Weakly Nonassociative Algebras and the KP Hierarchy

Folkert Müller-Hoissen
MPI for Dynamics and Self-Organization, Göttingen

joint work with
Aristophanes Dimakis
University of the Aegean, Chios, Greece

References: see arXiv 2006/7

ISLAND 3, July 2007

This talk is about

- Matrix KP equations (hierarchies) (Zakharov & Kuznetsov '86, Athorne & Fordy '87, ...). Some of its uses:
  - (complicated) solutions of the *scalar* KP hierarchy (and other integrable equations) arise from (simple) solutions of a matrix (or operator) version (Marchenko '88, Carl, Schiebold)
  - bridge to sdYM: *dispersionless limit* of matrix potential KP equation is pseudo-dual chiral model (dual to Ward's chiral model) (Dimakis & M-H '07)
- More generally: KP with dependent variable in any associative algebra (see also Olver & Sokolov '98, Kupershmidt '00).
  This point of view takes us away from the multi-component KP framework (Sato '81) which also covers matrix KP.
- Relation with a special type of *nonassociative* algebras.
  Older work on relations between nonassociative algebras and integrable systems: Svinolupov, Sokolov, ...

Plan of this talk

I. From nonassociativity to KP

II. What are WNA algebras ?

- free WNA algebra
- quasisymmetric functions

III. The hierarchy of ODEs on a WNA algebra

IV. A class of exact solutions

V. From WNA to Gelfand-Dickey-Sato

VI. Conclusions

I. From nonassociativity ...

$$
\begin{tikzcd}
 & & & f \arrow[d] & & & & \\
 & & & f^2 \arrow[dll] \arrow[drr] & & & & \\
 & f\,f^2 \arrow[dl] \arrow[drr] & & & & f^2 f \arrow[dll] \arrow[dr] & & \\
f(f\,f^2) & & & (f\,f^2)f \color{red}{=} f(f^2 f) & & & (f^2 f)f & f^2 f^2
\end{tikzcd}
$$

Look for **commuting derivations**:

$$
\begin{aligned}
\delta_1(f) &:= f^2 \\
\delta_2(f) &:= \alpha\, f\, f^2 + \beta\, f^2\, f \\
&\ \vdots
\end{aligned}
\qquad
[\delta_1, \delta_2] = 0 \iff
\begin{aligned}
&(\alpha - \beta)\,(f, f^2, f) \\
&= (\alpha + \beta)\, f^2\, f^2
\end{aligned}
$$

$$
(a, bc, d) = 0 \qquad \forall a, b, c, d \in \mathbb{A}
$$

$\curvearrowright$ Weak Non Associativity

... to KP

$$\begin{aligned}\delta_1(f) &= f^2\\ \delta_2(f) &= f\, f^2 - f^2\, f\\ \delta_3(f) &= f\,(f\, f^2) - f\, f^2\, f - f^2 f^2 + (f^2\, f)\, f\end{aligned}$$

$$\Rightarrow \quad \boxed{\delta_1\Big(4\,\delta_3(f) - \delta_1^3(f) + 6\,\delta_1(f)^2\Big) - 3\,\delta_2^2(f) \equiv 6\,[\delta_1(f), \delta_2(f)]}$$

This **identity** formally corresponds to **potential KP equation** via

$$\delta_n \quad \mapsto \quad \partial_{t_n}$$

This relation extends to the whole KP hierarchy !
Building law for the commuting derivations:

$$\boxed{\delta_n(f) := f \circ_n f}$$

where $a \circ_1 b := a\, b$ and

$$\boxed{a \circ_{n+1} b := a\,(f \circ_n b) - (a\, f) \circ_n b}$$

Consequence: (true for *any* WNA algebra)

$$\boxed{\partial_{t_n}(f) = f \circ_n f \quad n = 1, 2, \ldots}$$

$$\Longrightarrow \quad u := -\partial_{t_1}(f) \in \mathbb{A}' \text{ solves } \textbf{KP hierarchy}$$

II. What are WNA algebras ?

$\mathbb{A}$ WNA algebra. **Associative subalgebra** and ideal:

$$\boxed{\mathbb{A}' := \{ b \in \mathbb{A} \,|\, (a,b,c) = 0 \quad \forall a, c \in \mathbb{A} \}}$$

Construction of special WNA algebras:

- $\mathcal{A}$ associative algebra (over commutative ring)
- $g \in \mathcal{A}$ fixed
- linear maps $L, R \;:\; \mathcal{A} \to \mathcal{A}$ such that

$$[L, R] = 0\,, \qquad L(a\,b) = L(a)\,b\,, \qquad R(a\,b) = a\,R(b)$$

Augment $\mathcal{A}$ by an element f such that

$$f\,f = g\,, \quad f\,a = L(a)\,, \quad a\,f = R(a)$$

$\Longrightarrow$ WNA algebra $\mathbb{A}$ with $\dim(\mathbb{A}/\mathbb{A}') = 1$ and $\mathbb{A}' = \mathcal{A}$.

Generalization: $L_i, R_i, [L_i, R_j] = 0 \curvearrowright \dim(\mathbb{A}/\mathbb{A}') = N$

Any WNA algebra with $\dim(\mathbb{A}/\mathbb{A}') = N$ is isomorphic to one of these!

Example: free WNA algebra (with $\dim(\mathbb{A}/\mathbb{A}') = 1$)

$\mathcal{A}_{\text{free}}$ **associative** algebra freely generated by $c_{m,n}$, $m, n = 0, 1, 2, \ldots$.
Define

$$L(c_{m,n}) := c_{m+1,n}\,, \qquad R(c_{m,n}) := c_{m,n+1}$$

Augment $\mathcal{A}_{\text{free}}$ with an element f such that

$$f^2 := c_{0,0}\,, \qquad f\,a := L(a)\,, \qquad a\,f := R(a) \qquad \forall a \in \mathcal{A}_{\text{free}}$$

Writing $L_f(a) := f\,a$, $R_f(a) := a\,f$, we have

$$c_{m,n} = L_f^m R_f^n(f^2)$$

$\Longrightarrow$ WNA algebra $\mathbb{A}(f)_{\text{free}}$ **freely generated by** f

$\Longrightarrow$ derivations δ_n are defined by action on f and derivation rule
e.g. $\delta_3(f) = c_{2,0} - c_{1,1} + c_{0,2} - {c_{0,0}}^2$ (nonlinearity !)

$\Longrightarrow$ **'KP hierarchy identities'**

Example: Algebra of quasisymmetric functions

... in (commuting) variables $p_1, p_2, \ldots$ is spanned by elements

$$\sum_{i_1 < i_2 < \ldots < i_r} p_{i_1}^{n_1} \cdots p_{i_r}^{n_r}$$

Examples of quasisymmetric polynomials in three variables:

$$p_1 p_2^2 + p_1 p_3^2 + p_2 p_3^2 \,, \qquad p_1^3 p_2^2 + p_1^3 p_3^2 + p_2^3 p_3^2$$

Let $\mathcal{A} = \mathbb{Z}[[p_1, p_2, \ldots]]$. For a monomial $a = p_{i_1} \cdots p_{i_r}$ define

$$m(a) := \min\{i_1, \ldots, i_r\} \,, \qquad M(a) := \max\{i_1, \ldots, i_r\}$$

Introduce the new product $\boxed{a \circ_1 b = a\, b \, \sum_{M(a) < i \leq m(b)} p_i}$

Augment with f such that

$$f \circ_1 f := \sum_i p_i \,, \quad f \circ_1 a := a \sum_{i \leq m(a)} p_i \,, \quad a \circ_1 f := a \sum_{M(a) < i} p_i$$

$\Longrightarrow$ WNA algebra freely generated by f

$\Longrightarrow$ derivations δ_n exist $\Longrightarrow$ KP identities

We have

$$\sum_i p_i^m \circ_1 \sum_k p_k^n = \sum_{i<j\leq k} p_i^m p_j p_k^n$$

$$\delta_n(f) = f \circ_n f = \sum_i p_i^n$$

$$\delta_n(a) = \left(\sum_i p_i^n\right) a \qquad\qquad a \in \mathcal{A}$$

so that the **KP identity** becomes

$$\left(\sum_i p_i\right)\left[4\sum_i p_i^3 - \left(\sum_i p_i\right)^3 + 6\sum_{i<j\leq k} p_i p_j p_k\right] - 3\left(\sum_i p_i^2\right)^2$$
$$+6\sum_{i<j\leq k}\left(p_i^2 p_j p_k - p_i p_j p_k^2\right) \equiv 0$$

Such identities show up if one tries to solve the (potential) KP with a (formal) power series ansatz (Okhuma & Wadati '83)

See also: Dimakis & M-H, J. Phys. A **38** (2005) 5453

III. The hierarchy of ODEs on a WNA algebra $\mathbb{A}$

$$\boxed{\partial_{t_n}(f) = f \circ_n f \qquad n = 1, 2, \ldots}$$

Recall:

- $\mathbb{A}$ is WNA if $(a, bc, d) = 0$, i.e. $\mathbb{A}^2 \subset \mathbb{A}'$
  $\mathbb{A}' = \{b \in \mathbb{A} \,|\, (a, b, c) = 0 \quad \forall a, c \in \mathbb{A}\}$
- $a \circ_1 b := ab, \qquad a \circ_{n+1} b := a\,(f \circ_n b) - (a\,f) \circ_n b$
  Note: $\circ_n$ only depends on $[f] \in \mathbb{A}/\mathbb{A}'$
- If f solves the above hierarchy of ODEs, then $u = -\partial_{t_1}(f)$ solves the KP hierarchy in $\mathbb{A}'$

If there is a **constant** element $\nu \in \mathbb{A}$, $\nu \notin \mathbb{A}'$, with $[\nu] = [f]$, then

$$\boxed{\phi := \nu - f \in \mathbb{A}'}$$

and it solves the **potential KP hierarchy**.
The above hierarchy of ODEs then becomes

$$\boxed{\partial_{t_n}(\phi) = -\nu \circ_n \nu + \nu \circ_n \phi + \phi \circ_n \nu - \phi \circ_n \phi}$$

IV. A class of exact solutions

$\mathcal{A}$ = algebra of (complex) $N \times M$ matrices with product

$$\boxed{A \circ B := AQB} \qquad Q \quad \text{constant } M \times N \text{ matrix}$$

To turn it into a WNA algebra, augment by ν such that

$$\nu \circ \nu = -S\,, \qquad \nu \circ A = L\,A\,, \qquad A \circ \nu = -A\,R$$

with constant matrices S, L, R. Set

$$H := \begin{pmatrix} R & Q \\ S & L \end{pmatrix}, \qquad H^n =: \begin{pmatrix} R_n & Q_n \\ S_n & L_n \end{pmatrix}$$

The hierarchy of ODEs becomes the **matrix Riccati system**

$$\Longrightarrow \quad \boxed{\phi_{t_n} = S_n + L_n\,\phi - \phi\,R_n - \phi\,Q_n\,\phi} \qquad n = 1, 2, \ldots$$

which is solved in the Grassmannian way:

$$\curvearrowright \quad \boxed{Z_{t_n} = H^n Z}\,, \quad Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad \boxed{\phi = Y X^{-1}}$$

$$\Longrightarrow \quad \boxed{Z = e^{\xi(\mathbf{t},H)}\,Z_0} \quad \text{where} \quad \boxed{\xi(\mathbf{t}, H) = \sum_{n\geq 1} t_n\,H^n} \quad \curvearrowright \quad \phi$$

Some cases in which ϕ can be computed explicitly

1. Let $S = 0$, $Q = RK - KL$ with a matrix K

$$\phi = e^{\xi(\mathbf{t},L)}\phi_0\,(I_N + K\phi_0 - e^{-\xi(\mathbf{t},R)}Ke^{\xi(\mathbf{t},L)}\phi_0)^{-1}e^{-\xi(\mathbf{t},R)}$$

If $\boxed{\mathrm{rank}(Q) = 1}$ (and using the 'trace method' trick):

$$\varphi := \mathrm{tr}(Q\phi) = (\log\tau)_{t_1}$$

yields in particular scalar KP-II multi-solitons and resonances.
If $\boxed{\mathrm{rank}(Q) = m}$, *solutions of the $m \times m$ matrix KP hierarchy* are obtained via $\varphi := U^T\phi V$ where $Q = VU^T$.

2. $M = N$, $S = 0$, $R = L$, $Q = I_N + [L, K]$

$$\Longrightarrow \quad \phi = e^{\xi(\mathbf{t},L)}\phi_0\,(I_N + K\phi_0 + F)^{-1}e^{-\xi(\mathbf{t},L)}$$

where

$$F := \Big(\sum_{n\geq 1} n\,t_n\,L^{n-1} - e^{-\xi(\mathbf{t},L)}Ke^{\xi(\mathbf{t},L)}\Big)\,\phi_0$$

If $\mathrm{rank}(Q) = 1$ one easily recovers a tau function associated with *Calogero-Moser* systems (Shiota '94), and KP-I lump solutions.

V. From WNA to Gelfand-Dickey-Sato

$$\boxed{\mathfrak{L} = \partial + u_2\,\partial^{-1} + u_3\,\partial^{-2} + \ldots}$$

$\mathcal{U}$:= algebra of polynomials in $u_n^{(m)} = \partial^m(u_n)$, $m = 0, 1, \ldots$ with unit element I. Assume: $\{u_n^{(m)}\}$ algebraically independent. On the algebra of ΨDOs $\mathcal{V} = \sum_{i \ll \infty} w_i\,\partial^i$ with $w_i \in \mathcal{U}$, define

$$S(\mathcal{V}) := \mathfrak{L}\,\mathcal{V}\,, \qquad \pi_+ := \text{projection to diff. operator part}\,,$$

and $\pi_- := \mathrm{id} - \pi_+$. Furthermore,

$$\mathcal{O} := \operatorname{span}\{S, S\pi_\pm S\pi_\pm \cdots \pi_\pm S\} \qquad \text{product: } A \bullet B = A\pi_+ S\pi_- B$$

$$\mathcal{A} := \{w \in \mathcal{U} \mid w = \operatorname{res}(A(I))\,,\ A \in \mathcal{O}\}$$

Then: $\quad (\mathcal{O}, \bullet) \cong \mathcal{A}$

Augment $\mathcal{A}$ with f such that $\quad f\,f := -\operatorname{res}(\mathfrak{L}) \quad$ and

$$f\,\operatorname{res}(A(I)) := \operatorname{res}(\mathfrak{L}\pi_-(A(I)))\,, \qquad \operatorname{res}(A(I))\,f := -\operatorname{res}(\pi_-(A(I))\,\mathfrak{L})$$

$\curvearrowright$ **WNA**. Then $f_{t_n} = f \circ_n f = -\operatorname{res}(\mathfrak{L}^n)$ has integrability condition

$$\operatorname{res}(\mathfrak{L}^m)_{t_n} = \operatorname{res}([\pi_+(\mathfrak{L}^n), \mathfrak{L}^m]) \quad \curvearrowright \quad \boxed{\mathfrak{L}_{t_n} = [\pi_+(\mathfrak{L}^n), \mathfrak{L}]}$$

VI. Conclusions

The WNA framework constitutes a considerable *abstraction* from the usual KP setting. This allows to establish relations between seemingly unrelated structures.

- If the WNA subalgebra generated by $f \in \mathbb{A}$ admits *derivations* s.t. $\boxed{\delta_n(f) := f \circ_n f}$, then there are *'KP identities'*
  Example: *quasisymmetric functions*
  These actually appear in the Okhuma-Wadati method !
- Let $\mathbb{A}$ be WNA and f a solution of $\boxed{\partial_{t_n}(f) = f \circ_n f}$. Then $-\partial_{t_1}(f)$ solves the KP hierarchy in $\mathbb{A}'$
  (Instead of PDEs, we only have to solve ODEs.)
- Other realizations of WNA algebras and the δ_n ?

Needs clarification:

- (Further) relations with Grassmannians (and Sato theory)
- What about *other* hierarchies ?
  $\curvearrowright$ look for commuting derivations on other nonass. algebras

Thank you for your attention !