The ultra-discrete KdV equation and a box-ball system

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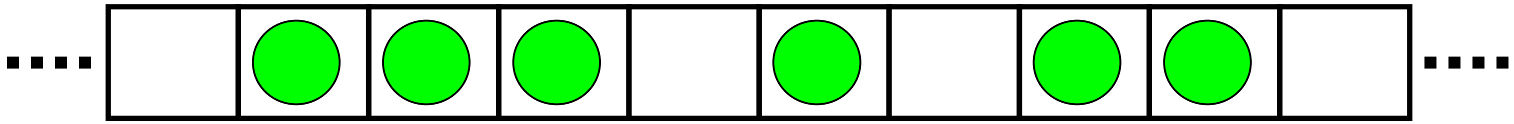
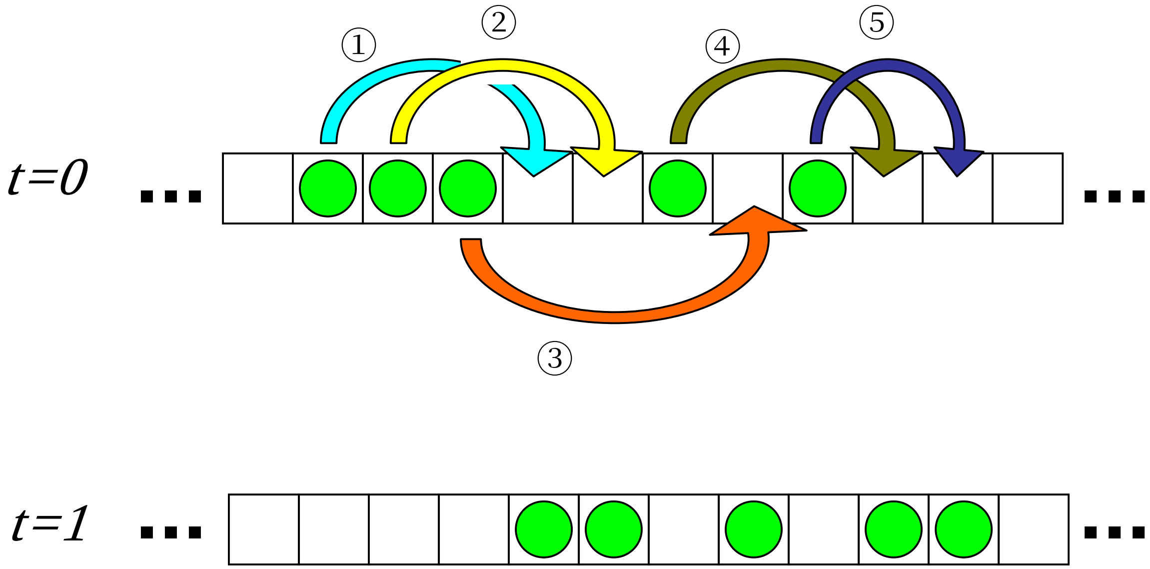
Box-ball system (BBS)

- Takahashi-Satsuma (1990)

  A dynamical system of balls in a one dimensional array of boxes



Time evolution rule



Example

□ = 0 ◉ = 1

t=0: …000111111001110000011111110000000011110011100000000…

t=1: …000000000110001111100000001111111100001100011111100…

t=2: …000000000001100000011111000000000011110011100000011…

t=3: …000000000000011000000000111110000000001100011111000…

t=4: …000000000000000110000000000001111100000011000000111…

t=5: …000000000000000001100000000000000011111000110000000…

⋮ ⋮

Def.

u_n^t : the number of balls in the n-th box at time t. $u_n^t \in \{0,1\}$

Fact:

For an initial state $\{u_n^0\}_{n=-\infty}^{\infty}$ with N solitons, we can determine their initial phases $\{b_i(0)\}_{i=1}^{N}$ and their amplitudes $\{P_i\}_{i=1}^{N}$ by a certain procedure.

Theorem (Initial value problem of the BBS)

Let $b_i(0)$ and P_i be as above.

Then the state $\left\{u_n^t\right\}_{n=-\infty}^{\infty}$ at time step t is given as

$$u_n^t = \left(\rho_{n+1}^{t-1} - \rho_{n+1}^{t}\right) - \left(\rho_n^{t-1} - \rho_n^{t}\right)$$

where

$$\rho_n^t = \max_{J \subseteq \{1,2,\ldots,N\}} \left[\sum_{i \in J} \left(\theta_i + tP_i - n\right) - 2 \sum_{\substack{i,j \in J \\ i<j}} \min\left[P_i, P_j\right] \right],$$

$$\theta_i := b_i(0) + 2 \sum_{j=i+1}^{N} \min[P_i, P_j]$$

and J runs over all the subsets of {1,2,…,N}.

Idea of the proof

d-KdV equation — N-soliton solution τ_n^t

Ultra-Discretization

Ultra-Discretization

ud-KdV equation — N-soliton solution

$$\rho_n^t := \lim_{\varepsilon \to +0} \varepsilon \log \tau_n^t(\varepsilon)$$

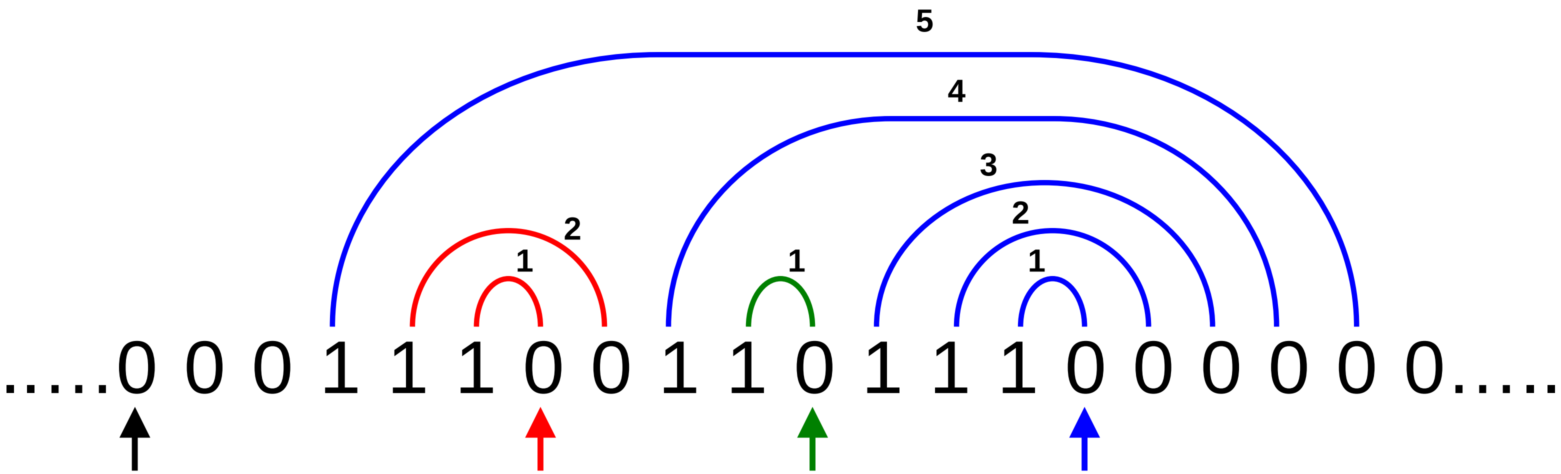
$$= \rho_n^t(\{P_i, \theta_i\}_{i=1}^N)$$

||

Time evolution rule of BBS

- $\{P_i, \theta_i\}_{i=1}^N$ is determined from $\left\{u_n^0\right\}_{n=-\infty}^{\infty}$.

Example



$n = 0$ $\quad$ $b_3(0) = 6$ $\quad$ $b_2(0) = 10$ $\quad$ $b_1(0) = 14$ ··· Initial phases

$P_3 = 2$ $\quad$ $P_2 = 1$ $\quad$ $P_1 = 5$ ··· Amplitudes

Proposition

For any state of the BBS $\{u_n^t\}_{n=-\infty}^{\infty}$, there exists a one parameter family of solutions $\tau_n^t(\varepsilon)$ of the d-KdV equation, which satisfies

$$u_n^t = \left(\rho_{n+1}^{t-1} - \rho_{n+1}^t\right) - \left(\rho_n^{t-1} - \rho_n^t\right),$$

$$\rho_n^t = \lim_{\varepsilon \to +0} \varepsilon \log \tau_n^t(\varepsilon) .$$

[Remark]

We can obtain the solution to the initial value problem of the periodic BBS in a similar manner.