A Remark on the Hankel Determinant Formula for the Solutions of the Toda Equation

Kenji Kajiwara

Faculty of Mathematics, Kyushu University

Based on the collaboration with:

Marta Mazzocco

Department of Mathematics, University of Manchester

Yasuhiro Ohta

Department of Mathematics, Kobe University

ISLAND 3 meeting, 5th July 2007

Introduction: Painlevé II Case

Determinant formula for Yablonski-Vorob'ev polynomials

$$\mathrm{P_{II}}: \quad \frac{d^2u}{dt^2} = 2u^3 - 4tu + 4\left(\alpha + \frac{1}{2}\right)$$

Yablonski-Vorob'ev polynomials T_n:

$$T_n''T_n - (T_n')^2 = T_{n+1}T_{n-1} - t\,T_n^2 : \quad \textbf{Toda equation}$$

$$\begin{aligned}
&T_0 = 1, \quad T_1 = t,\\
&T_2 = t^3 - 1,\\
&T_3 = t^6 - 5t^3 - 5,\\
&T_4 = t(t^9 - 15t^6 - 175), \cdots
\end{aligned}$$

$$\downarrow$$

Rational solutions : $u = \dfrac{d}{dt}\log\dfrac{T_{n+1}}{T_n}, \quad \alpha = n + \dfrac{1}{2}$

What is Y-V polynomials? $\to$ Determinant Formula (K-Ohta 1996)

Jacobi-Trudi Type:

$$T_n \propto \begin{vmatrix} p_N & p_{N+1} & \cdots & p_{2N-1} \\ p_{N-2} & p_{N-1} & \cdots & p_{2N-3} \\ \vdots & \vdots & \ddots & \vdots \\ p_{-N+2} & \cdots & p_0 & p_1 \end{vmatrix}$$

$$\sum_{k=0}^{\infty} p_n(t)\lambda^n = \exp\left[t\lambda + \frac{\lambda^3}{3}\right]$$

$$\downarrow$$

Specialization of Schur function!

Hankel Type:

$$T_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix}$$

$$a_0 = t, \quad a_n = a'_{n-1} + \sum_{k=0}^{n-2} a_k a_{n-2-k}$$

$$\downarrow$$

$$\boxed{\textbf{What does this formula mean?}}$$

Generating function for a_n: log derivative of the Airy fn. (Iwasaki-K-Nakamura, 2002)

$$\frac{\partial}{\partial\lambda}\log\theta(x,t) \sim \sum_{n=0}^{\infty} a_n(x)\,\lambda^{-n}, \quad \theta(x,\lambda) = \exp\left(-\frac{\lambda^3}{12}\right)\mathrm{Ai}\Big(\frac{\lambda^2}{4} - x\Big), \quad |\arg\lambda| < \frac{\pi}{2}.$$

Generic solutions for $\mathrm{P_{II}}$: (K-Masuda 1998, Joshi-K-Mazzocco 2004)

Hankel determinant formula:

$$\tau_n = \begin{cases} \det(a_{i+j-2})_{i,j\le n} & n>0, \\ 1 & n=0, \\ \det(b_{i+j-2})_{i,j\le |n|} & n<0, \end{cases}$$

$$a_n = a'_{n-1} + \psi_{-1}\sum_{k=1}^{n-2} a_k a_{n-2-k}, \quad a_0 = \psi_1,$$

$$b_n = b'_{n-1} + \psi_1\sum_{k=1}^{n-2} b_k b_{n-2-k}, \quad b_0 = \psi_{-1}$$

$\psi_{\pm 1}$:

$$\begin{cases} \dfrac{\psi''_{-1}}{\psi_{-1}} = \dfrac{\psi''_1}{\psi_1} = -2\psi_{-1}\psi_1 + 2t, \\ \psi'_1\psi_{-1} - \psi_1\psi'_{-1} = 2\alpha. \end{cases}$$

- ☞ $u_0 = (\log\psi_1)'$ satisies $\mathrm{P_{II}}[\alpha]$.
- ☞ $u_{-1} = -(\log\psi_{-1})'$ satisfies $\mathrm{P_{II}}[\alpha - 1]$.
- ☞ $u_N = \left(\log\frac{\tau_{N+1}}{\tau_N}\right)'$ satisfies $\mathrm{P_{II}}[\alpha + N]$.

Generating functions: linear problem

$$F(t,\lambda) = \sum_{n=0}^{\infty} a_n\,(-\lambda)^{-n}, \quad G(t,\lambda) = \sum_{n=0}^{\infty} b_n\,\lambda^{-n}.$$

$$\downarrow$$

G: Linear problem for $\mathrm{P_{II}}[\alpha]$ (Jimbo-Miwa)

$$\boxed{G = \lambda\,\frac{Y_2}{Y_1},} \quad \frac{\partial}{\partial\lambda}Y = AY, \quad \frac{\partial}{\partial t}Y = BY, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac14 & 0 \\ 0 & -\frac14 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & -\frac{\psi_1}{2} \\ \frac{\psi_{-1}}{2} & 0 \end{pmatrix}\lambda + \begin{pmatrix} -\frac{z+t}{2} & \frac{\psi_1 u_0}{2} \\ -\frac{\psi_{-1}u_{-1}}{2} & \frac{z+t}{2} \end{pmatrix},$$

$$B = \begin{pmatrix} -\frac12 & 0 \\ 0 & \frac12 \end{pmatrix}\lambda + \begin{pmatrix} 0 & \psi_1 \\ -\psi_{-1} & 0 \end{pmatrix}, \quad z = -\psi_{-1}\psi_1$$

F: Adjoint problem for $\mathrm{P_{II}}[\alpha]$

Toda equation: $\dfrac{dV_n}{dt} = V_n(I_n - I_{n+1}), \quad \dfrac{dI_n}{dt} = V_{n-1} - V_n$

Soliton solution: $n \in \mathbb{Z}$

$$V_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad I_n = \frac{d}{dt}\log\frac{\tau_{n-1}}{\tau_n}$$

$$\tau_n = \left.\begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}\right\} N$$

$$f_n^{(k)} = p_k^n e^{p_k t + p_{k,0}} + p_k^{-n} e^{\frac{1}{p_k}t + p_{k,1}}$$

bilinear equation:

$$\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1} - \tau_n^2$$

determinant size = **number of solitons**

Molcule type solution: $0 \le n \ (\le M)$

boundary condition: $V_0 = 0 \ (V_M = 0)$

$$\tau_n = \left.\begin{vmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix}\right\} n$$

$$\tau_0 = 1, \quad a_k' = a_{k+1} \quad \left(a_0 = \sum_{j=0}^{M} c_j e^{\mu_j t}\right)$$

bilinear equation:

$$\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1}$$

determinant size = **lattice site**

Determinant formula of soliton type : (incl. rational solution)

$$\boxed{\text{Meaning in the framework of the KP theory: Clear}}$$

Determinant formula of molecule type: usually exists for discrete variable

$$\boxed{\text{Meaning in the framework of the KP theory: ?}}$$

Physical meaning of semi-infinite(finite) Toda lattice:

$$\frac{dV_n}{dt} = V_n(I_n - I_{n+1}), \quad \frac{dI_n}{dt} = V_{n-1} - V_n, \quad V_0(= V_N) = 0$$

$\longrightarrow$ short-circuit the edge of LC ladder circuit $\longrightarrow V_n(\infty) = I_n(\infty) = 0$

- Not so much interesting physically
- Important in relation to orthogonal functions, Painlevé systems, numerical analysis,...

Toda equation: $\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1}, \quad n \in \mathbb{Z}$

Generalization of Hankel determinant formula of molecule type (KMNOY:1998)

$$\frac{\tau_{k+n}}{\tau_k} = \begin{cases} \det(a^{(k)}_{i+j-2})_{i,j=1,\dots,n} & n > 0, \\ 1 & n = 0, \\ \det(b^{(k)}_{i+j-2})_{i,j=1,\dots,|n|} & n < 0, \end{cases}$$

$$\begin{cases} a^{(k)}_i = a^{(k)\prime}_{i-1} + \dfrac{\tau_{k-1}}{\tau_k}\displaystyle\sum_{l=0}^{i-2} a^{(k)}_l a^{(k)}_{i-2-l}, & a^{(k)}_0 = \dfrac{\tau_{k+1}}{\tau_k}, \\ b^{(k)}_i = b^{(k)\prime}_{i-1} + \dfrac{\tau_{k+1}}{\tau_k}\displaystyle\sum_{l=0}^{i-2} b^{(k)}_l b^{(k)}_{i-2-l}, & b^{(k)}_0 = \dfrac{\tau_{k-1}}{\tau_k}, \end{cases}$$

$$\begin{array}{ccccccccc} \cdots & \tau_{k-3} & \tau_{k-2} & \tau_{k-1} & \tau_k & \tau_{k+1} & \tau_{k+2} & \tau_{k+3} & \cdots \\ \cdots & \frac{\tau_{k-3}}{\tau_k} & \frac{\tau_{k-2}}{\tau_k} & \frac{\tau_{k-1}}{\tau_k} & 1 & \frac{\tau_{k+1}}{\tau_k} & \frac{\tau_{k+2}}{\tau_k} & \frac{\tau_{k+3}}{\tau_k} & \cdots \\ \cdots & 3\times 3 & 2\times 2 & 1\times 1 & 0\times 0 & 1\times 1 & 2\times 2 & 3\times 3 & \cdots \end{array}$$

$\boxed{\tau_{k-1} = 0 \longrightarrow \text{molecule solution}}$

Linear problems and determinant formula

Auxiliary linear problem:

$$\Psi_n' = V_{n-1}\Psi_{n-1},$$
$$V_{n-1}\Psi_{n-1} + I_n\Psi_n + \Psi_{n+1} = \lambda\Psi_n,$$

Adjoint problem:

$$\Psi_n^{*\prime} = -V_n\Psi_{n+1}^*,$$
$$\Psi_{n-1}^* + I_n\Psi_n^* + V_{n+1}\Psi_{n+1}^* = \lambda\Psi_n^*$$

Theorem:

$$\left[\frac{\Psi_k}{\Psi_{k+1}}\right]^{(-1)} = \frac{1}{\lambda}\frac{\tau_k}{\tau_{k-1}}\sum_{i=0}^{\infty} b_i^{(k)}\,\lambda^{-i}$$
$$\left[\frac{\Psi_{k+1}^*}{\Psi_k^*}\right]^{(-1)} = \frac{1}{\lambda}\frac{\tau_k}{\tau_{k+1}}\sum_{i=0}^{\infty} a_i^{(k)}\,(-\lambda)^{-i}$$

Sketch of proof:

Riccati eq. for $\Xi_k = \dfrac{\Psi_k}{\Psi_{k+1}}$:

$$\frac{\partial\Xi_k}{\partial t} = -V_k\Xi_k^2 + (\lambda - I_k)\Xi_k - 1 \quad (\star)$$

Two kinds of series solutions:

$$\Xi_k^{(-1)} = \frac{1}{\lambda}\sum_{i=0}^{\infty} c_i\,\lambda^{-i}, \quad \Xi_k^{(1)} = \lambda\sum_{i=0}^{\infty} d_i\,\lambda^{-i}$$

Recurrence relation for b_i:

$$b_i^{(k)} = b_{i-1}^{(k)\prime} + \frac{\tau_{k+1}}{\tau_k}\sum_{l=0}^{i-2} b_l^{(k)} b_{i-2-l}^{(k)}, \quad b_0^{(k)} = \frac{\tau_{k-1}}{\tau_k}$$

$\rightarrow \; F = \dfrac{1}{\lambda}\dfrac{\tau_k}{\tau_{k-1}}\displaystyle\sum_{i=0}^{\infty} b_i^{(k)}\,\lambda^{-i}$ satisfies $(\star)$

Local Lax Pair and Painlevé equations(II,IV & V)

Linear problem for Toda eq.

$$\Psi'_{n+1} = V_n \Psi_n,$$
$$V_{n-1}\Psi_{n-1} + I_n \Psi_n + \Psi_{n+1} = \lambda \Psi_n$$

“Local Lax Pair” (Fadeev-Takhtajan)

$$\widetilde{L}_n \phi_n = \phi_{n+1}, \quad \frac{d\phi_n}{dt} = \widetilde{B}_n \phi_n,$$

$$\phi_n = \begin{pmatrix} \phi_n^{(1)} \\ \phi_n^{(2)} \end{pmatrix} = \mathrm{e}^{-\frac{1}{2}\lambda t} \begin{pmatrix} \Psi_n \\ \frac{\tau_{n-1}}{\tau_{n-2}} \Psi_{n-1} \end{pmatrix}$$

$$\widetilde{L}_n(t,\lambda) = \begin{pmatrix} -I_n + \lambda & -\frac{\tau_n}{\tau_{n-1}} \\ \frac{\tau_{n-1}}{\tau_n} & 0 \end{pmatrix},$$

$$\widetilde{B}_n(t,\lambda) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \lambda + \begin{pmatrix} 0 & \frac{\tau_n}{\tau_{n-1}} \\ -\frac{\tau_{n-2}}{\tau_{n-1}} & 0 \end{pmatrix}$$

$\mathbf{P_{II}}$:

$$\frac{d^2u}{dt^2} = 2u^3 - 4tu + 4\left(\alpha + \frac{1}{2}\right), \quad u = \frac{d}{dt}\log\frac{\tau_1}{\tau_0}$$

Auxiliary linear problem: ($\psi_{\pm 1} = \frac{\tau_{\pm 1}}{\tau_0}$)

$$\frac{\partial Y}{\partial \lambda} = AY, \quad \frac{\partial Y}{\partial t} = BY, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & -\frac{\psi_1}{2} \\ \frac{\psi_{-1}}{2} & 0 \end{pmatrix}\lambda + \begin{pmatrix} -\frac{z+t}{2} & \frac{\psi_1 u_0}{2} \\ -\frac{\psi_{-1}u_{-1}}{2} & \frac{z+t}{2} \end{pmatrix}, \quad z = -\frac{\tau_1\tau_{-1}}{\tau_0^2}$$

$$B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\lambda + \begin{pmatrix} 0 & \frac{\tau_1}{\tau_0} \\ -\frac{\tau_{-1}}{\tau_0} & 0 \end{pmatrix},$$

$$\boxed{\widetilde{B}_1(t,\lambda) = B}$$

Description in the framework of the KP theory

$\boldsymbol{\tau_n(x)}$: τ function of 1-dimensional Toda lattice hierarhchy and 1st modified KP hierarchy

$(x = (x_1, x_2, \ldots),\ x_1 = t)$

$$D_{x_1} p_{k+1}\left(\frac{1}{2}\tilde{D}\right) \tau_n \cdot \tau_n = p_k\left(\frac{1}{2}\tilde{D}\right) \tau_{n+1} \cdot \tau_{n-1}$$

$$\left[D_{x_1} p_k\left(\frac{1}{2}\tilde{D}\right) - p_{k+1}\left(\frac{1}{2}\tilde{D}\right) + p_{k+1}\left(-\frac{1}{2}\tilde{D}\right)\right]\tau_{n+1} \cdot \tau_n = 0$$

where $\displaystyle\sum_{j=0}^{\infty} p_j(x)\lambda^j = \exp\sum_{n=1}^{\infty} x_n\lambda^n, \quad \tilde{D} = \left(D_{x_1}, \frac{1}{2}D_{x_2}, \frac{1}{3}D_{x_3}, \cdots\right), \quad \tilde{\partial} = \left(\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \frac{1}{3}\partial_{x_3}, \cdots\right)$

$$\Downarrow$$

$$\boxed{\frac{\tau_{k+n}}{\tau_k} = \begin{cases} \det(a^{(k)}_{i+j-2})_{i,j=1,\ldots,n} & n > 0, \\ 1 & n = 0, \\ \det(b^{(k)}_{i+j-2})_{i,j=1,\ldots,|n|} & n < 0, \end{cases} \qquad \begin{aligned} a^{(k)}_n &= p_n(\tilde{\partial})\,\frac{\tau_{k+1}}{\tau_k}, \\ b^{(k)}_n &= (-1)^n p_n(-\tilde{\partial})\,\frac{\tau_{k-1}}{\tau_k} \end{aligned}}$$

Sketch of proof:

$$D_{x_1} p_{k+1}(\frac{1}{2}\tilde{D})\, \tau_n \cdot \tau_n = p_k(\frac{1}{2}\tilde{D})\, \tau_{n+1} \cdot \tau_{n-1}$$

$$\left[D_{x_1} p_k(\frac{1}{2}\tilde{D}) - p_{k+1}(\frac{1}{2}\tilde{D}) + p_{k+1}(-\frac{1}{2}\tilde{D})\right] \tau_{n+1} \cdot \tau_n = 0$$

$\downarrow$

$$\boxed{\text{Miwa transformation: } x_j = \frac{l}{j(-\lambda)^j}}$$

$\downarrow$

Bäcklund transformation of Toda equation:

$$D_{x_1} \tau_n(l+1) \cdot \tau_n(l) = -\frac{1}{\lambda} \tau_{n+1}(l+1)\tau_{n-1}(l)$$

$$\left(\frac{1}{\lambda} D_{x_1} + 1\right) \tau_{n+1}(l+1) \cdot \tau_n(l) - \tau_n(l+1)\tau_{n+1}(l) = 0$$

$\downarrow$

$$\boxed{\left\{\begin{aligned} V_n &= \frac{\tau_{n+1}(l)\tau_{n-1}(l)}{\tau_n(l)^2} \\ I_n &= \frac{d}{dt} \log \frac{\tau_{n-1}(l)}{\tau_n(l)} \end{aligned}\right. \qquad \left\{\begin{aligned} \Psi_{n+1} &= \lambda^n \frac{\tau_n(l-1)}{\tau_n(l)} \\ \Psi_n^* &= \lambda^{-n} \frac{\tau_n(l+1)}{\tau_n(l)} \end{aligned}\right.}$$

$\downarrow$

Linear and adjoint linear problems

Now:

$$\frac{\Psi_k}{\Psi_{k+1}} = \frac{1}{\lambda} \frac{\tau_k(l)}{\tau_{k-1}(l)} \sum_{i=0}^{\infty} b_i^{(k)} \lambda^{-i} = \frac{1}{\lambda} \frac{\tau_{k-1}(l-1)\tau_k(l)}{\tau_{k-1}(l)\tau_k(l-1)}$$

$$\longrightarrow \quad \sum_{i=0}^{\infty} b_i^{(k)} \lambda^{-i} = \frac{\tau_{k-1}(l-1)}{\tau_k(l-1)} = e^{-\partial_l} \frac{\tau_{k-1}}{\tau_k}$$

$$= \exp\left[-\sum_{n=0}^{\infty} \frac{(-\lambda)^{-n}}{n} \frac{\partial}{\partial x_n}\right] \frac{\tau_{k-1}}{\tau_k}$$

$$= \sum_{i=0}^{\infty} (-\lambda)^i p_i(-\tilde{\partial}) \frac{\tau_{k-1}}{\tau_k} \quad \rightarrow \quad \boxed{b_i^k = (-1)^i p_i(-\tilde{\partial}) \frac{\tau_{k-1}}{\tau_k}}$$

Comments:

☞ $\varphi = a_0 = \frac{\tau_{k+1}}{\tau_k}$, $\psi = b_0 = \frac{\tau_{k-1}}{\tau_k}$ satisfy the nonlinear Schrödinger hierarchy:

$$\begin{cases} \varphi_{x_2} = \varphi_{x_1x_1} + 2\psi\varphi^2, \\ \psi_{x_2} = -\psi_{x_1x_1} - 2\psi^2\varphi, \end{cases} \qquad \begin{cases} \varphi_{x_3} = \varphi_{x_1x_1x_1} + 6\psi\varphi\varphi_{x_1}, \\ \psi_{x_2} = \psi_{x_1x_1x_1} + 6\psi\varphi\psi_{x_1}, \end{cases} \qquad \ldots$$

☞ The determinant formula looks like:

$$\frac{\tau_{k+n}}{\tau_k} = \left.\begin{vmatrix} p_0(\tilde\partial)\,\frac{\tau_{k+1}}{\tau_k} & p_1(\tilde\partial)\,\frac{\tau_{k+1}}{\tau_k} & \cdots & p_{n-1}(\tilde\partial)\,\frac{\tau_{k+1}}{\tau_k} \\ p_1(\tilde\partial)\,\frac{\tau_{k+1}}{\tau_k} & p_2(\tilde\partial)\,\frac{\tau_{k+1}}{\tau_k} & \cdots & p_n(\tilde\partial)\,\frac{\tau_{k+1}}{\tau_k} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n-1}(\tilde\partial)\,\frac{\tau_{k+1}}{\tau_k} & p_n(\tilde\partial)\,\frac{\tau_{k+1}}{\tau_k} & \cdots & p_{2n-2}(\tilde\partial)\,\frac{\tau_{k+1}}{\tau_k} \end{vmatrix}\right\} n, \quad p_j(\tilde\partial) = \partial_{x_1}^j + \frac{1}{2}\partial_{x_2}\partial_{x_1}^{j-2} + \cdots$$

☞ Reduction to 1-dimensional Toda hierarchy is essential (so far) .

☞ The formula resembles the Darboux tranformation, but direct relationship is not known, since the Wronskian structure is essential to the Darboux transformation.

Summary: **Toda equation:** $\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1}, \quad n \in \mathbb{Z}$

☞ **Hankel determinant formula:**

$$\frac{\tau_{k+n}}{\tau_k} = \begin{cases} \det(a_{i+j-2}^{(k)})_{i,j=1,\dots,n} & n>0, \\ 1 & n=0, \\ \det(b_{i+j-2}^{(k)})_{i,j=1,\dots,|n|} & n<0, \end{cases} \qquad \begin{cases} a_i^{(k)} = a_{i-1}^{(k)\prime} + \dfrac{\tau_{k-1}}{\tau_k}\displaystyle\sum_{l=0}^{i-2} a_l^{(k)} a_{i-2-l}^{(k)}, & a_0^{(k)} = \dfrac{\tau_{k+1}}{\tau_k}, \\ b_i^{(k)} = b_{i-1}^{(k)\prime} + \dfrac{\tau_{k+1}}{\tau_k}\displaystyle\sum_{l=0}^{i-2} b_l^{(k)} b_{i-2-l}^{(k)}, & b_0^{(k)} = \dfrac{\tau_{k-1}}{\tau_k}, \end{cases}$$

☞ **Relation to linear problems:**

Ψ: solution of linear problem, Ψ^*: solution of adjoint problem

$$\frac{\Psi_k}{\Psi_{k+1}} = \frac{1}{\lambda}\frac{\tau_k}{\tau_{k-1}}\sum_{i=0}^{\infty} b_i^{(k)}\,\lambda^{-i}, \qquad \frac{\Psi_{k+1}^*}{\Psi_k^*} = \frac{1}{\lambda}\frac{\tau_k}{\tau_{k+1}}\sum_{i=0}^{\infty} a_i^{(k)}\,(-\lambda)^{-i}$$

☞ **KP theory:**

$$\frac{\tau_{k+n}}{\tau_k} = \begin{vmatrix} p_0(\tilde{\partial})\,\frac{\tau_{k+1}}{\tau_k} & \cdots & p_{n-1}(\tilde{\partial})\,\frac{\tau_{k+1}}{\tau_k} \\ \vdots & \cdots & \vdots \\ p_{n-1}(\tilde{\partial})\,\frac{\tau_{k+1}}{\tau_k} & \cdots & p_{2n-2}(\tilde{\partial})\,\frac{\tau_{k+1}}{\tau_k} \end{vmatrix}$$

☞ **Painlevé equations through the local Lax pair.**

☞ **Finite lattice**