Two-dimensional vector Yajima-Oikawa System
(2D vector Long wave-short wave resonant interaction equations)

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Island 3

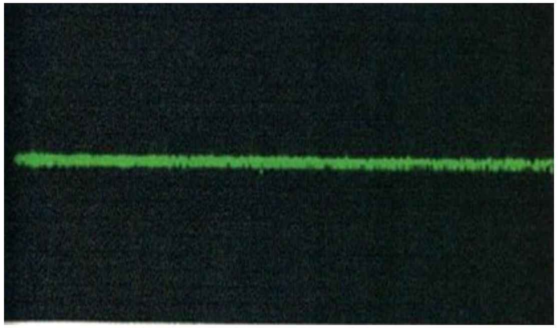
Outline

- Vector Solitons
- Resonance Interaction between long wave and short wave vector soliton (coupled soliton) equation
- (2+1)-dimensional vector soliton (Long wave-short wave resonant interaction)
- Multi-soliton solution

Optical Solitons

Theoretical prediction of (temporal) Soliton in optical fiber: Hasegawa and Tappert, 1973

Experimental realization: Mollenauer et al. 1980



Optical Soliton Communication: High-speed, large capacity

Application to optical devices (e.g. optical switching)

Nonlinear Schrödinger (NLS) equation: $$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+|\psi|^2\psi=0$$

Vector NLS (coupled NLS) equation(Manakov)

$$iq_t + q_{xx} + 2\|q\|^2 q = 0, \quad q = (q_1, q_2, \cdots, q_m).$$

Here $\|q\|^2 \equiv q \cdot q^\dagger = \sum_{j=1}^{m} |q_j|^2$

$$iq_{1,t} + q_{1,xx} + 2\left(\sum_{j=1m} |q_j|^2\right) q_1$$

$$iq_{2,t} + q_{2,xx} + 2\left(\sum_{j=1m} |q_j|^2\right) q_2$$

$$\cdots$$

$$iq_{m,t} + q_{m,xx} + 2\left(\sum_{j=1m} |q_j|^2\right) q_m$$

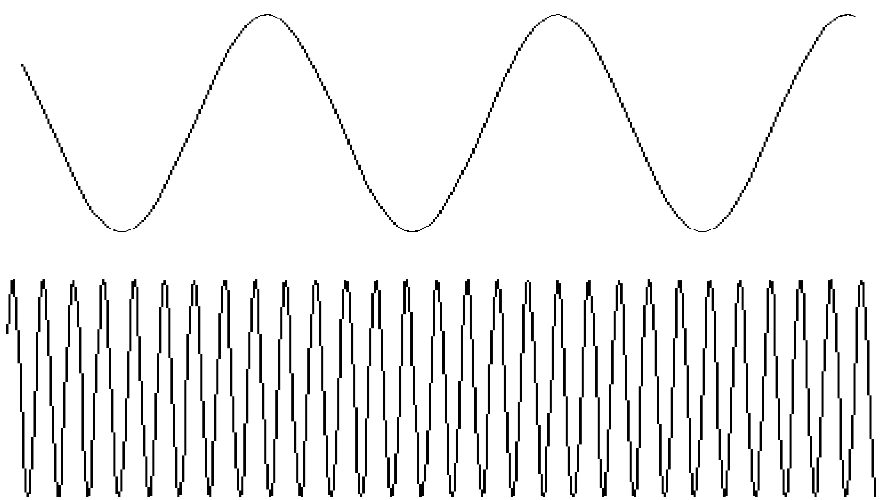
Question

- Is there any two-dimensional vector soliton equation having physical interpretation?
- If yes, investigate multi-soliton dynamics.
- Find some physical vector soliton equations (e.g. Laser physics) We can construct a new system of computing using soliton dynamics
(Construction of Logic Gate)

Physical Difference between KdV and NLS

KdV equation → **Long wave (e.g. Shallow water wave)**

NLS equation → **Short wave (e.g. Deep water wave)**

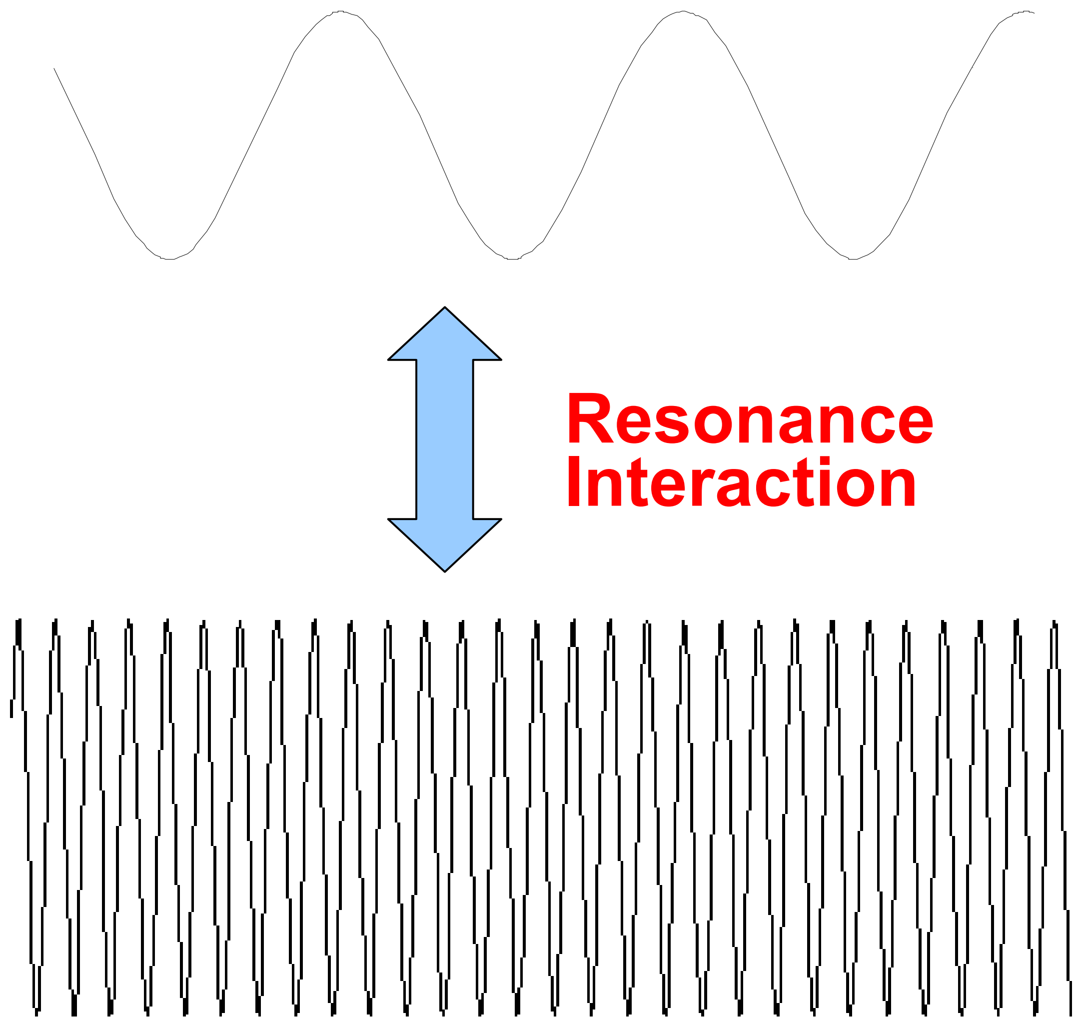


Long wave

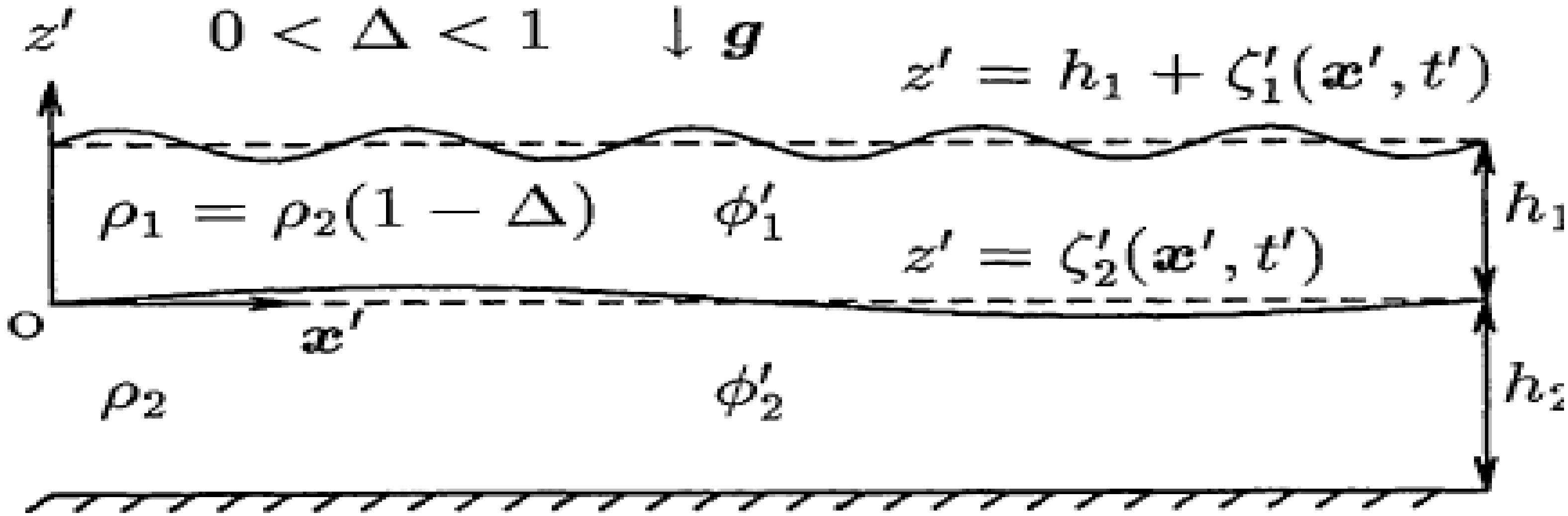
Short wave

Is there any physical phenomenon having both long wave and short wave?

Resonance Interaction between long wave and short wave



Example: Surface wave and internal wave (Oikawa & Funakoshi)



Yajima-Oikawa System
(Long wave- short wave resonance interaction eq.)

$$\mathrm{i}S_T - S_{XX} = -LS,$$

$$L_T = (|S|^2)_X,$$

Derivation of 2D vector Yajima-Oikawa system

Dispersion relation of weakly nonlinear wave

$$\omega_i = \omega_i(k_{x,i}, k_{y,i} : |A_1|^2, |A_2|^2, |A_3|^2), \quad \text{for} \quad i = 1, 2, 3$$

where ω_i and A_i are angular frequencies and amplitudes of each channel i, respectively Suppose that carrier wave is expressed by $\exp(i(k_{x,0}x + k_{y,0}y - \omega_0 t))$. Taylor expansion around $\mathbf{k}_0 = (k_{x,0}, k_{y,0})$, ω_0 and $|A_i| = 0$ makes

$$\begin{aligned}
\omega_i - \omega_0 = &\left(\frac{\partial \omega_i}{\partial k_{x,i}}\right)_0 (k_{x,i} - k_{x,0}) + \left(\frac{\partial \omega_i}{\partial k_{y,i}}\right)_0 (k_{y,i} - k_{y,0}) \\
&+ \frac{1}{2}\left(\frac{\partial^2 \omega_i}{\partial k_{x,i}^2}\right)_0 (k_{x,i} - k_{x,0})^2 + \frac{1}{2}\left(\frac{\partial^2 \omega_i}{\partial k_{y,i}^2}\right)_0 (k_{y,i} - k_{y,0})^2 \\
&+ \left(\frac{\partial^2 \omega_i}{\partial k_{x,i} k_{y,i}}\right)_0 (k_{x,i} - k_{x,0})(k_{y,i} - k_{y,0}) \\
&+ \left(\frac{\partial \omega_i}{\partial |A_1|^2}\right)_0 |A_1|^2 + \left(\frac{\partial \omega_i}{\partial |A_2|^2}\right)_0 |A_2|^2 + \left(\frac{\partial \omega_i}{\partial |A_3|^2}\right)_0 |A_3|^2 + \cdots, \\
&\text{for} \quad i = 1, 2, 3, \qquad (2.1)
\end{aligned}$$

where the subscript 0 of $(\quad)_0$ means setting $k_{x,i} = k_{x,0}$, $k_{y,i} = k_{y,0}$, $\omega_i = \omega_0$ and $|A_i| = 0$.

$$
\begin{aligned}
&i\frac{\partial A_1}{\partial t} + i v_{y,1}\frac{\partial A_1}{\partial y} + \alpha_1\frac{\partial^2 A_1}{\partial x^2} + \alpha_2\frac{\partial^2 A_1}{\partial y^2} + \alpha_3\frac{\partial^2 A_1}{\partial x \partial y} \\
&\quad + \alpha_4|A_1|^2 A_1 + \alpha_5|A_2|^2 A_1 + \alpha_6|A_3|^2 A_1 = 0\,, \\
&i\frac{\partial A_2}{\partial t} + i v_{y,2}\frac{\partial A_2}{\partial y} + \beta_1\frac{\partial^2 A_2}{\partial x^2} + \beta_2\frac{\partial^2 A_2}{\partial y^2} + \beta_3\frac{\partial^2 A_2}{\partial x \partial y} \\
&\quad + \beta_4|A_1|^2 A_2 + \beta_5|A_2|^2 A_2 + \beta_6|A_3|^2 A_2 = 0\,, \\
&i\frac{\partial A_3}{\partial t} + \gamma_1\frac{\partial^2 A_3}{\partial x^2} + \gamma_2\frac{\partial^2 A_3}{\partial y^2} + \gamma_3\frac{\partial^2 A_3}{\partial x \partial y} \\
&\quad + \gamma_4|A_1|^2 A_3 + \gamma_5|A_2|^2 A_3 + \gamma_6|A_3|^2 A_3 = 0\,.
\end{aligned}
$$

Assume that the channel 3 is normal dispersion and the channels 1 and 2 are anomalous dispersion. We study the dark pulses generated in the channel 3:[12]

$$A_1 = \psi_1 \exp(\mathrm{i}\delta_1 t)\,, \quad A_2 = \psi_2 \exp(\mathrm{i}\delta_2 t)\,, \quad A_3 = (u_0 + a(x,y,t))\exp(\mathrm{i}\Gamma t + \mathrm{i}\phi(x,y,t))\,,$$

$$\delta_1 = -\left(\frac{\partial \omega_1}{\partial |A_3|^2}\right)_0 u_0^2\,, \quad \delta_2 = -\left(\frac{\partial \omega_2}{\partial |A_3|^2}\right)_0 u_0^2\,, \quad \Gamma = -\left(\frac{\partial \omega_3}{\partial |A_3|^2}\right)_0 u_0^2\,,$$

Weakly nonlinear

Channel 3:Dark

Channel 1, 2: Bright

$$\frac{\partial a}{\partial t} + \gamma_1 u_0 \frac{\partial^2 \phi}{\partial x^2} + \gamma_2 u_0 \frac{\partial^2 \phi}{\partial y^2} + \gamma_3 u_0 \frac{\partial^2 \phi}{\partial x \partial y} = 0\,,$$

$$- u_0 \frac{\partial \phi}{\partial t} + \gamma_1 \frac{\partial^2 a}{\partial x^2} + \gamma_2 \frac{\partial^2 a}{\partial y^2} + \gamma_3 \frac{\partial^2 a}{\partial x \partial y} + \gamma_4 u_0 |\psi_1|^2 + \gamma_5 u_0 |\psi_2|^2 + 3\gamma_6 u_0^2 a = 0\,,$$

$$i\frac{\partial \psi_1}{\partial t} + i v_{y,1} \frac{\partial \psi_1}{\partial y} + \alpha_1 \frac{\partial^2 \psi_1}{\partial x^2} + \alpha_2 \frac{\partial^2 \psi_1}{\partial y^2} + \alpha_3 \frac{\partial^2 \psi_1}{\partial x \partial y} + \alpha_4 |\psi_1|^2 \psi_1 + \alpha_5 |\psi_2|^2 \psi_1 + 2\alpha_6 u_0 a \psi_1 = 0\,,$$

$$i\frac{\partial \psi_2}{\partial t} + i v_{y,2} \frac{\partial \psi_2}{\partial y} + \beta_1 \frac{\partial^2 \psi_2}{\partial x^2} + \beta_2 \frac{\partial^2 \psi_2}{\partial y^2} + \beta_3 \frac{\partial^2 \psi_2}{\partial x \partial y} + \beta_4 |\psi_1|^2 \psi_2 + \beta_5 |\psi_2|^2 \psi_2 + 2\beta_6 u_0 a \psi_2 = 0\,.$$

By

$$t' = \varepsilon t, \quad x' = \varepsilon^{1/2}(x + ct), \quad y' = \varepsilon y\,,$$

$(c = 3\gamma_1\gamma_6 u_0^2)$ with $a = \varepsilon a_0$, $\psi_1 = \varepsilon^{3/4}\Phi_1$, $\psi_2 = \varepsilon^{3/4}\Phi_2$ (ε is small), we obtain equations of lowest order of ε

$$2c\frac{\partial^2 a}{\partial x \partial t} + \gamma_1 \frac{\partial^2}{\partial x^2}(\gamma_4 u_0 |\psi_1|^2 + \gamma_5 u_0 |\psi_2|^2) = 0\,, \tag{2.14}$$

$$\boxed{i\frac{\partial \psi_1}{\partial t} + i v_{y,1}\frac{\partial \psi_1}{\partial y} + \alpha_1 \frac{\partial^2 \psi_1}{\partial x^2} + 2\alpha_6 u_0 a \psi_1 = 0\,,} \tag{2.15}$$

$$\boxed{i\frac{\partial \psi_2}{\partial t} + i v_{y,2}\frac{\partial \psi_2}{\partial y} + \beta_1 \frac{\partial^2 \psi_2}{\partial x^2} + 2\beta_6 u_0 a \psi_2 = 0\,.} \tag{2.16}$$

First equation

$$\boxed{2c\frac{\partial a}{\partial t} + \gamma_1 \frac{\partial}{\partial x}(\gamma_4 u_0 |\psi_1|^2 + \gamma_5 u_0 |\psi_2|^2) = 0\,.}$$

$v_{y,1} = v_{y,2}, \alpha_1 = \beta_1, \alpha_6 = \beta_6, \gamma_4 = \gamma_5$ ⇒ Integrable

2-dimensional 2-component Yajima-Oikawa system (2-dimensional 2-component long wave-short wave resonance interaction equations)

2-dimensional vector Yajima-Oikawa System (2-component)

$$i(S_t^{(1)} + S_y^{(1)}) - S_{xx}^{(1)} + LS^{(1)} = 0\,,$$
$$i(S_t^{(2)} + S_y^{(2)}) - S_{xx}^{(2)} + LS^{(2)} = 0\,,$$
$$L_t = 2(|S^{(1)}|^2)_x + 2(|S^{(2)}|^2)_x\,.$$

$$\tau_{nm}^{NM} =$$

$$\left|\begin{array}{cccccccccccc}
\varphi_1 & \varphi_1^{(1)} & \cdots & \varphi_1^{(N+M-1+n+m)} & \psi_1 & \psi_1^{(1)} & \cdots & \psi_1^{(N-1-n)} & 0 & 0 & \cdots & 0 \\
\varphi_2 & \varphi_2^{(1)} & \cdots & \varphi_2^{(N+M-1+n+m)} & \psi_2 & \psi_2^{(1)} & \cdots & \psi_2^{(N-1-n)} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\varphi_{2N} & \varphi_{2N}^{(1)} & \cdots & \varphi_{2N}^{(N+M-1+n+m)} & \psi_{2N} & \psi_{2N}^{(1)} & \cdots & \psi_{2N}^{(N-1-n)} & 0 & 0 & \cdots & 0 \\
\phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(N+M-1+n+m)} & 0 & 0 & \cdots & 0 & \chi_1 & \chi_1^{(1)} & \cdots & \chi_1^{(M-1-m)} \\
\phi_2 & \phi_2^{(1)} & \cdots & \phi_2^{(N+M-1+n+m)} & 0 & 0 & \cdots & 0 & \chi_2 & \chi_2^{(1)} & \cdots & \chi_2^{(M-1-m)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\phi_{2M} & \phi_{2M}^{(1)} & \cdots & \phi_{2M}^{(N+M-1+n+m)} & 0 & 0 & \cdots & 0 & \chi_{2M} & \chi_{2M}^{(1)} & \cdots & \chi_{2M}^{(M-1-m)}
\end{array}\right|$$

$$\varphi_i = e^{\xi_i}\,, \qquad \xi_i = p_i x_1 + p_i^2 x_2\,, \qquad \text{for} \quad i = 1, 2, \cdots, N$$

$$\varphi_{N+i} = e^{-\xi_i^*}\,, \qquad -\xi_i^* = -p_i^* x_1 + (-p_i^*)^2 x_2\,, \qquad \text{for} \quad i = 1, 2, \cdots, N$$

$$\phi_i = e^{\theta_i}\,, \qquad \theta_i = s_i x_1 + s_i^2 x_2\,, \qquad \text{for} \quad i = 1, 2, \cdots, M$$

$$\phi_{M+i} = e^{-\theta_i^*}\,, \qquad -\theta_i^* = -s_i^* x_1 + (-s_i^*)^2 x_2\,, \qquad \text{for} \quad i = 1, 2, \cdots, M$$

$$\psi_i = a_i e^{\eta_i}\,, \qquad \eta_i = q_i y_1 + \eta_{i0}\,, \qquad \text{for} \quad i = 1, 2, \cdots, N$$

$$\psi_{N+i} = a_{N+i} e^{-\eta_i^*}\,, \qquad -\eta_i^* = -q_i^* y_1 - \eta_{i0}^*\,, \qquad \text{for} \quad i = 1, 2, \cdots, N$$

$$\chi_i = b_i e^{\zeta_i}\,, \qquad \zeta_i = r_i z_1 + \zeta_{i0}\,, \qquad \text{for} \quad i = 1, 2, \cdots, M$$

$$\chi_{M+i} = b_{M+i} e^{-\zeta_i^*}\,, \qquad -\zeta_i^* = -r_i^* z_1 - \zeta_{i0}^*\,, \qquad \text{for} \quad i = 1, 2, \cdots, M$$

$$a_i = \left(\prod_{\substack{k=1\\k\neq i}}^{N} \frac{p_k - p_i}{q_k - q_i}\right)\left(\prod_{l=1}^{M}(s_l - p_i)\right), \qquad \text{for} \quad i = 1, 2, \cdots, N$$

$$a_{N+i} = \epsilon_i \left(\prod_{k=1}^{N} \frac{p_k + p_i^*}{q_k + q_i^*}\right)\left(\prod_{l=1}^{M}(s_l + p_i^*)\right), \qquad \text{for} \quad i = 1, 2, \cdots, N$$

$$b_i = \left(\prod_{k=1}^{N}(p_k - s_i)\right)\left(\prod_{\substack{l=1\\l\neq i}}^{M} \frac{s_l - s_i}{r_l - r_i}\right), \qquad \text{for} \quad i = 1, 2, \cdots, M$$

$$b_{M+i} = \delta_i \left(\prod_{k=1}^{N}(p_k + s_i^*)\right)\left(\prod_{l=1}^{M} \frac{s_l + s_i^*}{r_l + r_i^*}\right), \qquad \text{for} \quad i = 1, 2, \cdots, M$$

$$\varepsilon_i = \pm 1\,, \qquad \delta_i = \pm 1\,,$$

where * means complex conjugate and p_i, q_i $(1 \le i \le N)$ and s_i, r_i $(1 \le i \le M)$ are complex wave numbers, and η_{i0} $(1 \le i \le N)$ and ζ_{i0} $(1 \le i \le M)$ are complex phase parameters. In order to obtain regular solutions, we have to choose appropriate sign for ε_i and δ_i, which depend on parameters p_i, q_i, r_i, s_i. We take

$$x_1 = x\,, \qquad x_2 = -iy\,, \qquad y_1 = y - t\,, \qquad z_1 = y - t\,,$$

where x, y and t are real , (i.e. x_1, y_1 and z_1 are real and x_2 is pure imaginary).

Let

$$f = \tau_{00}\,, \qquad g = \tau_{10}\,, \qquad \bar{g} = \tau_{-1,0}\,, \qquad h = \tau_{01}\,, \qquad \bar{h} = \tau_{0,-1}\,.$$

These tau-functions satisfy the condition

$$\boxed{\left(\frac{g}{f}\right)^* = \frac{\bar{g}}{f}\,, \qquad \left(\frac{h}{f}\right)^* = \frac{\bar{h}}{f}\,, \qquad f\mathcal{G} : \text{real}\,.}$$

where $\mathcal{G}$ is an exponential factor which is a gauge function (see Appendix). Let $F = f\mathcal{G}$, $G = g\mathcal{G}$, $G^* = \bar{g}\mathcal{G}$, $H = h\mathcal{G}$, $H^* = \bar{h}\mathcal{G}$. The functions F, G and H satisfy the bilinear equations (3.5)-(3.7) and reality of F and complex conjugacy of G and H. The function $L = -2\frac{\partial^2}{\partial x^2}\log F$ represents $N+M$-soliton solution, $S_1 = G/F$ represents N-soliton solution, and $S_2 = H/F$ represents M-soliton solution.

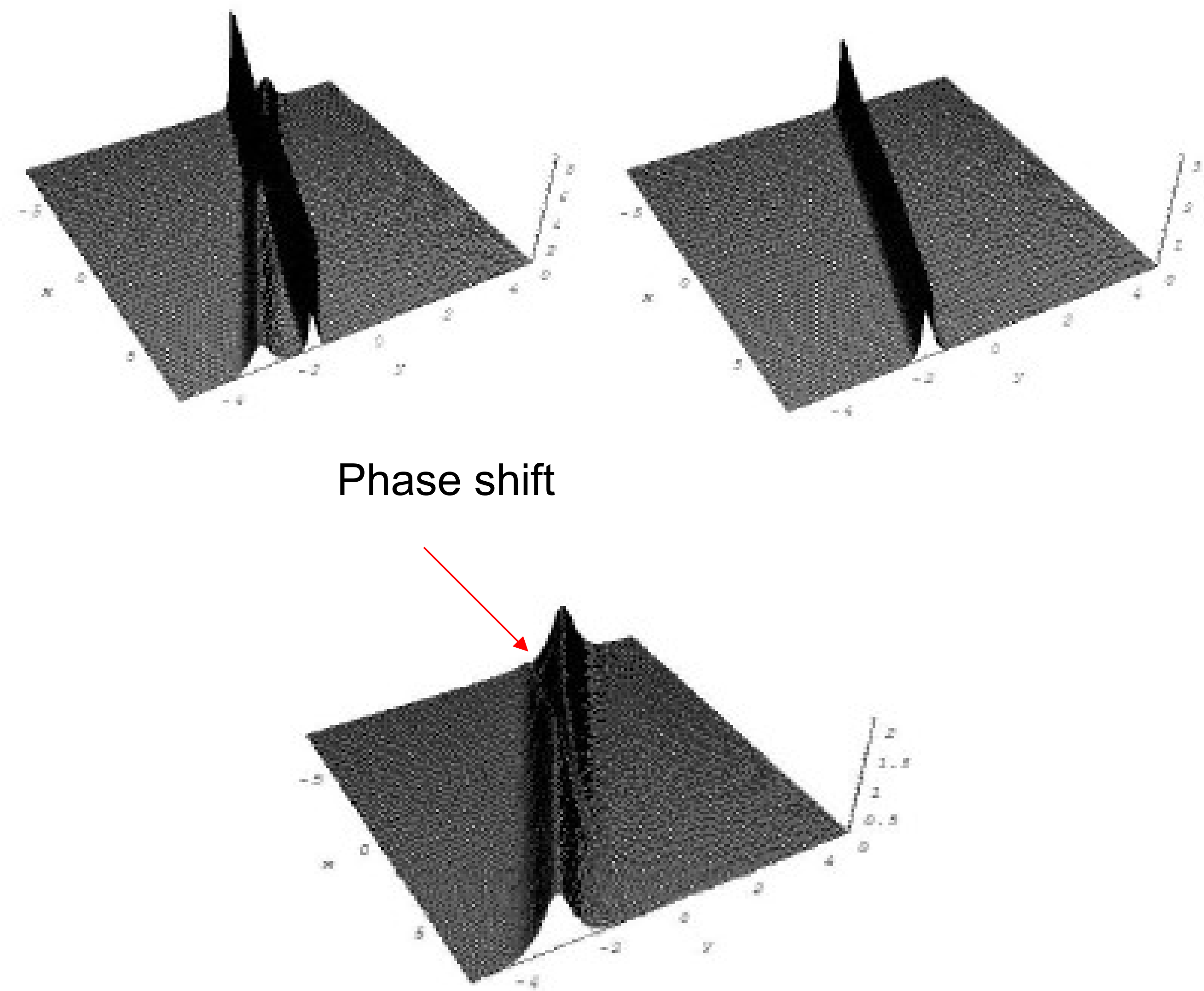


Figure 1. $(1, 1, 2)$-soliton solution. $p_1 = 2 + 2i$, $s_1 = -1 + i$, $q_1 = -2 + i$, $r_1 = 1 + i$, $\epsilon = \delta = 1$. The top left graph is $-L$, the top right graph is $S^{(1)}$ and the bottom graph is $S^{(2)}$ at $t = 0$.

Interaction of 2-line soliton and periodic soliton

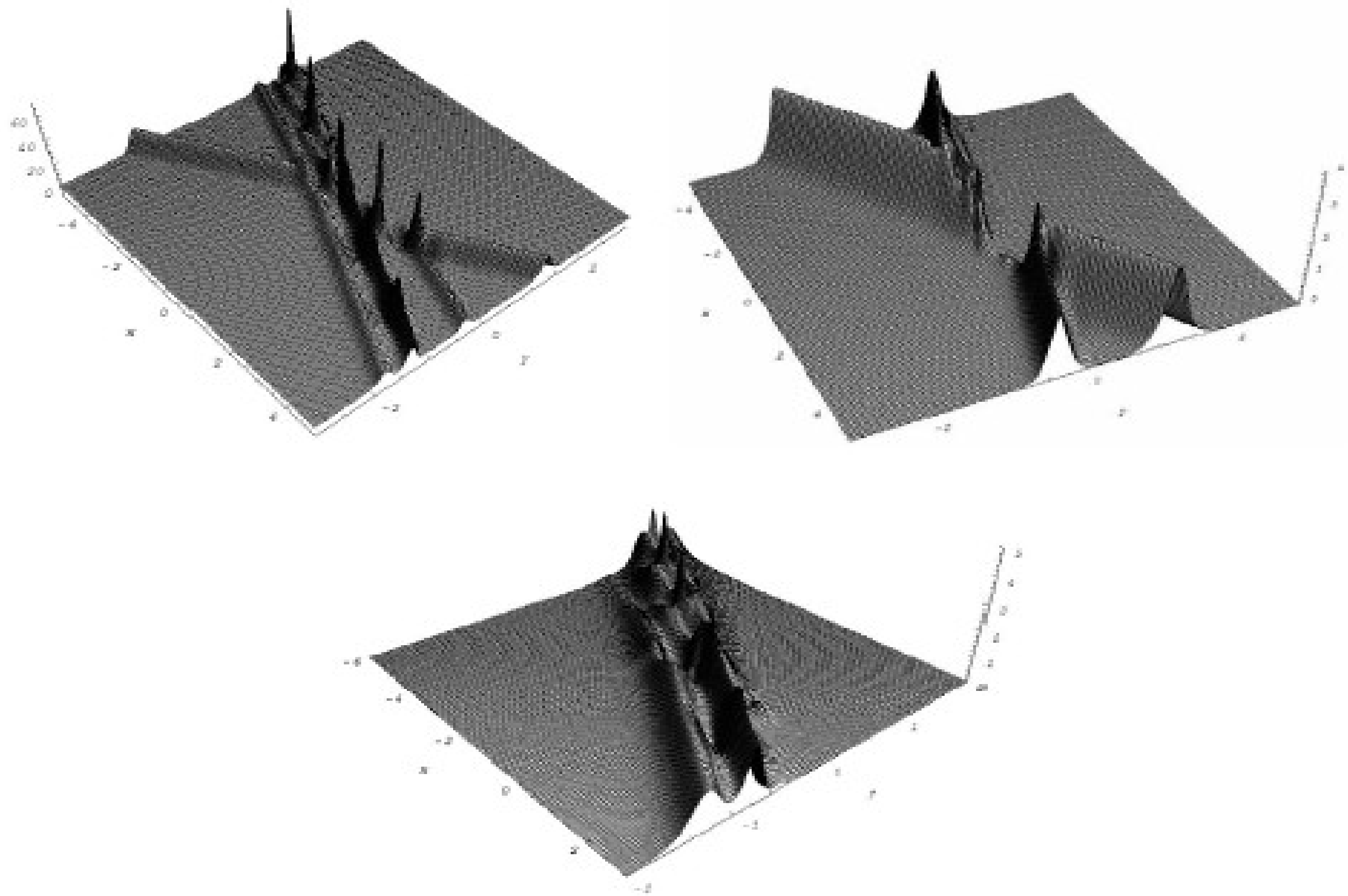


Figure 3. $(2, 2, 4)$-soliton solution. $p_1 = 2 + 3i$, $p_2 = 3 - i$, $p_3 = 2 - 3i$, $p_4 = 3 + i$, $s_1 = 2 + 2i$, $s_2 = 4 + 2i$, $s_3 = 2 - 2i$, $s_4 = -4 - 2i$, $q_1 = 2 + i$, $q_2 = 2.01 + i$, $q_3 = 2 - i$, $q_4 = 2.01 - i$, $r_1 = 1 + i$, $r_2 = 1.5 + i$, $r_3 = 1 - i$, $r_4 = 1.5 - i$, $\varepsilon_1 = \varepsilon_2 = \delta_1 = \delta_2 = -1$. The top left graph is $-L$, the top right graph is $S^{(1)}$ and the bottom graph is $S^{(2)}$ at $t = 0$.

Conclusion

- We derived 2-dimensional vector YO system in physical setting
- We constructed Wronskian solutions of 2-dimensional vector YO system
- Soliton interaction of vector YO system has some unusual properties.