Integrable Lattice Equations

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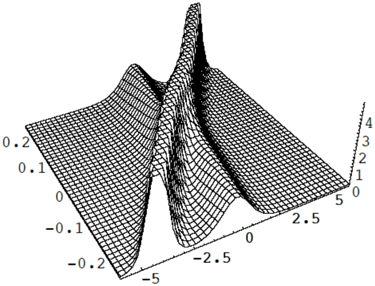
Soliton Equations

- Korteweg-de Vries Equation:

$$\frac{\partial u}{\partial t} + 6\,u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0\,.$$

  has N-soliton solutions.

- What discrete version preserves the special properties of the KdV?



The Lattice KdV Equation

- Consider two solutions of the KdV equation given by $u = w_x$ and $\widetilde{u} = \widetilde{w}_x$, related by

$$BT_\lambda: \ \left(\widetilde{w}+w\right)_x = 2\lambda - \frac{1}{2}\left(\widetilde{w}-w\right)^2$$

- Imagine two such transformations

$$BT_\lambda : w \overset{\lambda}{\mapsto} \widetilde{w}$$
$$BT_\mu : w \overset{\mu}{\mapsto} \widehat{w}$$

- Their compositions give

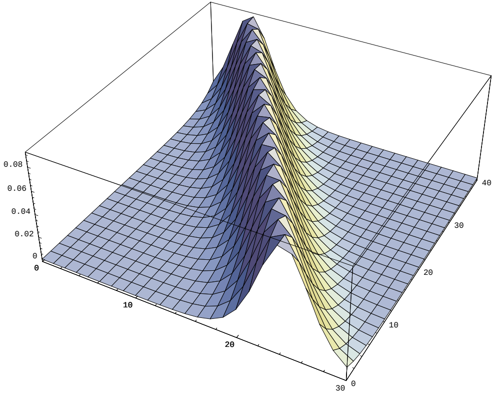
$$\widehat{\widetilde{w}} = BT_\mu \circ BT_\lambda w, \widetilde{\widehat{w}} = BT_\lambda \circ BT_\mu w$$

- Demanding $\widehat{\widetilde{w}} = \widetilde{\widehat{w}}$ leads to

$$\boxed{\left(\widehat{\widetilde{w}} - w\right)\left(\widehat{w} - \widetilde{w}\right) = 4(\mu - \lambda)}$$

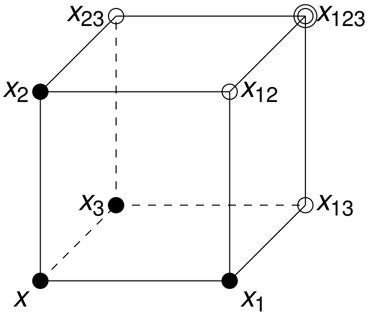
Evolves on a lattice with coordinates (n, m), where $w = w_{n,m}$, $\widehat{w} = w_{n,m+1}$, $\widetilde{w} = w_{n+1,m}$.

Discrete Solitons



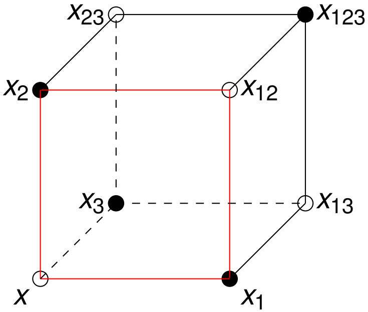
- $w = a\,m + b\,n + k\,\tanh\big(k\,x + \beta\,m + \gamma\,n + \xi\big)$
  $a^2 - b^2 = 4(\mu - \lambda)$
  $\beta = \frac{1}{2}\log\big((a+k)/(a-k)\big)$,
  $\gamma = \frac{1}{2}\log\big((b+k)/(b-k)\big)$
- The picture shows $\partial_x w$ (with $a = 1$, $b = 2$, $k = 0.3$, $x = 0$, $\xi = -7.5$).
- There are also multi-solitons.
- The continuum limit of the dKdV is $u_t = u_{xxx} + 3\,u_x^2$, $u = w_x$.
- The consistency condition: $\widehat{\widetilde{w}} = \widetilde{\widehat{w}}$ can be extended to many other integrable equations.

Multi-dimensional Consistency



- Start with ● at x, x_1, x_2, x_3.
- Calculate ○ x_{12}, x_{13}, and x_{23}.
- There are three ways of calculating x_{123}.
- Demand that these all give the same value ("Consistency around a cube"or CAC).

Conditions for Classification



Consider the base tile to be the red one, with lattice equation

$$Q(x, x_1, x_2, x_{12}; \alpha, \beta) = 0$$

- Q is linear in each variable.
- Q is (anti-)symmetric when $(x, x_1, x_2, x_{12}, \alpha, \beta) \mapsto (x, x_2, x_1, x_{12}, \beta, \alpha)$, and $(x, x_1, x_2, x_{12}, \alpha, \beta) \mapsto (x_1, x, x_{12}, x_2, \alpha, \beta)$.
- x_{123} does not depend on x (the "tetrahedron" property).

Classification Results

Nine canonical classes (equivalent under Möbius transformations) were obtained by Adler et al (2003). Four of these are

$$
\begin{aligned}
Q_1: \quad & \alpha(x_{n,m} - x_{n,m+1})(x_{n+1,m} - x_{n+1,m+1}) \\
& \quad + \beta(x_{n,m} - x_{n+1,m})(x_{n,m+1} - x_{n+1,m+1}) + \gamma = 0 \\
Q_2: \quad & \alpha(x_{n,m} - x_{n,m+1})(x_{n+1,m} - x_{n+1,m+1}) \\
& \quad + \beta(x_{n,m} - x_{n+1,m})(x_{n,m+1} - x_{n+1,m+1}) \\
& \quad + \gamma(x_{n,m} + x_{n,m+1} + x_{n+1,m} + x_{n+1,m+1}) + \delta = 0 \\
Q_3: \quad & \alpha(x_{n,m}x_{n+1,m+1} + x_{n,m+1}x_{n+1,m}) \\
& \quad + \beta(x_{n,m}x_{n+1,m} + x_{n,m+1}x_{n+1,m+1}) \\
& \quad + \gamma(x_{n,m}x_{n,m+1} + x_{n+1,m}x_{n+1,m+1}) + \delta = 0 \\
Q_4: \quad & sn\,\alpha\,(x_{n,m}x_{n+1,m+1} + x_{n,m+1}x_{n+1,m}) \\
& \quad - sn\,\beta\,(x_{n,m}x_{n+1,m} + x_{n,m+1}x_{n+1,m+1}) \\
& \quad - sn(\alpha - \beta)\,(x_{n,m}x_{n,m+1} + x_{n+1,m}x_{n+1,m+1}) \\
& \quad + sn\,\alpha\, sn\,\beta\, sn(\alpha - \beta)\,(1 + k^2 x_{n,m}x_{n,m+1}x_{n+1,m}x_{n+1,m+1}) = 0
\end{aligned}
$$

Without the Tetrahedron Property

- Hietarinta (2004) showed that CAC without the tetrahedron property leads to other equations such as

$$\frac{(x_{n,m}+b)}{(x_{n,m}+a)}\,\frac{(x_{n+1,m+1}+d)}{(x_{n+1,m+1}+c)}=\frac{(x_{n+1,m}+b)}{(x_{n+1,m}+c)}\,\frac{(x_{n,m+1}+d)}{(x_{n,m+1}+a)}$$

- This is linearizable! (Ramani, J., Grammaticos and Tamizhmani, 2006) to

$$R_{n+1,m+1}-AR_{n,m+1}-R_{n+1,m}+(A-B)R_{n,m}=0$$

where $B=(d-b)/(b-c)$, $A=(d-a)/(a-c)$, and $x_{n,m}$ is found by taking

$$x_{n,m}=-\frac{a\,(c-d)R_{n,m}+c\,(a-c)R_{n+1,m}}{(c-d)R_{n,m}+(a-c)R_{n+1,m}}$$

- It is now believed that all CAC systems without the tetrahedron property are linearizable.

Reductions

- Q_4 is the generic equation from which all others can be obtained as limits.
- Impose $x_{n,m+1} = x_{n+1,m}$:

$$(\operatorname{sn}\alpha - \operatorname{sn}\beta)x_n(x_{n+1} + x_{n-1}) - \operatorname{sn}(\alpha-\beta)(x_{n+1}x_{n-1} + x_n^2)$$
$$+\operatorname{sn}\alpha\operatorname{sn}\beta\operatorname{sn}(\alpha-\beta)(1 + k^2x_n^2x_{n+1}x_{n-1}) = 0$$

- This is integrable, because there is a “conserved”quantity

$$K = \frac{((1 + k^2x_{n+1}^2x_n^2)\operatorname{sn}\alpha\operatorname{sn}\beta - x_{n+1}^2 - x_n^2)\operatorname{sn}(\alpha-\beta) + 2x_{n+1}x_n(\operatorname{sn}\alpha - \operatorname{sn}\beta)}{((1 + k^2x_{n+1}^2x_n^2)\operatorname{sn}\alpha\operatorname{sn}\beta + x_{n+1}^2 + x_n^2)(\operatorname{sn}\alpha - \operatorname{sn}\beta) + 2x_{n+1}x_n\operatorname{sn}(\alpha-\beta)(k^2\operatorname{sn}^2\alpha\operatorname{sn}^2\beta - 1)}$$

where $K(x_n, x_{n-1}) = -K(x_{n+1}, x_n)$.

First Surprise

- Reductions of integrable systems are usually integrable with conserved quantities of the form

$$K(x,y) = \frac{\alpha_0 y^2 x^2 + \beta_0 yx(y+x) + \gamma_0(y^2+x^2) + \epsilon_0 yx + \zeta_0(y+x) + \mu_0}{\alpha_1 y^2 x^2 + \beta_1 yx(y+x) + \gamma_1(y^2+x^2) + \epsilon_1 yx + \zeta_1(y+x) + \mu_1}$$

  where $K(x_n, x_{n-1}) = K(x_{n+1}, x_n)$, called QRT invariants. This gives the iteration of the difference equation as iteration along an elliptic curve.
- Instead we have invariant curves that are products of two curves of QRT-type.

Second Surprise

- The conservation of the Q_4 reduction holds even if the system is of the form

$$(A-B)x_n(x_{n+1}+x_{n-1}) - C\,(x_{n+1}x_{n-1}+x_n^2) \\ +A\,B\,C\,(1+k^2x_n^2x_{n+1}x_{n-1}) = 0$$

  where A, B and C do not have to lie on an elliptic curve.

- Recently, Viallet has also found that the algebraic entropy of Q_4 is bounded regardless of whether its parameters lie on an elliptic curve.

- But CAC does not recognize this generality. Why not?

Lax Pairs

- The Lattice modified KdV

$$LMKdV: \quad x_{l+1,m+1} = x_{l,m}\,\frac{\left(x_{l+1,m} - r\,x_{l,m+1}\right)}{\left(x_{l,m+1} - r\,x_{l+1,m}\right)}$$

has a non-autonomous form given by $r(l,m) = \mu(m)/\lambda(l)$.

- This has Lax pair

$$\begin{aligned} v(l+1,m) &= L(l,m)v(l,m), \\ v(l,m+1) &= M(l,m)v(l,m). \end{aligned}$$

where, using the notation $\bar{v} = v(l+1,m)$ and $\hat{v} = v(l,m+1)$,

$$\begin{aligned} L &= \begin{pmatrix} \bar{x}/x & -\lambda/(\nu x) \\ -\lambda\bar{x}/\nu & 1 \end{pmatrix}, \\ M &= \begin{pmatrix} \hat{x}/x & -\mu/(\nu x) \\ -\mu\hat{x}/\nu & 1 \end{pmatrix}. \end{aligned}$$

Nonautonomous Reductions

- $\hat{x} = \bar{\bar{x}}$ reduces the LMKdV equation to qP_{II}

$$\bar{y}\underline{y} = \frac{1 - ry}{y(y - r)}.$$

where $y = \bar{\bar{x}}/\bar{x}$, and $\log r = al + b + c(-1)^l$.

- $\hat{x} = 1/\bar{x}$ reduces the LMKdV to qP_{III}

$$\bar{\bar{x}}x = \frac{\beta\gamma^l\bar{x}^2 - 1}{\beta\gamma^l - \bar{x}^2}$$

where $r = \beta\gamma^l$

- Many, many other reductions are possible, including cases of higher-order.

Reduced Lax Pairs

- The above reductions provide 2×2 Lax pairs for q-Painlevé equations.
- The Lax pair

$$L = \begin{pmatrix} \frac{\bar{x}}{x} & -\frac{\lambda}{\nu x} \\ \frac{-\lambda \bar{x}}{\nu} & 1 \end{pmatrix},$$

and

$$N = \begin{pmatrix} -\frac{1}{\nu}(\lambda\beta x\bar{x} + \frac{\alpha\bar{x}}{\lambda\sigma x}) & \beta x + \frac{\alpha}{\nu^2\sigma x} \\ \frac{\gamma}{x} + \frac{\alpha x}{\nu^2\sigma} & -\frac{1}{\nu}(\frac{\lambda\gamma}{x\bar{x}} + \frac{\alpha x}{\lambda\sigma\bar{x}}) \end{pmatrix}$$

is the first known 2×2 Lax pair for

$$qP_{III}: \quad x\bar{\bar{x}} = \frac{\mu_1 q^l \bar{x}^2 + \mu_2}{\mu_3 q^l + \bar{x}^2}$$

Summary

- A class of two-dimensional lattice equations have been derived through the property of multidimensional consistency.
  Are these complete?
- Reductions of these lead to difference equations with unexpected properties. These suggest that much more could be done.
  Can Q_4 be generalized?
- Reductions of Lax pairs are also possible.
  How do such reductions fit into the consistency around a cube property?

The Continuum Limit

- $\boxed{\left(\widehat{\widetilde{w}} - w\right)\left(\widehat{w} - \widetilde{w}\right) = 4(\mu - \lambda)}$ has a continuum limit.

$$\left.\begin{array}{l} \mu - \lambda = \delta\nu \\ \tau = \delta m \\ l = n + m \\ w(n,m) = v(l,\tau) \\ \widehat{w} = v + \delta\,\partial_\tau v + \dots \end{array}\right\} \Rightarrow\ v_\tau\left(\overline{\overline{v}} - v\right) = 2\nu$$

where $\overline{v} = v(l+1,\tau)$.

- Now take $v(l,\tau) = \tau + l\,p + u(l,\tau)$, $2\nu = -2\,p$. Then

$$\left.\begin{array}{l} p = 1/\epsilon, \epsilon \to 0 \\ 2n\epsilon + 2\tau\epsilon^2 \to x \\ 2n\epsilon^3/3 + 2\tau\epsilon^4 \to t \end{array}\right\} \Rightarrow\ u_t = u_{xxx} + 3\,u_x^2$$

- The consistency condition: $\widehat{\widetilde{w}} = \widetilde{\widehat{w}}$ can be extended to many other integrable equations.