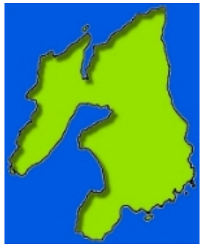
ISLAND 3 : Algebraic aspects of integrable systems



# Local Darboux transformations and geometric crystals (II):

## Transformations of the dKP equation and their reduction to the dKdV case

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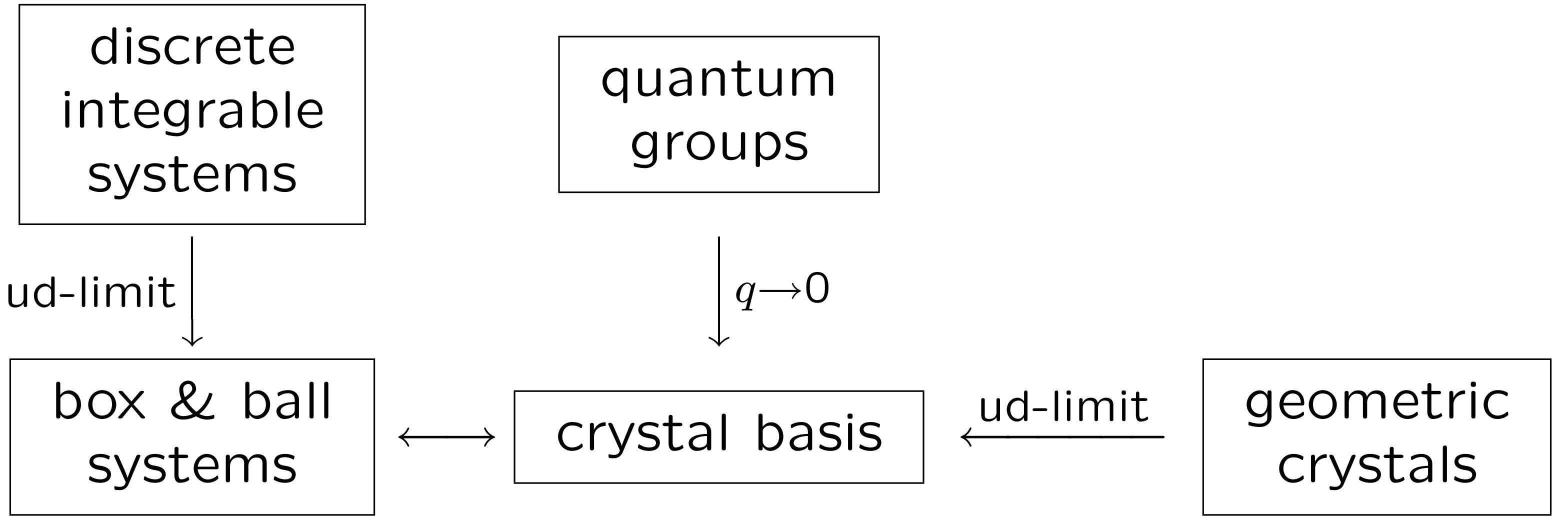
Joint work with J.J.C. Nimmo and R. Willox

# Motibation



discrete integrable systems
quantum groups
ud-limit
$q \longrightarrow 0$
box & ball systems
crystal basis
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geometric crystals

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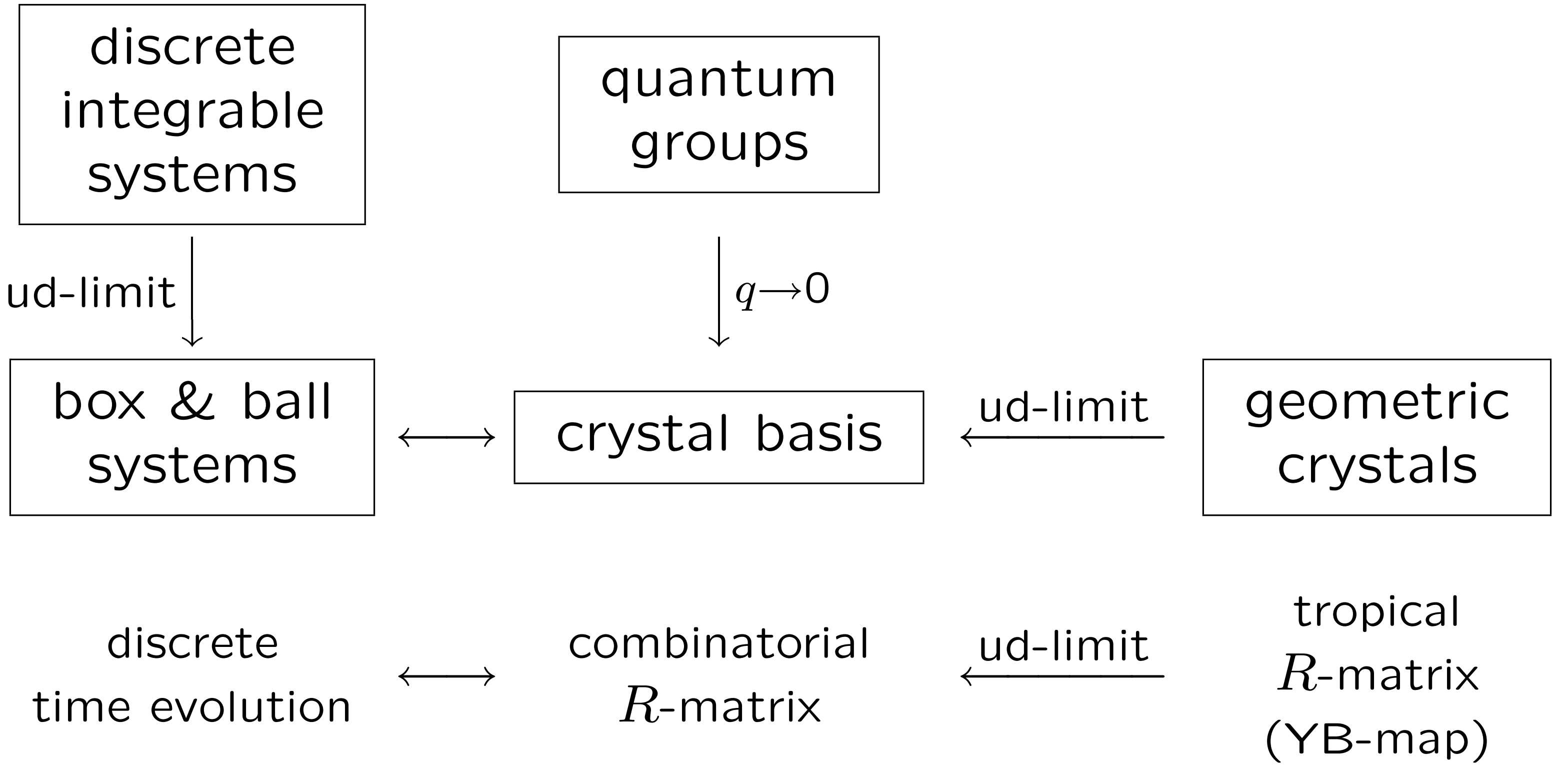
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tropical $R$-matrix (YB-map)

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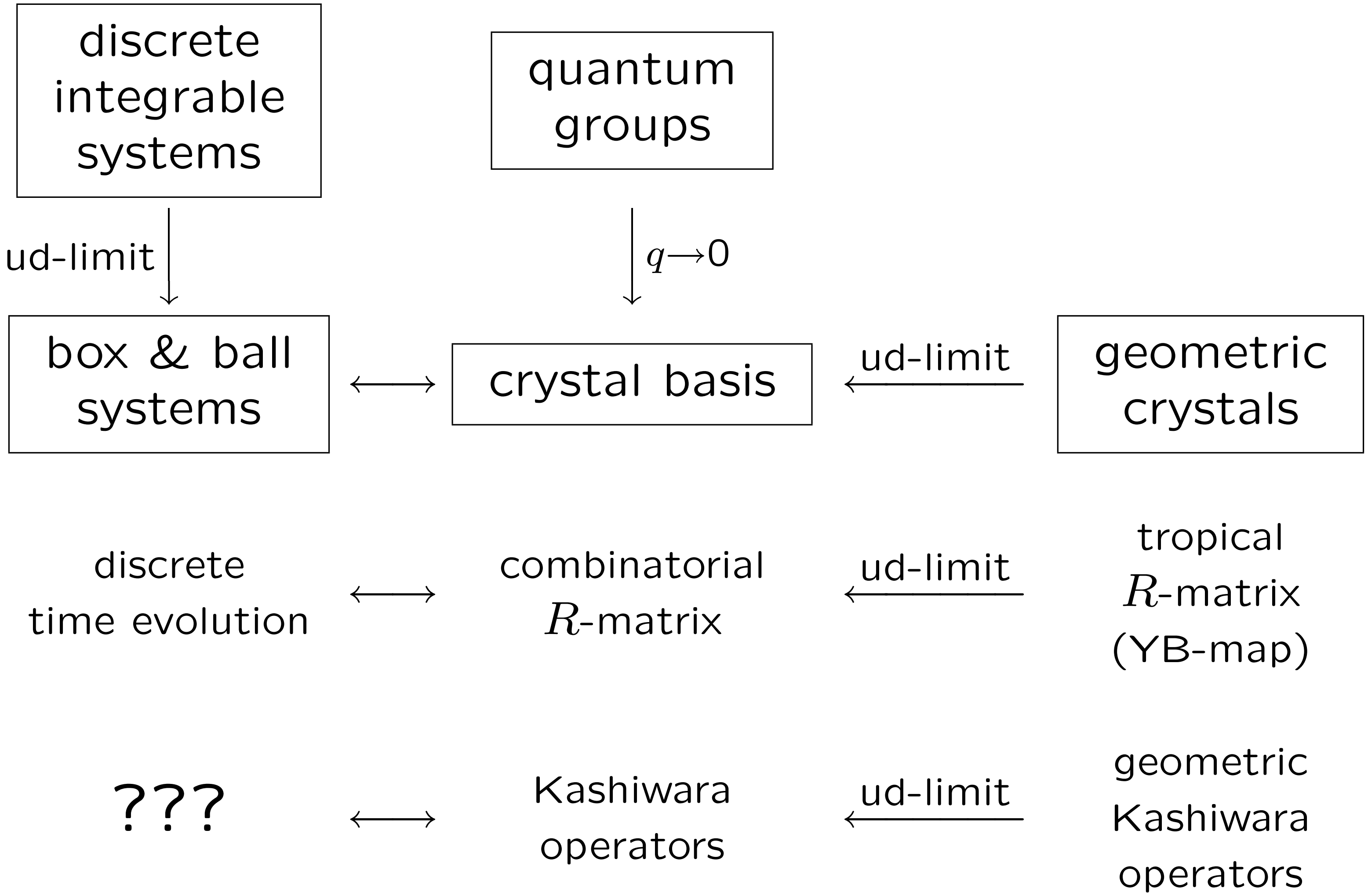
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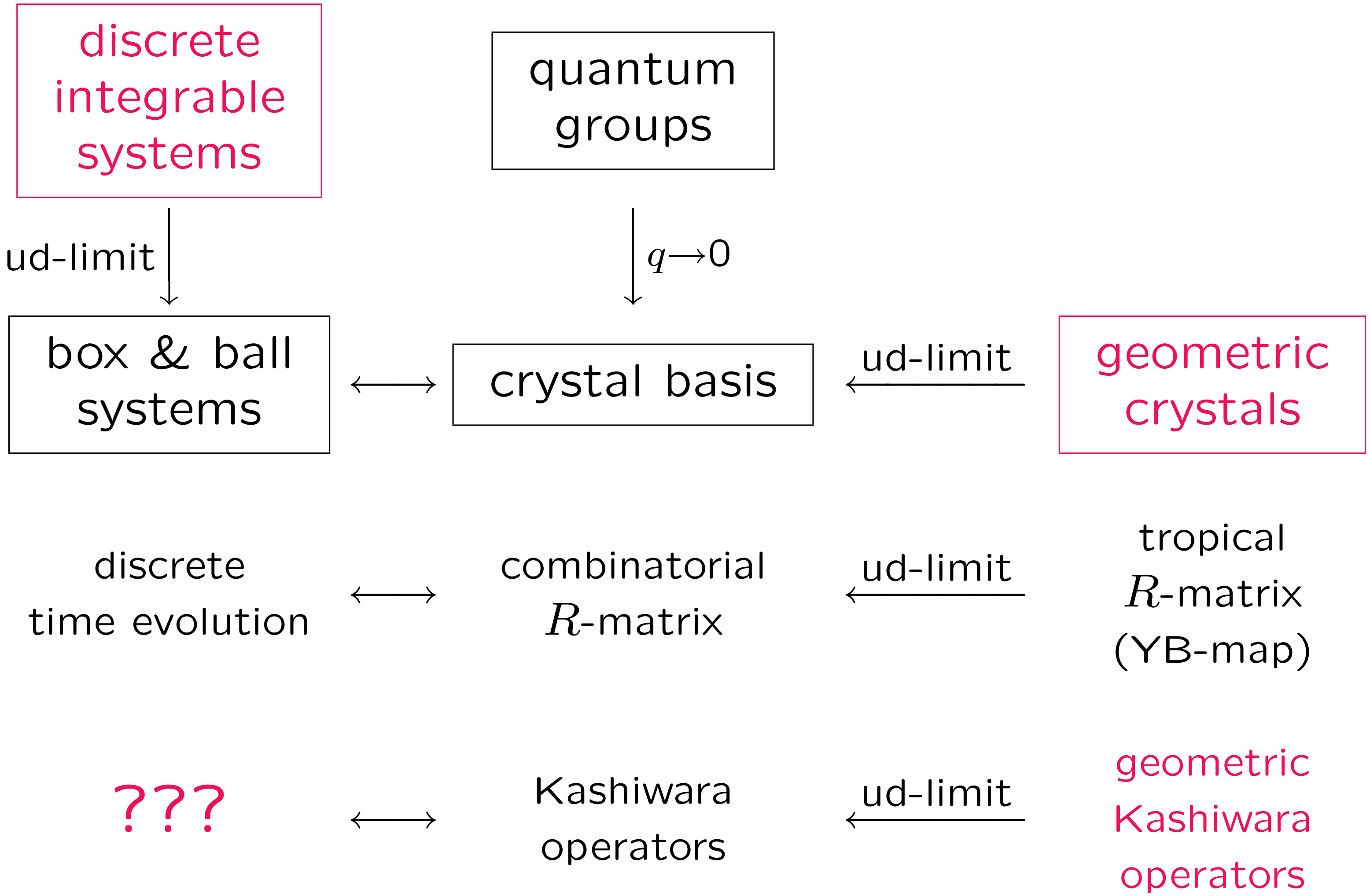
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$\Delta$ : root system

$\Pi := \{\alpha_i\}_{i\in\mathcal{I}}$ : simple roots

$A = \left(a_{ij}\right)_{i,j\in\mathcal{I}}$ , $a_{ij} := \dfrac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ : Cartan matrix

$X$ : algebraic variety/$\mathbb{C}$

$e_i$ : $\mathbb{C}^* \times X \to X$ (rational $\mathbb{C}^*$-actions)
For $c \in \mathbb{C}^*$, $x \in X$, we will use the notation $e_i^c(x) := e_i(c,x)$.

$\gamma_i$ : $X \to \mathbb{C}$ (rational functions)

$\varepsilon_i$ : $X \to \mathbb{C}$ (rational functions)

## §1. Geometric Crystals (Berenstein-Kazhdan, 2000)

**Def.** $(X, \{e_i\}_{i\in\mathcal{I}}, \{\gamma_i\}_{i\in\mathcal{I}}, \{\varepsilon_i\}_{i\in\mathcal{I}})$ is a geometric crystal if

(i) $\gamma_i(e_j^c(x)) = c^{a_{ji}}\gamma_i(x)$,

(ii) $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$,

(iii) (0) $a_{ij} = a_{ji} = 0 \quad \Rightarrow \quad e_i^{c_1}e_j^{c_2} = e_j^{c_2}e_i^{c_1}$,

(1) $a_{ij} = a_{ji} = -1 \quad \Rightarrow \quad e_i^{c_1}e_j^{c_1c_2}e_i^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1}$,

(2) $a_{ij} = -2,\ a_{ji} = -1$

$$\Rightarrow \quad e_i^{c_1}e_j^{c_1^2c_2}e_i^{c_1c_2}e_j^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^2c_2}e_i^{c_1},$$

(3) $a_{ij} = -3,\ a_{ji} = -1$

$$\Rightarrow \quad e_i^{c_1}e_j^{c_1^3c_2}e_i^{c_1^2c_2}e_j^{c_1^3c_2^2}e_i^{c_1c_2}e_j^{c_2}$$
$$= e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^3c_2^2}e_i^{c_1^2c_2}e_j^{c_1^3c_2}e_i^{c_1}.$$

## Unipotent Crystal ($A_{n-1}$-case)

$$GL_n(\mathbb{C}) \begin{array}{l} \cup \ \mathcal{B}^+ : \text{ upper triangular matrices} \\ \cap \ \mathcal{U}^- : \text{ lower triangular \& diagonal entries} = 1 \end{array}$$

Notation:

- $E_{ij} \in GL_n$, $\left(E_{ij}\right)_{kl} = \delta_{ik}\delta_{kl}$
- $G_j(a) := \sum_i E_{ii} + aE_{j+1,j} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & a & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \in \mathcal{U}^-$

For $X = \sum_{i \leq j} x_{ij} E_{ij} \in \mathcal{B}^+$ and $c \in \mathbb{C}^*$, we define

- $e_i^c(X) := G_i\left((c-1)x_{ii}/x_{i,i+1}\right) X \, G_i\left((c^{-1}-1)x_{i+1,i+1}/x_{i,i+1}\right)$,
- $\gamma_i(X) := x_{i+1,i+1}/x_{ii}$

Consider $X \in \mathcal{B}^+$ of the following form:

$$X = \begin{bmatrix} x_1 & 1 & & \\ & x_2 & \ddots & \\ & & \ddots & 1 \\ & & & x_n \end{bmatrix}.$$

We can calculate the action of $e_i^c$ explicitly:

$$e_i^c \ : \ \begin{bmatrix} x_1 & 1 & & \\ & x_2 & \ddots & \\ & & \ddots & 1 \\ & & & x_n \end{bmatrix} \mapsto \begin{bmatrix} x_1 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & cx_i & 1 & & \\ & & & c^{-1}x_{i+1} & \ddots & \\ & & & & \ddots & 1 \\ & & & & & x_n \end{bmatrix}$$

$$\wr\!| \qquad\qquad \wr\!|$$

$$e_i^c \ : \ (x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) \mapsto (x_1, \ldots, cx_i, c^{-1}x_{i+1}, \ldots, x_n)$$

Consider a product $X \times Y$ $(X, Y \in \mathcal{B}^+)$.
We can define an action of $e_i^c$ on $X \times Y$:

$$e_i^c(X \times Y) = e_i^{c_1}(X) \times e_i^{c_2}(Y)$$

where $c_1$, $c_2$ are some rational functions w.r.t. $x_i$, $y_i$ $(i = 1, \ldots, n)$

## Tropical $R$-matrix

$$R \ : \ \mathcal{B}^+ \times \mathcal{B}^+ \ \rightarrow \ \mathcal{B}^+ \times \mathcal{B}^+$$

which satisfies

$$[R,\ e_i^c] = 0, \quad \forall i$$

Cf. Solvable lattice models $\sim$ Quantum $R$

$$[R,\ U_q(\mathfrak{g})] = 0$$

## §2. Local Darboux transformation for the discrete KP hierarchy

For $f(\ell, \mathbf{m}) = f(\ell, m_1, \ldots, m_M)$:

$$T_\ell f(\ell, \mathbf{m}) = f(\ell+1, \mathbf{m}),$$
$$T_j f(\ell, \mathbf{m}) = f(\ell, m_1, \ldots, m_j + 1, \ldots, m_M), \quad j = 1, \ldots, M$$

**Sato-Wilson operators :**

$$W(\ell, \mathbf{m}) := I + w_1 T_\ell^{-1} + w_2 T_\ell^{-2} + \cdots,$$
$$\overline{W}(\ell, \mathbf{m}) := \overline{w}_0 + \overline{w}_1 T_\ell^{1} + \overline{w}_2 T_\ell^{2} + \cdots,$$

**discrete Sato equations :**

$$(T_j \widetilde{W})(1 - \alpha_j + \alpha T_\ell) = B_j \widetilde{W} \quad (\widetilde{W} = W \text{ or } \overline{W},\ j = 1, \ldots, M),$$
$$B_j = \alpha T_\ell + (1 - \alpha_j) u_j, \quad u_j := \frac{T_j \overline{w}_0}{\overline{w}_0}$$

**(Formal) Baker-Akhiezer function :**

$$\Psi_\lambda(\ell,\mathbf{m}) := W(\ell,\mathbf{m})\left(1-\frac{\lambda}{a}\right)^{\ell}\prod_{j=1}^{M}\left(1-\frac{\lambda}{b_j}\right)^{m_j}$$

that satisfies

$$T_j\Psi_\lambda(\ell,\mathbf{m}) = B_j\Psi_\lambda(\ell,\mathbf{m}) = \left\{\alpha_j T_\ell + (1-\alpha_j)u_j\right\}\Psi_\lambda(\ell,\mathbf{m})$$

$$\Leftrightarrow \quad \Psi_\lambda = \frac{1}{a-b_j}\frac{1}{u_j}\left(aT_\ell - b_jT_j\right)\Psi_\lambda$$

[J.J.C. Nimmo, J. Phys. A (1997)]

**Compatibility condition :** $(T_jB_k)B_j = (T_kB_j)B_k$

$$\begin{cases} u_j(T_ju_k) = u_k(T_ku_j), \\ \alpha_j(1-\alpha_k)(T_ju_k) + \alpha_k(1-\alpha_j)(T_\ell u_j) \\ \qquad = \alpha_k(1-\alpha_j)(T_ku_j) + \alpha_j(1-\alpha_k)(T_\ell u_k) \end{cases}$$

(discrete KP equation)

## Linear equations in infinite matrix form

$$T_j \Psi_\lambda(\ell, \mathbf{m}) = \left\{ \alpha_j T_\ell + (1-\alpha_j) u_j \right\} \Psi_\lambda(\ell, \mathbf{m})$$

$$\Leftrightarrow \quad T_j \begin{bmatrix} \vdots \\ \psi(\ell-1) \\ \psi(\ell) \\ \psi(\ell+1) \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \ddots & \ddots & & & \\ & (1-\alpha_j)u_j(\ell-1) & \alpha_j & & \\ & & (1-\alpha_j)u_j(\ell) & \alpha_j & \\ & & & (1-\alpha_j)u_j(\ell+1) & \ddots \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \psi(\ell-1) \\ \psi(\ell) \\ \psi(\ell+1) \\ \vdots \end{bmatrix}$$

Consider $2 \times 2$ subsystem :

$$T_j \begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix} = \begin{bmatrix} (1-\alpha_j)u_j(k) & \alpha_j \\ 0 & (1-\alpha_j)u_j(k+1) \end{bmatrix} \begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix}.$$

Compatibility of the $2 \times 2$-system follows from that of the infinite case.

## Local Darboux transformation

**Data :**

- Potentials $\left\{u_j(\ell, \mathbf{m})\right\}$ that satisfy the dKP equation.
- Eigenfunctions $\{\phi(j)\}_{j\in\mathbb{Z}}$ that satisfy

$$T_j \begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix} = \begin{bmatrix} (1-\alpha_j)u_j(k) & \alpha_j \\ 0 & (1-\alpha_j)u_j(k+1) \end{bmatrix} \begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix}.$$

Fix an integer $k$ and define $\left\{\widehat{u}_j(\ell, \mathbf{m})\right\}$ as

$$\widehat{u}_j(\ell, \mathbf{m}) = \begin{cases} u_j(\ell, \mathbf{m}) & (\ell \neq k, k+1) \\ u_j(k, \mathbf{m})c_j(k, \mathbf{m}) & (\ell = k) \\ u_j(k+1, \mathbf{m})/c_j(k, \mathbf{m}) & (\ell = k+1) \end{cases}$$

where $c_j(k, \mathbf{m}) := 1 + \phi(k+1)/\left\{(\alpha_j - 1)\phi(k)u_j(k, \mathbf{m})\right\}$.

**Proposition :**

The transformed potentials $\left\{\widehat{u}_j(\ell, \mathbf{m})\right\}$ satisfy the dKP equation.

# §3. Reduction to dKdV

**Reduction :** $$\begin{aligned} W(\ell, m_1+1, m_2+1) &= W(\ell, m_1, m_2),\\ \overline{W}(\ell, m_1+1, m_2+1) &= \overline{W}(\ell, m_1, m_2) \end{aligned}$$

$$\Rightarrow \quad \Psi_\lambda(\ell, , m_1+1, m_2+1) = \lambda \Psi_\lambda(\ell, , m_1, m_2)$$

Define $\Phi_\lambda(\ell, , m_1, m_2) := \Psi_\lambda(\ell, , m_1, m_2+1)$.
Then $\Phi_\lambda(\ell, , m_1+1, m_2) = \lambda \Psi_\lambda(\ell, , m_1, m_2)$.

$\Rightarrow$ $2 \times 2$ Lax pair

$$\begin{cases} T_\ell \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \end{bmatrix} = \begin{bmatrix} (1-\alpha_2^{-1})u & \alpha_2^{-1} \\ \alpha_1^{-1}\lambda & (1-\alpha_1^{-1})u^{-1} \end{bmatrix} \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \end{bmatrix}, \\ T_1 \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \end{bmatrix} = \begin{bmatrix} (1-\alpha_1)v & \alpha_1\alpha_2^{-1} \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \end{bmatrix}. \end{cases}$$

$2 \times 2$ Lax pair

$$\begin{cases} T_\ell \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \end{bmatrix} = \begin{bmatrix} (1-\alpha_2^{-1})u & \alpha_2^{-1} \\ \alpha_1^{-1}\lambda & (1-\alpha_1^{-1})u^{-1} \end{bmatrix} \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \end{bmatrix}, \\ T_1 \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \end{bmatrix} = \begin{bmatrix} (1-\alpha_1)v & \alpha_1\alpha_2^{-1} \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \end{bmatrix}. \end{cases}$$

Compatibility condition :

$$\begin{cases} (T_1 u)v = u(T_\ell v), \\ v = \dfrac{1}{T_1 u} - \dfrac{1-\alpha_2^{-1}}{1-\alpha_1^{-1}} u \end{cases}$$

Eliminating $v$, we obtain the **discrete KdV equation** :

$$\frac{1}{T_1 T_\ell u} - \frac{1}{u} = \delta \left\{ (T_\ell u) - (T_1 u) \right\}, \quad \delta = \frac{1-\alpha_2^{-1}}{1-\alpha_1^{-1}}$$

# dKdV as a Box and Ball System

$\ell \leftrightarrow t, \quad m_1 \leftrightarrow n$

$$\begin{cases} u_n^{t+1} v_n^t = u_n^t v_{n+1}^t \\ u_n^{t+1} = \dfrac{1}{v_n^t + \delta u_n^t} \end{cases} \quad \Leftrightarrow \quad \frac{1}{u_{n+1}^{t+1}} - \frac{1}{u_n^t} = \delta \left\{ u_{n+1}^t - u_n^{t+1} \right\}$$

Ultra-discrete limit [Tokihiro-Takahashi-Matsukidaira-Satsuma, 1996]

$$\lim_{\varepsilon \to +0} \varepsilon \log \left( e^{A/\varepsilon} + e^{B/\varepsilon} \right) = \max[A, B]$$

$u_n^t = \exp\left[U_n^t/\varepsilon\right], \; v_n^t = \exp\left[V_n^t/\varepsilon\right], \; \delta = \exp[-1/\varepsilon]$

$$\begin{cases} e^{(U_n^{t+1} + V_n^t)/\varepsilon} = e^{(U_n^t + V_{n+1}^t)/\varepsilon} \\ e^{U_n^{t+1}/\varepsilon} = \dfrac{1}{e^{V_n^t/\varepsilon} + e^{(U_n^t - 1)/\varepsilon}} \end{cases} \xrightarrow{\text{ud-lim}} \begin{cases} U_n^{t+1} + V_n^t = U_n^t + V_{n+1}^t \\ U_n^{t+1} = -\max\left[U_n^t - 1, V_n^t\right] \end{cases}$$

## dKdV as a Yang-Baxter map

$$\text{dKdV} \quad \Leftrightarrow \quad \begin{cases} (T_1 u)v = u(T_\ell v) \\ v = \dfrac{1}{T_1 u} - \delta u \end{cases} \quad \Leftrightarrow \quad \begin{cases} T_1 u = \dfrac{1}{v + \delta u} \\ T_\ell v = \dfrac{v}{u(v + \delta u)} \end{cases}$$

Yang-Baxter map

$$R : (u, v) \mapsto \left( \frac{1}{v + \delta u}, \frac{v}{u(v + \delta u)} \right)$$

This satisfies Yang-Baxter equation :

$$\widetilde{R}_{uv} R_{uw} R_{vw} = R_{vw} R_{uw} \widetilde{R}_{uv},$$

where $\widetilde{R} : (u, v) \mapsto (v, u)$ and

$$R_{vw}(u, v, w) = (u, R(v, w)), \quad \text{etc.}$$

## discrete KdV

$$\begin{cases} (T_1 u)v = u(T_\ell v) \\ v = \dfrac{1}{T_1 u} - \delta u \end{cases} \quad \Leftrightarrow \quad \frac{1}{T_1 T_\ell u} - \frac{1}{u} = \delta \left\{ (T_\ell u) - (T_1 u) \right\}$$

## Cf. [Papageorgiou-Tongas-Veselov, JMP, 2006]

$$\begin{cases} uv = xy \\ u - \dfrac{\beta_1}{x} = y - \dfrac{\beta_1}{v} \end{cases} \quad \Leftrightarrow \quad \begin{cases} u = y\dfrac{\beta_1 + xy}{\beta_2 + xy} \\ v = x\dfrac{\beta_2 + xy}{\beta_1 + xy} \end{cases}$$

$R : (x, y) \mapsto (u, v)$ satisfies the YB relation.

# Local Darboux transformation for dKdV

**Proposition :**

Assume that the local eigenfunction $\begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix}$ satisfies

$$T_1 T_2 \begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix} = \kappa \begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix}, \quad \kappa \neq 0.$$

Then the local Darboux transformation preserves the reduction condition for the dKdV.

$$\frac{1}{T_1 T_\ell u} - \frac{1}{u} = \delta \left\{ (T_\ell u) - (T_1 u) \right\}$$

$$\{u_j(\ell, \mathbf{m})\} \mapsto \widehat{u}_j(\ell, \mathbf{m}) = \begin{cases} u_j(\ell, \mathbf{m}) & (\ell \neq k, k+1) \\ u_j(k, \mathbf{m}) c_j(k, \mathbf{m}) & (\ell = k) \\ u_j(k+1, \mathbf{m}) / c_j(k, \mathbf{m}) & (\ell = k+1) \end{cases}$$

$$c_j(k, \mathbf{m}) := 1 + \phi(k+1) / \left\{ (\alpha_j - 1) \phi(k) u_j(k, \mathbf{m}) \right\}$$

# §4. Reduction to dBoussinesq

**Reduction 1:** $\Psi_\lambda(\ell+1, m_1+1, m_2+1) = \lambda\Psi_\lambda(\ell, m_1, m_2)$

Define $\Phi_\lambda(\ell, m_1, m_2, m_3) := \Psi_\lambda(\ell, m_1, m_2+1)$,
$\Theta_\lambda(\ell, m_1, m_2, m_3) := \Psi_\lambda(\ell, m_1+1, m_2+1)$,
Then $\Theta_\lambda(\ell+1, m_1, m_2) = \lambda\Psi_\lambda(\ell, m_1, m_2)$.

$\Rightarrow$ $3\times 3$ Lax pair

$$\begin{cases} T_\ell \begin{bmatrix}\Psi_\lambda\\ \Phi_\lambda\\ \Theta_\lambda\end{bmatrix} = \begin{bmatrix}(1-\alpha_2^{-1})u_2 & \alpha_2^{-1} & 0\\ 0 & (1-\alpha_1^{-1})(T_1^{-1}T_\ell^{-1}u_1) & \alpha_1^{-1}\\ \lambda & 0 & 0\end{bmatrix}\begin{bmatrix}\Psi_\lambda\\ \Phi_\lambda\\ \Theta_\lambda\end{bmatrix}, \\ T_1 \begin{bmatrix}\Psi_\lambda\\ \Phi_\lambda\\ \Theta_\lambda\end{bmatrix} = \begin{bmatrix}(1-\alpha_1)v & \alpha_1\alpha_2^{-1} & 0\\ 0 & 0 & 1\\ \alpha_1\lambda & 0 & (1-\alpha_1)(T_\ell^{-1}u_1)\end{bmatrix}\begin{bmatrix}\Psi_\lambda\\ \Phi_\lambda\\ \Theta_\lambda\end{bmatrix}. \end{cases}$$

Compatibility condition

$$\begin{cases} u_2(T_\ell v) = (T_1 u_2)v \\ T_\ell v = T_1^{-1}T_\ell^{-1}u_2 - \delta(T_1u_2) \\ v = u_1 - \delta u_2 \end{cases} \quad\Rightarrow\quad \begin{cases} \dfrac{T_1u_2}{u_2} = \dfrac{T_\ell u_1 - \delta(T_\ell u_2)}{u_1-\delta u_2} \\ u_1(T_1u_2) = u_2(T_1^{-1}T_\ell^{-1}u_1) \end{cases}$$

**Reduction 2:**

$$\Psi_\lambda(\ell, m_1+1, m_2+1, m_3+1) = \lambda\Psi_\lambda(\ell, m_1, m_2, m_3)$$

Define $\Phi_\lambda(\ell, m_1, m_2, m_3) := \Psi_\lambda(\ell, m_1, m_2, m_3+1)$,

$\Theta_\lambda(\ell, m_1, m_2, m_3) := \Psi_\lambda(\ell, m_1, m_2+1, m_3+1)$,

Then $\Theta_\lambda(\ell, m_1+1, m_2, m_3) = \lambda\Psi_\lambda(\ell, m_1, m_2, m_3)$.

$\Rightarrow$ $3\times 3$ Lax pair

$$\begin{cases} T_\ell \begin{bmatrix}\Psi_\lambda\\ \Phi_\lambda\\ \Theta_\lambda\end{bmatrix} = \begin{bmatrix}(1-\alpha_3^{-1})u_3 & \alpha_3^{-1} & \\ & (1-\alpha_2^{-1})(T_3u_2) & \alpha_2^{-1}\\ \alpha_1^{-1}\lambda & & (1-\alpha_1^{-1})(T_2T_3u_1)\end{bmatrix}\begin{bmatrix}\Psi_\lambda\\ \Phi_\lambda\\ \Theta_\lambda\end{bmatrix}, \\ T_1 \begin{bmatrix}\Psi_\lambda\\ \Phi_\lambda\\ \Theta_\lambda\end{bmatrix} = \begin{bmatrix}(1-\alpha_1)v_3 & \alpha_1\alpha_3^{-1} & \\ & (1-\alpha_1)v_2 & \alpha_1\alpha_2^{-1}\\ \lambda & & \end{bmatrix}\begin{bmatrix}\Psi_\lambda\\ \Phi_\lambda\\ \Theta_\lambda\end{bmatrix}. \end{cases}$$

$3 \times 3$ Lax pair

$$\begin{cases} T_\ell \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \\ \Theta_\lambda \end{bmatrix} = \begin{bmatrix} (1-\alpha_3^{-1})u_3 & \alpha_3^{-1} & \\ & (1-\alpha_2^{-1})/(u_1u_3) & \alpha_2^{-1} \\ \alpha_1^{-1}\lambda & & (1-\alpha_1^{-1})u_1 \end{bmatrix} \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \\ \Theta_\lambda \end{bmatrix}, \\ T_1 \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \\ \Theta_\lambda \end{bmatrix} = \begin{bmatrix} (1-\alpha_1)v_3 & \alpha_1\alpha_3^{-1} & \\ & (1-\alpha_1)v_2 & \alpha_1\alpha_2^{-1} \\ \lambda & & \end{bmatrix} \begin{bmatrix} \Psi_\lambda \\ \Phi_\lambda \\ \Theta_\lambda \end{bmatrix}. \end{cases}$$

Compatibility condition :

$$\begin{cases} (T_1u_3)v_3 = u_3(T_\ell v_3), \quad u_1u_3v_2 = (T_1u_1)(T_1u_3)(T_\ell v_2), \\ v_3 = (T_1u_1) - \delta_3 u_3, \quad v_2 = \dfrac{(T_1u_1)(T_1u_3)}{u_3} - \dfrac{\delta_2}{u_1u_3}. \end{cases}$$

(modified d-Boussinesq equation (?))

## m-dBoussinesq as Yang-Baxter map

(Cf. Papageorgiou-Tongas-Veselov, JMP, 2006)

$$\text{Compatibility} \Leftrightarrow \begin{cases} T_1 u_1 = v_2 + \delta_2 u_2, \quad T_1 u_2 = \dfrac{u_1 u_2 v_1 + \delta_1}{u_1 (v_2 + \delta_2 u_2)}, \\ T_\ell v_1 = \dfrac{u_1 u_2 v_1 v_2 + \delta_1 v_2}{u_1 u_2 (v_2 + \delta_2 u_2)}, \quad T_\ell v_2 = \dfrac{u_1^2 u_2 v_1}{u_1 u_2 v_1 + \delta_1} \end{cases}$$

Yang-Baxter map

$$R : (u_1, u_2, v_1, v_2) \mapsto \left( v_2 + \delta_2 u_2, \ \frac{u_1 u_2 v_1 + \delta_1}{u_1 (v_2 + \delta_2 u_2)}, \frac{u_1 u_2 v_1 v_2 + \delta_1 v_2}{u_1 u_2 (v_2 + \delta_2 u_2)}, \frac{u_1^2 u_2 v_1}{u_1 u_2 v_1 + \delta_1} \right)$$

This satisfies Yang-Baxter equation :

$$\widetilde{R}_{uv} R_{uw} R_{vw} = R_{vw} R_{uw} \widetilde{R}_{uv},$$

where $\widetilde{R} : (u_1, u_2, v_1, v_2) \mapsto (v_1, v_2, u_1, u_2)$ and

$$R_{vw}(u, v, w) = (u, R(v, w)), \quad \text{etc.}$$

## Local Darboux transformation for m-dBoussinesq

**Proposition :**

Assume that the local eigenfunction $\begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix}$ satisfies

$$T_1T_2T_3 \begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix} = \kappa \begin{bmatrix} \phi(k) \\ \phi(k+1) \end{bmatrix}, \quad \kappa \neq 0.$$

Then the local Darboux transformation preserves the reduction condition for the dBoussinesq.

$$\{u_j(\ell, \mathbf{m})\} \mapsto \widehat{u}_j(\ell, \mathbf{m}) = \begin{cases} u_j(\ell, \mathbf{m}) & (\ell \neq k, k+1) \\ u_j(k, \mathbf{m}) c_j(k, \mathbf{m}) & (\ell = k) \\ u_j(k+1, \mathbf{m}) / c_j(k, \mathbf{m}) & (\ell = k+1) \end{cases}$$

$$c_j(k, \mathbf{m}) := 1 + \phi(k+1) / \{(\alpha_j - 1)\phi(k) u_j(k, \mathbf{m})\}$$