systems, solvable lattice lattice models

- Partition function, correlation function (under parametric family of deformations in measures)

- Hankel, Toeplitz or Fredholm *Determinant*

- Boltzmann weight on statistical ensemble

3. Random processes

- Weight on path space

- Generating function for transition probabilities

Sato-Segal-Wilson interpretation of the KP-τ function:

An infinite **Fredholm determinant** of a projection operator on a Hilbert space Grassmannian

$$\tau_g(\mathbf{t}) = \det(\pi_+ : W(\mathbf{t}) \to \mathcal{H}_+), \quad \mathbf{t} = (t_1, t_2, \ldots)$$

$$W(\mathbf{t}) = \gamma(\mathbf{t})(g(\mathcal{H}_+)) \subset \mathcal{H}$$

$$\mathcal{H} = \mathcal{H}_- + \mathcal{H}_+ \quad \text{(Hilbert space; e.g. } L^2(S^1))$$

$$e^{\sum_{i=1}^{\infty} t_i z^i} =: \gamma(\mathbf{t}) : \mathcal{H} \to \mathcal{H}, \quad g \in GL(\mathcal{H})$$

(linearly evolving subspace under an abelian group action)

Fermionic Fock space VEV's (KP τ function)

$$\tau_{N,g}(\mathbf{t},\tilde{\mathbf{t}}) :=< N|\gamma(\mathbf{t})g|N >$$
$$\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i H_i}$$
$$H_i := \sum_{j\in\mathbf{Z}} f_i \bar{f}_{j+i}, \quad g = e^{\mathcal{A}}, \quad \mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j$$

2-D Toda τ function

$$\tau_{N,g}(\mathbf{t},\tilde{\mathbf{t}}) :=< N|\gamma(\mathbf{t})g\tilde{\gamma}(\tilde{\mathbf{t}})|N >$$
$$\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i H_i}, \quad \tilde{\gamma}(\tilde{\mathbf{t}}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i H_{-i}}$$
$$H_i := \sum_{j\in\mathbf{Z}} f_i \bar{f}_{j+i}, \quad g = e^{\mathcal{A}}, \quad \mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j$$

1. Classical integrable systems:

The τ function determines Hamilton's principle function on Lagrangian leaves

$$
\begin{aligned}
S(\mathbf{q}(\mathbf{t}), \mathbf{C}) &= \int_{\mathbf{P}=\mathbf{C}} \mathbf{p} \cdot d\mathbf{q} \\
&= \mathcal{D}(\ln\tau(t_1, t_2, \ldots))
\end{aligned}
$$

$\mathcal{D}$ = Linear differential operator in the flow parameters $(t_1, t_2, \ldots)$)

Example: Finite dimensional isospectral quasi-periodic flows:

Rational Lax matrix $L(\lambda)$ ($\lambda =$ "spectral parameter")

$$\frac{dL(\lambda)}{dt} = [A(L, \lambda), L(\lambda]$$

Lagrangian leaves ($\mathbf{P} = \mathbf{C}$) $\leftrightarrow$ **spectral curve**

$$\det(L(\lambda) - z\mathbf{I}) = 0$$

$$\tau(\mathbf{t}) = \Theta(\mathbf{Q}(\mathbf{t})), \quad \mathbf{Q}(\mathbf{t}) = \mathbf{Q_0} + (\nabla_{\mathbf{P}} H, \mathbf{t})$$

This is an **infinite** determinant. Degeneration to rational curves with cusp singularities give **solitons**:

$$\tau \sim \det(e^{(\Lambda_{ij}, \mathbf{t}) + \kappa_{\mathbf{ij}}})$$

2. Random Matrices: Two matrix models

Most **statistical properties** of the spectrum are expressible as expectation values

$$< F >= \frac{1}{\mathbf{Z}_N^{(2)}} \int F(M_1, M_2) d\Omega(M_1, M_2)$$

where the **Partition function** is

$$\mathbf{Z}_N^{(2)} := \int d\Omega(M_1, M_2)$$

For some **conjugation invariant** F's, unitarily diagonalizable matrices M_1, M_2, and certain matrix measures $d\Omega(M_1, M_2)$ this reduces to:

$$< F > \propto \prod_{i=1}^{N} \int\int_{\kappa\Gamma} d\mu(x_i, y_i) \Delta_N(x) \Delta_N(y) \tilde{F}(x_1, \ldots, x_N, y_1, \ldots y_N)$$

Example: (Itzykson-Zuber (1980)

$$d\Omega(M_1, M_2) = d\mu_0(M_1)d\mu_0(M_2)e^{\mathrm{Tr}(V_1(M_1)+V_2(M_2)+M_1M_2)}$$

Reduced 2−matrix integrals

$$\mathbf{Z}_N^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}) := \prod_{i=1}^{N} \int\int_{\kappa\Gamma} d\mu(x_i, y_i)\gamma(x_i, \mathbf{t})\tilde{\gamma}(y_i, \tilde{\mathbf{t}})\Delta_N(\mathbf{x})\Delta_N(\mathbf{y})$$

where $d\mu(x, y)$ is some two-variable measure

General form

$$d\mu(x,y) = d\mu(x)d\tilde{\mu}(y)h(xy)\sum_{a=1}^{k}\sum_{b=1}^{l} z_{ab}\chi_a(x)\tilde{\chi}_b(y)$$

$$\gamma(x,\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i x^i}, \quad \tilde{\gamma}(y,\tilde{\mathbf{t}}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i y^i}$$

$$\mathbf{t} = (t_1, t_2, \ldots), \quad \tilde{\mathbf{t}} = (\tilde{t}_1, \tilde{t}_2, \ldots)$$

$$\kappa = \{\kappa_{\alpha\beta}\}\big|_{\substack{1\le\alpha\le d_1\\ 1\le\beta\le d_2}}. \quad \Gamma = \{\Gamma_\alpha \times \tilde{\Gamma}_\beta\}$$

$$\int\!\!\int_{\kappa\Gamma} := \sum_{\alpha\beta} \kappa_{\alpha\beta} \int_{\Gamma_\alpha} \int_{\tilde{\Gamma}_\beta}$$

Examples

- Two-matrix partition function:

$$k = l = 1, \quad z_{11} = 1, \quad \chi_1 = \tilde{\chi}_1 = 1$$

$$d\mu(x) = e^{V_1(x)}dx, \quad d\tilde{\mu}(y) = e^{V_2(y)}, \quad h(xy) = e^{xy}$$

- Generating function for (k, l) point correlators (marginal distributions) of eigenvalues

$$\chi_a = \delta(x - X_a), \quad \chi_b = \delta(y - Y_b)$$

- Generating function for *gap probabilities*

$$\chi_a = \chi_{[\alpha_{2a-1}, \alpha_{2a}]}(x) \quad \tilde{\chi}_b = \chi_{[\beta_{2b-1}, \beta_{2b}]}(x)$$

- Generating function for *Janossy distributions* (Combine the above two.)

Two matrix determinantal correlators:

$$\mathbf{I}_N^{(2)} := \left\langle \prod_{a=1}^{N} \frac{\prod_{\alpha=1}^{L_1} \det(\xi_\alpha \mathbf{I} - M_1) \prod_{\beta=1}^{L_2} \det(\zeta_\beta \mathbf{I} - M_2)}{\prod_{j=1}^{M_1} \det(\eta_j \mathbf{I} - M_1) \prod_{k=1}^{M_2} \det(\mu_k \mathbf{I} - M_2)} \right\rangle$$

For suitable measures, this reduces to **Integrals of rational symmetric functions** in $2N$ variables:

$$\mathbf{I}_N^{(2)}(\xi,\zeta,\eta,\mu) := \frac{1}{\mathbf{Z}_N^{(2)}} \prod_{i=1}^{N} \iint_{\kappa\Gamma} d\mu(x_i, y_i) \Delta_N(x) \Delta_N(y) \prod_{a=1}^{N} \frac{\prod_{\alpha=1}^{L_1} (\xi_\alpha - x_a) \prod_{\beta=1}^{L_2} (\zeta_\beta - y_a)}{\prod_{j=1}^{M_1} (\eta_j - x_a) \prod_{k=1}^{M_2} (\mu_k - y_a)}$$

$$\mathbf{Z}_N^{(2)} := \prod_{i=1}^{N} \iint_{\kappa\Gamma} d\mu(x_i, y_i) \Delta_N(x) \Delta_N(y)$$

Assuming generic conditions on the matrix of **bimoments**:

$$B_{jk} := \iint_{\kappa\Gamma} d\mu(x,y) x^j y^k < \infty, \quad 0, \quad \forall j,k \in \mathbf{N}$$
$$\det(B_{jk})_{0\le j,k\le N} \neq 0, \quad \forall N \in \mathbf{N}$$

implies the existence of a unique sequence of

Biorthogonal polynomials

$$\iint_{\kappa\Gamma} d\mu(x,y) P_j(x) S_k(y) = \delta_{jk},$$

normalized to have leading coefficients that are equal:

$$P_j(x) = \frac{x^j}{\sqrt{h_j}} + O(x^{j-1}), \qquad S_j(x) = \frac{y^j}{\sqrt{h_j}} + O(y^{j-1}).$$

Also, assume existence of their **Hilbert transforms**

$$\tilde{P}_j(\mu) := \iint_{\kappa\Gamma} d\mu(x,y)\frac{P_j(x)}{\mu - y},$$
$$\tilde{S}_j(\eta) := \iint_{\kappa\Gamma} d\mu(x,y)\frac{S_j(x)}{\eta - x}$$

and their **Hilbert transforms**

$$\tilde{P}_j(\mu) := \iint_{\kappa\Gamma} d\mu(x,y)\frac{P_j(x)}{\mu - y},$$
$$\tilde{S}_j(\eta) := \iint_{\kappa\Gamma} d\mu(x,y)\frac{S_j(x)}{\eta - x}$$

Determinantal expression for correlator (Assume $N+L_2-M_2 \geq N+L_1-M_1 \geq 0$)

$$\mathbf{I}_N^{(2)} = \epsilon(L_1, L_2, M_2, M_2) \frac{\prod_{n=0}^{N+L_2-M_2-1} \sqrt{h_n} \, \prod_{n=0}^{N+L_1-M_1-1} \sqrt{h_n}}{\prod_{n=0}^{N-1} h_n}$$
$$\times \frac{\prod_{\alpha=1}^{L_1} \prod_{j=1}^{M_1} (\xi_\alpha - \eta_j) \prod_{\beta=1}^{L_2} \prod_{k=1}^{M_2} (\zeta_\beta - \mu_k)}{\Delta_{L_1}(\xi) \Delta_{L_2}(\zeta) \Delta_{M_1}(\eta) \Delta_{M_2}(\mu)} \, \det G$$

where

$$\epsilon(L_1, L_2, M_2, M_2) := (-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)} (-1)^{L_1 M_2}$$

and G is the $(L_2+M_1) \times (L_2+M_1)$ matrix:

$$G = \begin{pmatrix} \overset{N+L_1-M_1}{K_{11}}(\xi_\alpha, \eta_j) & \overset{N+L_1-M_1}{K_{12}}(\xi_\alpha, \zeta_\beta) \\ \overset{N+L_1-M_1}{K_{21}}(\mu_k, \eta_j) & \overset{N+L_1-M_1}{K_{22}}(\mu_k, \zeta_\beta) \\ \tilde{S}_{N+L_1-M_1}(\eta_j) & S_{N+L_1-M_1}(\zeta_\beta) \\ \vdots & \vdots \\ \tilde{S}_{N+L_2-M_2-1}(\eta_j) & S_{N+L_2-M_2-1}(\zeta_\beta) \end{pmatrix}$$

where the kernels $K^J_{11}, K^J_{11}, K^J_{11}, K^J_{11}$ are defined by:

$$\overset{J}{K}_{11}(\xi, \eta) := \sum_{n=0}^{J-1} P_n(\xi)\tilde{S}_n(\eta) + \frac{1}{\xi - \eta}, \qquad \overset{J}{K}_{12}(\xi, \zeta) := \sum_{n=0}^{J-1} P_n(\xi)S_n(\zeta),$$

$$\overset{J}{K}_{21}(\mu, \eta) := \sum_{n=0}^{J-1} \tilde{P}_n(\mu)\tilde{S}_n(\eta) - \iint_{\kappa\Gamma} \frac{d\mu(x, y)}{(\eta - x)(\mu - y)}, \quad \overset{J}{K}_{22}(\mu, \zeta) := \sum_{n=0}^{J-1} \tilde{P}_n(\mu)S_n(\zeta) + \frac{1}{\zeta - \mu}$$

Two methods of derivation.

Direct method: J. Harnad and A. Yu. Orlov, "Integrals of rational symmetric functions, two-matrix models and bi-orthogonal polynomials", *J. Math. Phys.* **48** (in press, Sept. 2007)

Fermionic vacuum state expectation values: J. Harnad and A.Yu. Orlov, "Fermionic approach to the evaluation of integrals of rational symmetric functions", arXiv:0704.1150)

Previous work.

Rational symmetric integrals; polynomial case: G. Akemann and G. Vernizzi, "Characteristic polynomials of complex random matrix models", *Nucl. Phys.* **B 660**, 532–556 (2003).

Complex matrix model; rational case: M. Bergère (hep-th/0404126)

Analogous results for one-matrix models: V.B. Uvarov, (1969) (general case), E. Brezin and S. Hikami, (2000), (polynomial integrals), Y.V. Fyodorov and E. Strahov (2003) ($N \geq M$),
J. Baik, P. Deift and E. Strahov, (2003) ($N \geq M$), A. Borodin and E. Strahov (2006))

Relation to integrable systems:

Deform the measure

$$d\Omega(M_1, M_2) \to d\Omega(M_1, M_2) e^{\operatorname{tr}(\sum_{j=0}^{\infty}(t_j M_1^j + \tilde{t}_j M_2^j)}$$
$$:= d\Omega(M_1, M_2, \mathbf{t}, \tilde{\mathbf{t}})$$

The deformed partition function

$$\mathbf{Z}_N^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}) := \int d\Omega(M_1, M_2, \mathbf{t}, \tilde{\mathbf{t}})$$

is a **2-Toda τ function**. The **biorthogonal polynomials** and the Hilbert transforms

$$\{P_j(x, \mathbf{t}, \tilde{\mathbf{t}}), \tilde{P}_j(y, \mathbf{t}, \tilde{\mathbf{t}})\}, \{\tilde{S}_j(x, \mathbf{t}, \tilde{\mathbf{t}}), S_j(y, \mathbf{t}, \tilde{\mathbf{t}})\}_{j \in \mathbf{N}}$$

are **Baker-Akhiezer** and **dual Baker-Akhiezer functions**.

Second result: Double Schur function perturbation expansion

$$\mathbf{Z}_N^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}) = N! \sum_{\lambda,\mu} B_{\lambda\mu} s_\lambda(\mathbf{t}) s_\mu(\tilde{\mathbf{t}})$$

where $s_\lambda(\mathbf{t})$, $s_\mu(\tilde{\mathbf{t}})$ are Schur functions corresponding to partitions $\lambda := (\lambda_1, \dots \lambda_{\ell(\lambda)})$, $\mu := (\mu_1, \dots \mu_{\ell(\mu)})$ of lengths $\ell(\lambda), \ell(\mu) \leq N$, and

$$B_{\lambda,\mu} = \det(B_{\lambda_i - i + N, \mu_j - j + N})\big|_{i,j=1,\dots,N},$$

Two methods of derivation.

Direct method:

J. Harnad and A.Yu. Orlov, "Scalar products of symmetric functions and matrix integrals", *Theor. Math. Phys.* **137**, 1676–1690 (2003).
J. Harnad and A. Yu. Orlov, "Matrix integrals as Borel sums of Schur function expansions", In: Symmetries and Perturbation theory SPT2002, eds. S. Abenda and G. Gaeta, World Scientific, Singapore, (2003).

Fermionic vacuum state expectation values:

J. Harnad and A. Yu. Orlov, "Fermionic construction of partition functions for two-matrix models and double Schur function expansions", *J. Phys. A* **39**, 8783–8809 (July 2006) math-phys/0512056

Early work:

V. A. Kazakov, M. Staudacher and T. Wynter "Character Expansion Methods for Matrix Models of Dually Weighted Graphs", *Commun. Math. Phys.* **177**, 451-468 (1996)

Direct method.

The key tools for the Schur function expansion are:

1. Cauchy Littlewood identity

$$e^{\sum_{i=1}^{\infty} it_i\tilde{t}_i} = \sum_{\lambda} s_\lambda(\mathbf{t})s_\lambda(\tilde{\mathbf{t}})$$

2. Andreief identity

$$\prod_{a=1}^{N} \iint d\mu(x_a, y_a)\mathrm{det}\phi_i(x_j)\mathrm{det}\psi_k(y_l)$$

$$= N!\mathrm{det}\left(\int\int d\mu(x,y)\phi_i(x)\psi_j(y)\right)$$

$(1 \leq i, j, k, l \leq N)$

For the integrals of rational symmetric functions:

3. Multivariable partial fraction expansions

For $N \geq M$:

$$\frac{\Delta_N(x)\Delta_M(\eta)}{\prod_{a=1}^N \prod_{j=1}^M (\eta_j - x_a)} =$$

$$(-1)^{MN} \sum_{\sigma \in S_M} \mathrm{sgn}(\sigma) \sum_{a_1 < \ldots < a_M}^{N} (-1)^{\sum_{j=1}^M a_j} \frac{\Delta_{N-M}(x[\mathbf{a}])}{\prod_{j=1}^M (\eta_{\sigma_j} - x_{a_j})}$$

For $N \leq M$:

$$\frac{\Delta_N(x)\Delta_M(\eta)}{\prod_{a=1}^N \prod_{j=1}^M (\eta_j - x_a)} =$$

$$\frac{(-1)^{\frac{1}{2}N(N-1)}}{(M-N)!} \sum_{\sigma \in S_M} \mathrm{sgn}(\sigma) \frac{\Delta_{M-N}(\eta_{\sigma_{N+1}}, \ldots, \eta_{\sigma_M})}{\prod_{a=1}^N (\eta_{\sigma_a} - x_a)}$$

4. Cauchy-Binet identity

If V is an oriented Euclidean vector space with volume form Ω and $(P^1, \ldots, P^L)$, $\quad (S^1, \ldots, S^L)$ are two sets of L vectors, then the scalar product of their exterior products $(\wedge_{\alpha=1}^{L} P^{\alpha}, \wedge_{\beta=1}^{L} S^{\beta})$ is equal to the determinant of the matrix formed from the scalar products:

$$(\wedge_{\alpha=1}^{L} P^{\alpha}, \wedge_{\beta=1}^{L} S^{\beta}) = \det G$$

$$G^{\alpha\beta} := (P^{\alpha}, S^{\beta}), \quad 1 \leq i, j \leq L$$

Remark: In the fermionic approach, this is just the **Wick theorem**

Fermionic approach: vacuum state expectation values (VEV)

Two-component fermions.

$$[f_n^{(\alpha)}, f_m^{(\beta)}]_+ = [\bar{f}_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = 0, \quad [f_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = \delta_{\alpha,\beta}\delta_{nm}, \qquad \alpha = 1,2$$

Fermionic fields.

$$f^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^k f_k^{(\alpha)} \ , \quad \bar{f}^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^{-k-1} \bar{f}_k^{(\alpha)}$$

Right and left vacuum vectors. $|0,0\rangle$, $\langle 0,0|$

$$f_m^{(\alpha)}|0,0\rangle = 0 \qquad (m<0), \qquad \bar{f}_m^{(\alpha)}|0,0\rangle = 0 \qquad (m \geq 0),$$

$$\langle 0,0|f_m^{(\alpha)} = 0 \qquad (m \geq 0), \qquad \langle 0,0|\bar{f}_m^{(\alpha)} = 0 \qquad (m<0)$$

Wick's theorem implies, for linear elements of the Clifford algebra

$$\langle 0,0|w_1 \cdots w_N \bar{w}_N \cdots \bar{w}_1|0,0\rangle = \det\left(\langle 0,0|w_i \bar{w}_j|0,0\rangle\right)|_{i,j=1,\dots,N}$$

Charged vacuum states

$$|n^{(1)}, n^{(2)}\rangle := \bar{C}_{n^{(2)}} \bar{C}_{n^{(1)}} |0, 0\rangle$$

where

$$\bar{C}_{n^{(\alpha)}} := f^{(\alpha)}_{n^{(\alpha)}-1} \cdots f^{(\alpha)}_0 \quad \text{if} \quad n^{(\alpha)} > 0$$
$$\bar{C}_{n^{(\alpha)}} := \bar{f}^{(\alpha)}_{n^{(\alpha)}} \cdots \bar{f}^{(\alpha)}_{-1} \quad \text{if} \quad n^{(\alpha)} < 0, \quad \bar{C}_0 := 1$$

$gl(\infty)$ operators

$$\mathcal{A} := \int\int f^{(1)}(x) \bar{f}^{(2)}(y) d\mu(x, y), \qquad H(\mathbf{t}, \tilde{\mathbf{t}}) := \sum_{k=1}^{\infty} H_k^{(1)} t_k - \sum_{k=1}^{\infty} H_k^{(2)} \tilde{t}_k$$

where

$$H_k^{(\alpha)} := \sum_{n=-\infty}^{+\infty} f_n^{(\alpha)} \bar{f}^{(\alpha)}_{n+k} \,, \quad k \neq 0, \quad \alpha = 1, 2.$$

are two sequences of commuting operators.

$\mathbf{Z}_N^{(2)}(\mathbf{t},\tilde{\mathbf{t}})$ **is a 2-Toda τ function** (VEV)

$$\begin{aligned}\tau_N(\mathbf{t},\tilde{\mathbf{t}}) &:= \langle N,-N|e^{H(\mathbf{t},\tilde{\mathbf{t}})}e^{A}|0,0\rangle \\ &= \frac{1}{N!}(-1)^{\frac{1}{2}N(N+1)}\mathbf{Z}_N^{(2)}(\mathbf{t},\tilde{\mathbf{t}})\end{aligned}$$

To prove this, we use:

$$\langle N,-N|\prod_{i=1}^{N} f^{(1)}(x_i)\bar{f}^{(2)}(y_i)|0,0\rangle = (-1)^{\frac{1}{2}N(N+1)}\Delta_N(x)\Delta_N(y)$$

$$\langle N,-N|\prod_{i=1}^{k} f^{(1)}(x_i)\bar{f}^{(2)}(y_i)|0,0\rangle = 0 \quad \text{if } k\neq N, \qquad \langle N,-N|A^k|0,0\rangle = 0 \quad \text{if} \quad k\neq N$$

and

$$\langle N,-N|e^{H(\mathbf{t},\tilde{\mathbf{t}})} f^{(1)}_{h_1}\bar{f}^{(2)}_{-h'_1-1}\cdots f^{(1)}_{h_N}\bar{f}^{(2)}_{-h'_N-1}|0,0\rangle = (-1)^{\frac{1}{2}N(N+1)}s_\lambda(\mathbf{t})\mathbf{s}_\mu(\tilde{\mathbf{t}})$$

$$h_i := \lambda_i - i + N, \quad h'_j := \mu_j - j + N$$

3. Random Processes (J. H. and A. Yu. Orlov, "Fermionic construction of tau functions and random processes", arXiv:0704.1157)

3.1. Maya diagrams as basis for Fock space $\mathcal{F} := \Lambda\mathcal{H}$.

Choose a basis $\{e_i\}_{i\in\mathbf{Z}}$ for $\mathcal{H}$, dual basis $\{\tilde{e}_i\}_{i\in\mathbf{Z}}$ and represent f_i (creation) and $\bar{f}_i$ (annihilation) operators on $\mathcal{F}$ by:

$$f_j := i(\tilde{e}_j), \quad \bar{f}_j := e_j\wedge$$

For each integer N, and partition λ of length $\ell(\lambda)$

$$\lambda := \lambda_1 \geq \lambda_2 \geq \dots \quad \lambda_i \in \mathbf{N} \qquad \lambda_{\ell(\lambda)} \neq 0, \quad \lambda_i = \forall i > \ell(\lambda)$$

define **particle positions** (levels): $l_i := \lambda_i - i + N$ to form a "Maya Diagram"
and a **basis vector**: $|\lambda, N >= e_{-l_1-1} \wedge e_{-l_2-1} \wedge \cdots,$ $\qquad |N >:= |0, N >$ (N charge vacuum)

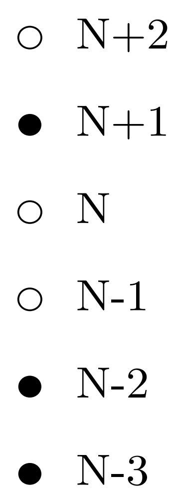


Fig.1 Maya diagram for $|(2,1);N>$

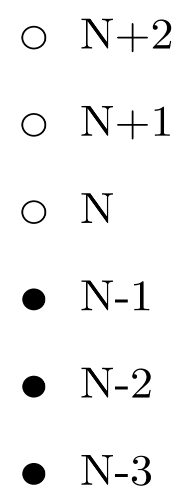


Fig.2 Dirac sea of level N. $|0;N>$

$gl(\infty)$ **action on** $\mathcal{F}$

$$gl(\infty) : \mathcal{F} \to \mathcal{F}$$

$$gl(\infty) = \mathrm{span}\{E_{ij} := f_i \bar{f}_j\}_{i,j \in \mathbf{Z}}$$

This determines weighted actions on Maya diagrams

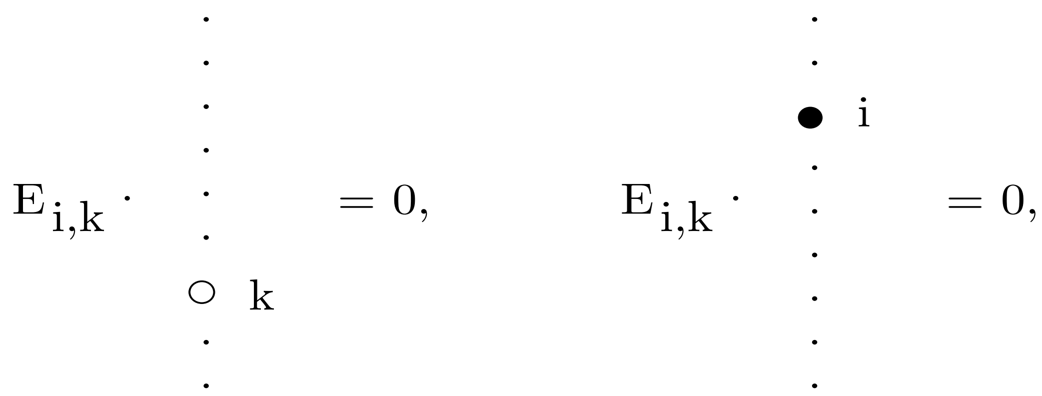
$$\mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j$$

$$\mathcal{A} : |\lambda; N> \to \sum_{ij} a_{ij} f_i \bar{f}_j |\lambda; N> = \sum_{N',\mu} C_{\mu\lambda}^{N'N} |\mu, N'>$$

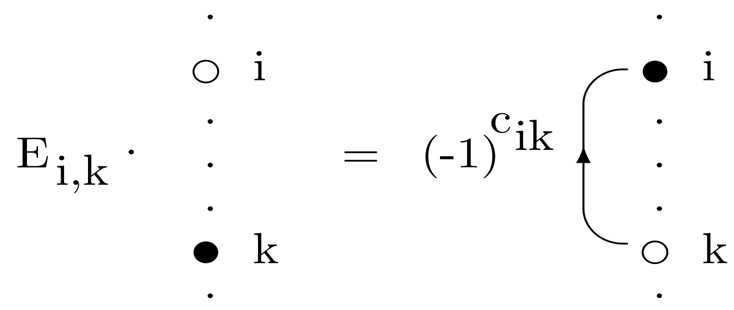
For positive coefficients a_{ij}, we can view

$$< \lambda, N'|\mathcal{A}^k|\mu, N >$$

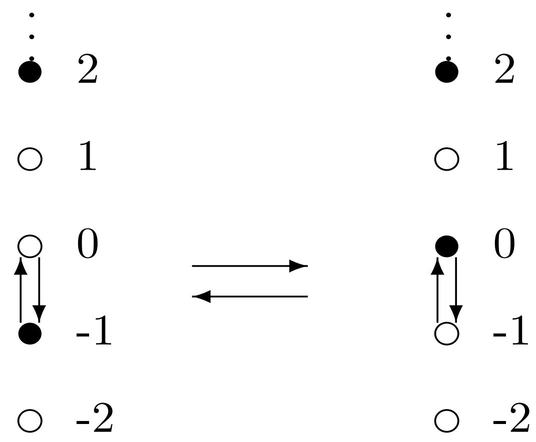
as an (unnormalized) transition rate in k (discrete) time steps.



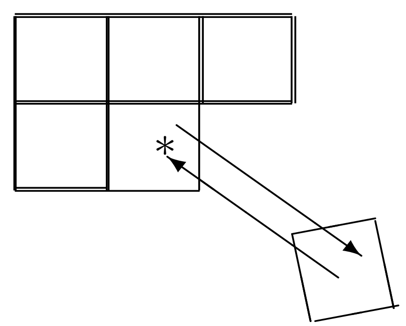
1. Elimination of Maya diagrams



2. Nontrivial action



1. Upward/downward hop
of a particle on a Maya diagram



2. Adding/removing boxes on a Young diagram

3.2 The τ function as a generating function for transition probabilities

Recall

$$\tau_{N,g}(\mathbf{t},\tilde{\mathbf{t}}) :=< N|\gamma(\mathbf{t})e^{\mathcal{A}}\tilde{\gamma}(\tilde{\mathbf{t}})|N >$$

$$\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i H_i}, \quad \tilde{\gamma}(\tilde{\mathbf{t}}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i H_{-i}}$$

$$H_i := \sum_{j\in\mathbf{Z}} f_i \bar{f}_{j+i}, \quad \mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j$$

Assuming $\mathcal{A}$ preserves N, and use

$$< N|\tilde{\gamma}(\tilde{\mathbf{t}})|\mu, N >= s_\mu(\tilde{\mathbf{t}}) \quad < \lambda, N|\gamma(\mathbf{t})|N >= s_\lambda(\mathbf{t})$$

The τ function may be expanded:

$$\tau_{N,g}(\mathbf{t},\tilde{\mathbf{t}}) = \sum_{k=0}^{\infty} \frac{1}{k!} < \lambda, N|\mathcal{A}^k|\mu, N > s_\mu(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}})$$

$$< \lambda, N|\mathcal{A}^k|\mu, N > := \frac{1}{k!} < \lambda, N|\mathcal{A}_+^k + \mathcal{A}_-^k|\mu, N > \qquad (\pm \text{ permutations})$$

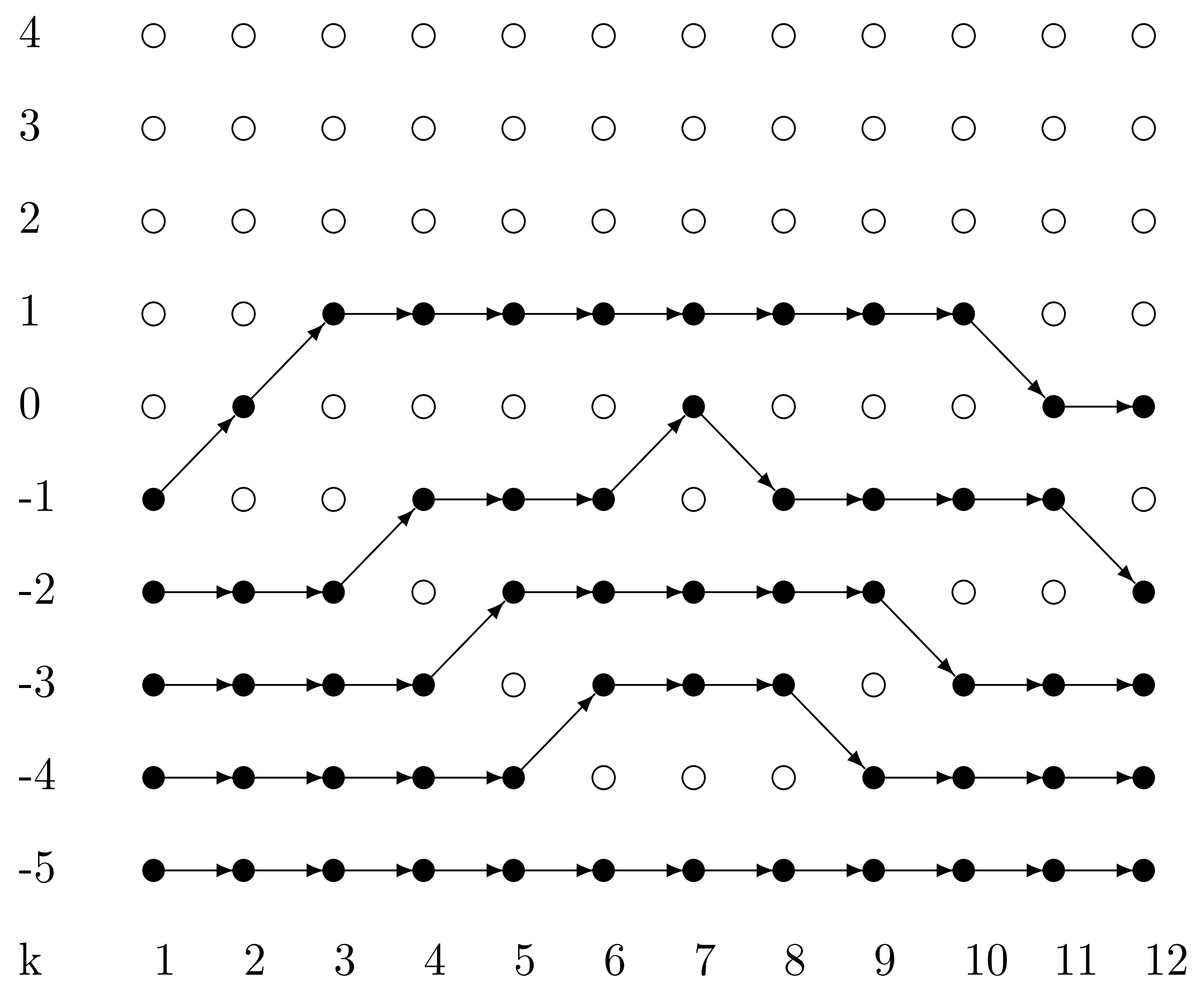
Transition probability in k time steps:

$$P_k((\mu, N) \to (\lambda, N)) = \frac{W_{N,g}^k(\lambda, \mu)}{\sum_\nu W_{N,g}^k(\nu, \mu)},$$

$$W_{N,g}^k(\nu, \mu) := < \lambda, N|\mathcal{A}_+^k|\mu, N > - < \lambda, N|\mathcal{A}_-^k|\mu, N >$$

Example. Random turn non-intersecting walkers

$$\mathcal{A} := \sum_{i \in \mathbf{Z}} (p_l f_{i-1} \bar{f}_i + p_r f_{i+1} \bar{f}_i), \qquad p_l, p_r \geq 0, \quad p_l + p_r = 1$$



Random turn non-intersecting walkers.

$$\mathcal{A} = \sum_i (p_l f_{i-1} \bar{f}_i + p_r f_{i+1} \bar{f}_i)$$

Asymmetric Exclusion Process (ASEP)

Recent results of **Tracy and Widom** (ArXiv0704.2633v2 [math.PR]):

Continuous time limit (ASEP) for a finite number of particles.

Particle positions:

$$X = (x_1, \dots x_n), \qquad Y = (y_1, \dots y_n)$$

Let $u_Y(X;t) :=$ probability of being in state X at time t, given they are in state Y at $t = 0$.

Master equation of ASEP

$$\frac{du_Y}{dt} = \sum_{i=1}^{n} \big(p_r u_Y(X_i^-;t) + p_l u_Y(X_i^+;t) - u(X;t)\big)$$

$$X_i^- := (x_1, \dots x_{i-1}, x_i - 1, \dots, x_n), \quad X_i^+ := (x_1, \dots x_{i-1}, x_i + 1, \dots, x_n)$$

Initial condition and **boundary** conditions

$$u_Y(X;0) = \delta_{X,Y}$$

$$u_Y((x_1, \dots x_i, x_i + 1, \dots x_n);t) = p_r u_Y((x_1, \dots, x_i, x_i, \dots x_n);t) + p_l u_Y((x_1, \dots, x_i + 1, x_i + 1, \dots x_n);t)$$

Bethe ansatz solution:

$$u_Y(X;t) = \sum_{\sigma\in S_n} \left(\frac{1}{2\pi i}\right)^n \oint_{\xi 1=0} \cdots \oint_{\xi_n=0} A_\sigma \prod_{i=1}^n \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\sum_{i=1}^n \epsilon(\xi_i)t} d\xi_i$$

$$A_\sigma := \prod_{\sigma|\text{inversion}} \prod_{(\alpha,\beta)\in\sigma} \left(-\frac{p_r + p_l\xi_\alpha\xi_\beta - \xi_\alpha}{p_r + p_l\xi_\alpha\xi_\beta - \xi_\beta}\right)$$

$$\epsilon(\xi) := p\xi^{-1} + q\xi - 1$$

Other relations of random processes to integrable systems and τ functions

Bethe ansatz solution of ASEP using equivalence with spin models

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Tau functions as weights on 2-D partitions (path space weight for 1-D partitions

A. Okounkov, N. Reshetikhin, "Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram "*J. Amer. Math. Soc.* **16** 581-603 (2003);, "Random skew plane partitions and the Pearcey process " *Commun. Math. Phys.* **269**, (2007);

Asymptotics of random partitions, growth problems, limiting shapes

A. Borodin and G. Olshanski "Z-measures on partitions and their scaling limits " *Eyr. J. Comb.* **26**, 795-834 (2005); A. Borodin and G. Olshanski Random partitions and the gamma kernel *Adv. Math.* **194** , 141-202 (2005)
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