Kernel functions for Koornwinder's q-difference operators

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Kernel function

Consider two sets of variables $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$, and two operators $\mathcal{A}_x$ and $\mathcal{B}_y$ which act on functions in the x variables and the y variables, respectively. Then a function $\Phi(x; y)$ in $m + n$ variables $(x; y)$ is called a *kernel function* associated with the pair of operators $(\mathcal{A}_x, \mathcal{B}_y)$ if it satisfies the functional equation

$$\mathcal{A}_x \, \Phi(x; y) = \mathcal{B}_y \, \Phi(x; y).$$

Koornwinder's q-difference operator

Koornwinder's q-difference operator $\mathcal{D}_x = \mathcal{D}_x(a, b, c, d|q, t)$ in m variables $x = (x_1, \dots, x_m)$ is defined by

$$\mathcal{D}_x = \sum_{i=1}^{m} A_i(x)(T_{q,x_i} - 1) + \sum_{i=1}^{m} A_i(x^{-1})(T_{q,x_i}^{-1} - 1)$$

where

$$A_i(x) = \frac{(1 - ax_i)(1 - bx_i)(1 - cx_i)(1 - dx_i)}{(1 - x_i^2)(1 - qx_i^2)} \prod_{j \neq i} \frac{(tx_i - x_j)(1 - tx_i x_j)}{(x_i - x_j)(1 - x_i x_j)}.$$

The Koornwinder polynomials $P_\lambda(x) = P_\lambda(x; a, b, c, d|q, t)$ (parametrized by partitions λ) are characterized as such eigenfunctions of $\mathcal{D}_x$ that are invariant Laurent polynomials under the action of the hyperoctahedral group of degree n.

Summary of the talk

We introduce some kernel functions which intertwine Koornwinder's q-difference operators in different sets of variables. As an application we derive explicit formulas for those Koornwinder polynomials attached to single columns and single rows. This talk is based on discussion with Jun'ichi Shiraishi (Tokyo) and Yasushi Komori (Nagoya).

General remarks: Why kernel functions?

Consider two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$. A function $\Phi(x; y)$ in $(x; y)$ is called a *kernel function* associated with a pair of operators $(\mathcal{A}_x, \mathcal{B}_y)$ if it satisfies

$$\mathcal{A}_x \, \Phi(x; y) = \mathcal{B}_y \, \Phi(x; y).$$

1. Expansion of a kernel function in terms of eigenfunctions

Suppose that there exists a family $\varphi_k(y)$ $(k = 1, 2, \ldots)$ of eigenfunctions of $\mathcal{B}_y$ with distinct eigenvalues,

$$\mathcal{B}_y \, \varphi_k(y) = \lambda_k \, \varphi_k(y) \qquad (k = 1, 2, \ldots),$$

and that the kernel function $\Phi(x; y)$ has an expansion by the eigenfunctions $\varphi_k(y)$ in the form

$$\Phi(x; y) = \sum_{k=1}^{\infty} f_k(x) \, \varphi_k(y)$$

with nonzero functions $f_k(x)$ in x. Then each coefficient $f_k(x)$ must be an eigenfunction of the operator $\mathcal{A}_x$ with the same eigenvalue λ_k as that of $\varphi_k(y)$:

$$\mathcal{A}_y \, f_k(x) = \lambda_k \, f_k(x) \qquad (k = 1, 2, \ldots).$$

2. Integral transformations defined by a kernel function

Consider a measure $d\mu(y)$ on the affine space $\mathbb{C}^n$ with canonical coordiates $y = (y_1, \ldots, y)$, and suppose that in an appropriate function space the operator $\mathcal{B}_y$ has an adjoint operator $\mathcal{B}_y^*$ with respect to $d\mu(y)$:

$$\int \mathcal{B}_y \, \varphi(y) \, \psi(y) d\mu(y) = \int \varphi(y) \, \mathcal{B}_y^* \, \psi(y) \, d\mu(y).$$

Then the integral transformation

$$f(x) = \int \Phi(x; y) \, \varphi(y) \, d\mu(y)$$

defined by the kernel $\Phi(x; y)$ transforms each eigenfunction $\varphi(y)$ of the adjoint operator $\mathcal{B}_y^*$ into an eigenfunction $f(x)$ of $\mathcal{A}_x$:

$$\begin{aligned}
\mathcal{A}_x f(x) &= \int \mathcal{A}_x \Phi(x; y) \, \varphi(y) \, d\mu(y) \\
&= \int \mathcal{B}_y \Phi(x; y) \, \varphi(y) \, d\mu(y) \\
&= \int \Phi(x; y) \, \mathcal{B}_y^* \, \varphi(y) \, d\mu(y).
\end{aligned}$$

Macdonald's q-difference operator (of type A_{m-1})

Macdonald's q-difference operator (of first order) in m variables $x = (x_1, \ldots, x_m)$ is defined by

$$D_x = D_x(q,t) = \sum_{i=1}^{m} \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}.$$

The Macdonald polynomials $P_\lambda(x|q,t)$ are symmetric polynomials, parameterized by partitions $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m$, $(\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$, such that

$$D_x \, P_\lambda(x|q,t) = d_\lambda \, P_\lambda(x|q,t), \qquad d_\lambda = \sum_{i=1}^{m} t^{m-i} q^{\lambda_i}.$$

In the case of Macdonald operators there are two types of known kernel functions (as already described in Macdonald's book).

Kernel function of type I

$$\Phi(x;y) = \prod_{j=1}^{m} \prod_{l=1}^{n} \frac{(tx_j y_l; q)_\infty}{(x_j y_l; q)_\infty} = \sum_{l(\lambda) \leq m \wedge n} b_\lambda(q,t) \, P_\lambda(x|q,t) \, P_\lambda(y|q,t),$$

where $|q| < 1$ and $(x;q)_\infty = \prod_{i=0}^{\infty}(1 - q^i x)$. This kernel function intertwines two Macdonald operators in x variables and y variables:

$$t^{-m} \left(D_x - \frac{t^m - 1}{t - 1} \right) \Phi(x;y) = t^{-n} \left(D_y - \frac{t^n - 1}{t - 1} \right) \Phi(x;y).$$

Kernel function of type II

$$\Psi(x;y) = \prod_{j=1}^{m} \prod_{l=1}^{n} (x_j - y_l) = \sum_{\lambda \subset (n^m)} (-1)^{|\lambda^*|} \, P_\lambda(x|q,t) \, P_{\lambda^*}(y|t,q),$$

where the summation is taken over all partition λ contained in the $m \times n$ rectangle, and $\lambda^* = (m - \lambda'_n, m - \lambda'_{n-1}, \ldots, m - \lambda'_1)$ stands for the partition *complementary* to λ in (n^m). This function $\Psi(x;y)$ satisfies the functional equation

$$(t-1) D_x \Psi(x;y) + (q-1) \widetilde{D}_y \Psi(x;y) = (t^m q^n - 1) \Psi(x;y),$$

where $\widetilde{D}_y = D_y(t,q)$.

The case of Schur functions ($t = q$)

The Schur functions are recovered from the Macdoanld polynomials as the special case $t = q$:

$$s_\lambda(x) = P_\lambda(x; q, q).$$

Kernel function of type I

$$\Phi(x; y) = \prod_{j=1}^{m} \prod_{l=1}^{n} \frac{1}{1 - x_j y_l} = \sum_{l(\lambda) \le m \wedge n} s_\lambda(x)\, s_\lambda(y).$$

This formula follows from Cauchy's lemma

$$\det \left[\frac{1}{1 - x_i y_j} \right]_{i,j=1}^{N} = \frac{\prod_{i<j} (x_i - x_j)(y_i - y_j)}{\prod_{i,j=1}^{N} (1 - x_i y_j)}.$$

Kernel function of type II

$$\Psi(x; y) = \prod_{j=1}^{m} \prod_{l=1}^{n} (x_j - y_l) = \sum_{\lambda \subset (n^m)} (-1)^{|\lambda^*|}\, s_\lambda(x)\, s_{\lambda^*}(y).$$

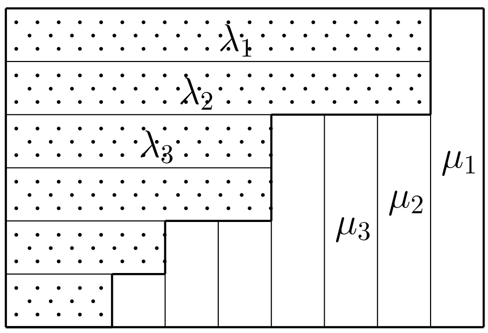
This formula is equivalent to Vandermonde's determinant formula

$$\det \begin{bmatrix} x_j^{i-1} & (i = 1, \ldots, m) \\ y_j^{i-m-1} & (i = m+1, \ldots, m+n) \end{bmatrix}_{i,j=1}^{m+n}$$

$$= \prod_{1 \le i < j \le m} (x_j - x_i) \prod_{1 \le k < l \le n} (y_l - y_k) \prod_{j=1}^{m} \prod_{l=1}^{n} (y_l - x_j).$$

- Partition $\lambda^* = (m - \lambda'_n, m - \lambda'_{n-1}, \ldots, m - \lambda'_1)$ *complementary* to λ in the $m \times n$ rectangle (n^m).

λ



$\mu = \lambda^*$

Koornwinder's q-difference operator (of type BC_m)

Koornwinder's q-difference operator $\mathcal{D}_x = \mathcal{D}_x(a,b,c,d|q,t)$ in m variables $x = (x_1,\ldots,x_m)$ is defined by

$$\mathcal{D}_x = \sum_{i=1}^{m} A_i(x)(T_{q,x_i} - 1) + \sum_{i=1}^{m} A_i(x^{-1})(T_{q,x_i}^{-1} - 1)$$

where

$$A_i(x) = \frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{(1-x_i^2)(1-qx_i^2)} \prod_{j\neq i} \frac{(tx_i - x_j)(1-tx_ix_j)}{(x_i - x_j)(1-x_ix_j)}.$$

For generic values of the parameters, the Koornwinder polynomial $P_\lambda(x) = P_\lambda(x;a,b,c,d|q,t)$ associated with a partition λ is characterized as a unique eigenfunction of $\mathcal{D}_x$ in the ring of W_m-invariant Laurent polynomials, $W_m = \{\pm 1\}^m \rtimes \mathfrak{S}_m$ being the Weyl group, in the form

$$P_\lambda(x) = x^\lambda + (\text{lower order terms}).$$

We use the following notations:

the set of partitions with length $\leq m$

$$P^+ = \{ \lambda = (\lambda_1,\ldots,\lambda_m) \in \mathbb{Z}^m \mid \lambda_1 \geq \cdots \geq \lambda_m \geq 0 \},$$

the *dominace ordering* $\preceq$ in P^+ defined by

$$\mu \preceq \lambda \iff \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \quad (i = 1,\ldots,m),$$

the *orbit sum* $m_\lambda(x) = \sum_{\mu \in W_m\lambda} x^\mu$ attached to $\lambda \in P^+$.

Theorem [Koornwinder, 1992, Contemp. Math. **138**]
For each $\lambda \in P^+$, there exists a unique W_m-invariant Laurent polynomial

$$P_\lambda(x) \in \mathbb{K}[x^{\pm 1}]^{W_m}, \qquad \mathbb{K} = \mathbb{C}(a,b,c,d,q,t),$$

such that

$$(1) \qquad P_\lambda(x) = m_\lambda(x) + \sum_{\mu \prec \lambda} a_{\lambda\mu}\, m_\mu(x) \quad (a_{\lambda\mu} \in \mathbb{K}),$$

$$(2) \qquad \mathcal{D}_x\, P_\lambda(x) = d_\lambda\, P_\lambda(x) \quad \text{for some } \ d_\lambda \in \mathbb{K}.$$

The eigenvalue d_λ of $P_\lambda(x)$ is given by

$$d_\lambda = \sum_{i=1}^{m} abcdq^{-1}t^{2m-i-1}(q^{\lambda_i} - 1) + \sum_{i=1}^{m} t^{i-1}(q^{-\lambda_i} - 1).$$

Kernel function of type II for Koornwinder's $\mathcal{D}_x$

Take two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ and define

$$\Psi(x;y) = \prod_{j=1}^{m}\prod_{l=1}^{n}(x_j + x_j^{-1} - y_l - y_l^{-1}).$$

Theorem [Mimachi, 2001, Duke Math. J. **107**]
(1) The function $\Psi(x;y)$ defined as above satisfies the functional equation

$$\begin{aligned} &t^{-m}(t-1)\mathcal{D}_x\Psi(x;y) + q^{-n}(q-1)\widetilde{\mathcal{D}}_y\Psi(x;y) \\ &\qquad = t^{-m}q^{-n}(t^m-1)(q^n-1)(abcdt^{m-1}q^{n-1}-1)\Psi(x;y), \end{aligned}$$

where $\widetilde{\mathcal{D}}_y = \mathcal{D}_y(a,b,c,d|t,q)$.
(2) The kernel function $\Psi(x;y)$ has the following eigenfunction expansion:

$$\Psi(x;y) = \sum_{\lambda\subset(n^m)} (-1)^{|\lambda^*|}\, P_\lambda(x)\, \widetilde{P}_{\lambda^*}(y)$$

where $\widetilde{P}_\lambda(y) = P_\lambda(y;a,b,c,d|t,q)$.

In particular the Koorwinder polynomial $P_{(n^m)}(x)$ of m variables $x = (x_1, \ldots, x_m)$, attached to the $m \times n$ rectangle for $n = 0, 1, 2, \ldots$, has an integral representation of Selberg type if $\max\{|a|, |b|, |c|, |d|, |q|, |t|\} < 1$:

$$P_{(n^m)}(x) = \text{const.} \int_{T^n} \Psi(x;y)\, w(y)\, \frac{dy_1\cdots dy_n}{y_1\cdots y_n},$$

where $w(y)$ is the weight function

$$w(y) = \prod_{k=1}^{n}\frac{(y_k^{\pm 2};t)_\infty}{(ay_k^{\pm1}, by_k^{\pm1}, cy_k^{\pm1}, dy_k^{\pm1};t)_\infty}\prod_{1\le k<l\le n}\frac{(y_k^{\pm1}y_l^{\pm1};t)_\infty}{(qy_k^{\pm1}y_l^{\pm1};t)_\infty}.$$

What about kernel functions of type I?

Comments on known results concerning the BC_m case

• Many things are known in the one variable case.
⇒ Askey-Wilson polynomials

• Question of integrability [van Diejen]:
There are sufficiently many independent q-difference operators commuting with $\mathcal{D}_x$.

• Relation to affine Hecke algerbras [Macdonald, Noumi, Sahi, Stokman]:
The commuting family of q-difference operators arise from representations of affine Hecke algebras.
⇒ q-Dunkl operators, duality, scalar products, nonsymmetric Koornwinder polynomials ...

• Elliptic setting [Ruijsenaars, van Diejen, Komori, Hikami, ...]

In spite of those progresses, however, no one seems to have *seen* (explicitly written down) the Koorwinder polynomials $P_\lambda(x)$ themselves during the fifteen years after Koornwinder's 1992 paper, even those attached to single columns and single rows ?!

Kernel functions of type I for Koornwinder's $\mathcal{D}_x$

We take the two sets of variables

$$x = (x_1, \ldots, x_m) \quad \text{and} \quad y = (y_1, \ldots, y_n)$$

as before, and consider x and y as canonical coordinates of the algebraic tori $\mathbb{T}_x^m = (\mathbb{C}^*)^m$ and $\mathbb{T}_y^n = (\mathbb{C}^*)^n$, respectively.

Consider the the following system of q-difference equations of rank one for a holomorphic function $\Phi(x; y)$ on $\mathbb{T}_x^m \times \mathbb{T}_y^n$ (or its universal covering):

$$(*)\quad \begin{cases} T_{q,x_i}\Phi(x;y) = t^n \displaystyle\prod_{l=1}^{n} \frac{(1-\sqrt{q/t}\,x_i\,y_l)(1-\sqrt{q/t}\,x_i/y_l)}{(1-\sqrt{q\,t}\,x_i\,y_l)(1-\sqrt{q\,t}\,x_i/y_l)}\,\Phi(x;y) \\ T_{q,y_k}\Phi(x;y) = t^m \displaystyle\prod_{j=1}^{m} \frac{(1-\sqrt{q/t}\,y_k\,x_j)(1-\sqrt{q/t}\,y_k/x_j)}{(1-\sqrt{q\,t}\,y_k\,x_j)(1-\sqrt{q\,t}\,y_k/x_j)}\,\Phi(x;y) \end{cases}$$

for $i = 1, \ldots, m$ and $k = 1, \ldots, n$. It is directly checked that $(*)$ satisfies the compatiblity condition.

Assuming that $|q| < 1$, we define a function $\Phi_0(x; y)$ in $m+n$ variables $(x; y)$ by

$$\Phi_0(x;y) = (x_1 \cdots x_m)^{\beta} \prod_{j=1}^{m}\prod_{l=1}^{n} \frac{(\sqrt{q\,t}\,x_j\,y_l;q)_\infty(\sqrt{q\,t}\,x_j/y_l;q)_\infty}{(\sqrt{q/t}\,x_j\,y_l;q)_\infty(\sqrt{q/t}\,x_j/y_l;q)_\infty},$$

where $t = q^{\beta}$. Then it turns out that:

- $\Phi_0(x; y)$ satisfies the system of q-difference equations $(*)$.
- Any solution $\Phi(x; y)$ of $(*)$ is a multiple of $\Phi_0(x; y)$ by some function on $\mathbb{T}_x^m \times \mathbb{T}_y^n$ which is q-periodic with respect to all x_i and y_k:
$$\Phi(x;y) = f(x;y)\,\Phi_0(x;y),$$
where $T_{q,x_i} f = f$ $(i = 1, \ldots, m)$, and $T_{q,y_k} f = f$ $(k = 1, \ldots, n)$.

By choosing such *q-periodic factors* appropriately, one can produce many solutions of $(*)$ with different analytic properties.

Theorem Suppose that a holomorphic function $\Phi(x; y)$ on $\widetilde{\mathbb{T}}_x^m \times \widetilde{\mathbb{T}}_y^n$ satisfies the q-difference system $(*)$ above. Then $\Phi(x; y)$ is a kernel function for the pair of Koornwinder's q-difference operators

$$\mathcal{D}_x = \mathcal{D}_x(a, b, c, d \,|\, q, t) \quad \text{and}$$
$$\widetilde{\mathcal{D}}_y = \mathcal{D}_y(\sqrt{qt}/a, \sqrt{qt}/b, \sqrt{qt}/c, \sqrt{qt}/d \,|\, q, t).$$

Kernel functions of type I (continued)

Suppose that a holomorphic function $\Phi(x;y)$ on $\widetilde{\mathbb{T}}_x^m \times \widetilde{\mathbb{T}}_y^n$ is a solution to the following q-difference system of rank one:

$$(*)\qquad \begin{cases} T_{q,x_i}\Phi(x;y) = t^n \displaystyle\prod_{l=1}^{n} \frac{(1-\sqrt{q/t}\,x_i\,y_l)(1-\sqrt{q/t}\,x_i/y_l)}{(1-\sqrt{q\,t}\,x_i\,y_l)(1-\sqrt{q\,t}\,x_i/y_l)}\,\Phi(x;y) \\ \quad (i=1,\ldots,m), \\ T_{q,y_k}\Phi(x;y) = t^m \displaystyle\prod_{j=1}^{m} \frac{(1-\sqrt{q/t}\,y_k\,x_j)(1-\sqrt{q/t}\,y_k/x_j)}{(1-\sqrt{q\,t}\,y_k\,x_j)(1-\sqrt{q\,t}\,y_k/x_j)}\,\Phi(x;y) \\ \quad (k=1,\ldots,n). \end{cases}$$

Then $\Phi(x;y)$ satisfies the following functional equation of a kernel function for a pair of Koornwinder's q-difference operators:

$$(**)\qquad \begin{aligned} &(1-t)\mathcal{D}_x\,\Phi(x;y) - (1-t)abcdq^{-1}t^{m-n-1}\widetilde{\mathcal{D}}_y\,\Phi(x;y) \\ &\quad = -(1-t^m)(1-t^n)(1-abcdq^{-1}t^{m-n-1})t^{m-n-1}\,\Phi(x;y), \end{aligned}$$

where

$$\mathcal{D}_x = \mathcal{D}_x(a,b,c,d\,|\,q,t),$$
$$\widetilde{\mathcal{D}}_y = \mathcal{D}_y(\sqrt{qt}/a,\sqrt{qt}/b,\sqrt{qt}/c,\sqrt{qt}/d\,|\,q,t).$$

$$\mathcal{D}_x(a,b,c,d\,|\,q,t) = \sum_{i=1}^{m} A_i(x)(T_{q,x_i}-1) + \sum_{i=1}^{m} A_i(x^{-1})(T_{q,x_i}^{-1}-1),$$
$$A_i(x) = \frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{(1-x_i^2)(1-qx_i^2)} \prod_{j\neq i} \frac{(tx_i-x_j)(1-tx_ix_j)}{(x_i-x_j)(1-x_ix_j)}.$$

Some special cases:

In the following two cases, we obtain *rational* kernel functions.

(1) For $t = q^k$ $(k = 0, 1, 2, \ldots)$,

$$\Phi^{(k)}(x; y) = \frac{(x_1 \cdots x_m)^{kn}}{\displaystyle\prod_{j=1}^{m}\prod_{l=1}^{n} \left(q^{\frac{1}{2}(1-k)} x_j y_l; q\right)_k \left(q^{\frac{1}{2}(1-k)} x_j / y_l; q\right)_k},$$

(2) For $t = q^{-k}$ $(k = 0, 1, 2, \ldots)$,

$$\Phi^{(-k)}(x; y) = (x_1 \cdots x_m)^{-kn} \prod_{j=1}^{m}\prod_{l=1}^{n} \left(q^{\frac{1}{2}(1-k)} x_j y_l; q\right)_k \left(q^{\frac{1}{2}(1-k)} x_j / y_l; q\right)_k.$$

The last kernel function with $t = q^{-k}$ will be used to determine Koornwinder polynomials $P_{(l)}(x)$ $l = 0, 1, 2, \ldots$ attached to single rows.

Application to explicit formulas

- Kernel function of type I
  $\Longrightarrow$ Explicit formulas for Koornwinder polynomials $P_{(l)}(x)$ attached to single rows $(l = 0, 1, 2, \ldots)$
- Kernel function of type II
  $\Longrightarrow$ Explicit formulas for Koornwinder polynomials $P_{(1^r)}(x)$ attached to single columns $(r = 0, 1, \ldots, m)$

For comparison recall the case of type A_{m-1}:

- Macdonald polynomials $P^A_{(l)}(x)$ for single rows $(l = 0, 1, 2, \ldots)$:

$$\frac{(t; q)_l}{(q; q)_l} P^A_{(l)}(x) = Q^A_{(l)}(x) = \sum_{\nu_1 + \cdots + \nu_m = l} \frac{(t; q)_{\nu_1} \cdots (t; q)_{\nu_m}}{(q; q)_{\nu_1} \cdots (q; q)_{\nu_m}} x_1^{\nu_1} \cdots x_m^{\nu_m}.$$

- Macdonald polynomials $P^A_{(1^r)}(x)$ for single columns $(r = 1, 2, \ldots, m)$:

$$P^A_{(1^r)}(x) = e_r(x) = \sum_{1 \le i_1 < \cdots < i_r \le m} x_{i_1} \cdots x_{i_r}.$$

Remarks on fundamental invariants of type BC_m

The ring of invariant Laurent polynomials

$$\mathbb{K}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]^{W_m}, \qquad W_m = \{\pm 1\}^m \rtimes \mathfrak{S}_m$$

is generated by the orbit sums $m_{(1^r)}(x)$ $(r = 1, \ldots, m)$:

$$\begin{aligned} m_{(1^r)}(x) &= e_r(x+x^{-1}) \\ &= \sum_{1\le i_1<\cdots<i_r\le m} (x_{i_1} + x_{i_1}^{-1}) \cdots (x_{i_r} + x_{i_r}^{-1}) \\ &= \sum_{1\le i_1<\cdots<i_r\le m} \sum_{\epsilon_1,\ldots,\epsilon_r=\pm 1} x_{i_1}^{\epsilon_1} \cdots x_{i_r}^{\epsilon_r}. \end{aligned}$$

The orbit sums $m_{(1^r)}(x)$, however, do *not* seem to be the right polynomials by which Koorwinder polynomials should be expanded.

We modify the expansion formula

$$\prod_{j=1}^{m} (y + y^{-1} - x_j - x_j^{-1}) = \sum_{r=0}^{m} (-1)^r e_r(x+x^{-1})\,(y + y^{-1})^{m-r},$$

by replacing $(y + y^{-1})^n$ with the q-shifted product

$$(y + y^{-1} - a - a^{-1})(y + y^{-1} - qa - q^{-1}a^{-1}) \cdots (n \text{ factors})$$

with base point a.

We define a set of fundamental invariants $E_r(x; a|q)$ $(r = 1, \ldots, m)$ as the expansion coefficients in

$$\prod_{j=1}^{m} \langle y; x_j \rangle = \sum_{r=0}^{m} (-1)^r E_r(x; a|q)\,\langle y; a\rangle_{m-r},$$

where

$$\langle y; x\rangle = y + y^{-1} - x - x^{-1} = -x^{-1}(1 - xy)(1 - x/y),$$

$$\langle y; a\rangle_n = \langle y; a\rangle\langle y; qa\rangle \cdots \langle y; q^{n-1}a\rangle = (-1)^n q^{-\binom{n}{2}} a^{-n} (ay; q)_n (a/y; q)_n$$

These $E_r(x; a|q)$ are determined explicitly as

$$E_r(x; a|q) = \sum_{1\le i_1<i_2<\cdots<i_r\le m} \langle x_{i_1}; q^{i_1-1}a\rangle\langle x_{i_2}; q^{i_2-2}a\rangle \cdots \langle x_{i_r}; q^{i_r-r}a\rangle$$

(They are W_m-invariant despite of the appearance).

Koornwinder polynomials attached to single columns

For each $r = 0, 1, \ldots, m$, the Koornwinder polynomial $P_{(1^r)}(x) = P_{(1^r)}(x_1, \ldots, x_m)$ attached a single column (1^r) is expressed in the form

$$P_{(1^r)}(x) = \sum_{l=0}^{r} \frac{\langle t^{m-r+1}, t^{m-r}ab, t^{m-r}ac, t^{m-r}ad\rangle_{t,l}}{\langle t, t^{2(m-r)}abcd\rangle_{t,l}} E_{r-l}(x),$$

where

$$\begin{aligned} E_r(x) &= \sum_{1\le i_1<\cdots<i_r\le m} \langle x_{i_1}; t^{i_1-1}a\rangle\langle x_{i_2}; t^{i_2-2}a\rangle \cdots \langle x_{i_r}; t^{i_r-r}a\rangle \\ &= e_r(x+x^{-1}) + c_{r,r-1}\, e_{r-1}(x+x^{-1}) + \cdots + c_{r,0}1. \end{aligned}$$

Here we have used the notations

$$\begin{aligned} &\langle x; a\rangle = x + x^{-1} - a - a^{-1}, \quad \langle a\rangle = a^{\frac{1}{2}} - a^{-\frac{1}{2}}, \\ &\langle a\rangle_{t,n} = \langle a\rangle\langle ta\rangle \cdots \langle t^{n-1}a\rangle, \quad \langle a_1, \cdots, a_r\rangle_{t,n} = \langle a_1\rangle_{t,n} \cdots \langle a_r\rangle_{t,n}. \end{aligned}$$

For example, the first nontrivial Koornwinder polynomial is given by

$$\begin{aligned} &P_{(1)}(x) = E_1(x) + \tfrac{(1-t^m)(1-t^{m-1}ab)(1-t^{m-1}ac)(1-t^{m-1}ad)}{at^{m-1}(1-t)(1-abcdt^{2(m-1)})} \\ &= \sum_{j=1}^{m}(x_j + x_j^{-1}) - \tfrac{(1-t^m)(1+a^2t^{m-1})}{at^{m-1}(1-t)} + \tfrac{(1-t^m)(1-t^{m-1}ab)(1-t^{m-1}ac)(1-t^{m-1}ad)}{at^{m-1}(1-t)(1-abcdt^{2(m-1)})}. \end{aligned}$$

The explicit formula above is derived from

Mimachi's kernel function (of type II) $(n = 1)$

$$\prod_{j=1}^{m}\langle y; x_j\rangle = \prod_{j=1}^{m}(y + y^{-1} - x_j - x_j^{-1}) = \sum_{r=0}^{m}(-1)^r P_{(1^r)}(x) p_{m-r}(y|t),$$

where $p_l(y|t) = p_l(y; a, b, c, d|t)$ are the Askey-Wilson polynomials with base t.

Kernel fuction of type II $\Longrightarrow$ $P_{(1^r)}(x)$ (single columns)

Goal:
To expand the kernel function by Askey-Wilson polynomials with base t:

$$\prod_{j=1}^{m}\langle y;x_j\rangle=\sum_{r=0}^{m}(-1)^{m-r}P_{(1^{m-r})}(x)\,p_r(y|t)$$

(1) Expand the kernel function in terms of $\langle y;a\rangle_{t,l}=\langle y;a\rangle\langle y;ta\rangle\cdots\langle y;t^{l-1}a\rangle$ and $E_l(x)=E_l(x;a|t)$ $(l=0,1,\ldots,m)$:

$$\prod_{j=1}^{m}\langle y;x_j\rangle=\sum_{l=0}^{m}(-1)^{m-l}E_{(1^{m-l})}(x)\,\langle y;a\rangle_{t,l}$$

(2) Express $\langle y;a\rangle_{t,l}$ in terms of Askey-Wilson polynomials $p_r(y|t)$:

$$\langle y;a\rangle_{t,l}=\sum_{r=0}^{l}(-1)^{l-r}\frac{\langle t^{r+1},t^rab,t^rac,t^rad\rangle_{t,l-r}}{\langle t,abcdt^{2r}\rangle_{l-r}}\,p_r(y|t).$$

(3) By substituting (2) into (1), we obtain

$$\prod_{j=1}^{m}\langle y;x_j\rangle=\sum_{0\le r\le l\le m}(-1)^{m-r}E_{(1^{m-l})}(x)\,\frac{\langle t^{r+1},t^rab,t^rac,t^rad\rangle_{t,l-r}}{\langle t,abcdt^{2r}\rangle_{l-r}}\,p_r(y|t).$$

This implies

$$P_{(1^{m-r})}(x)=\sum_{l=r}^{m}\frac{\langle t^{r+1},t^rab,t^rac,t^rad\rangle_{t,l-r}}{\langle t,abcdt^{2r}\rangle_{l-r}}\,E_{m-l}(x),$$

namely,

$$P_{(1^r)}(x)=\sum_{l=0}^{r}\frac{\langle t^{m-r+1},t^{m-r}ab,t^{m-r}ac,t^{m-r}ad\rangle_{t,r-l}}{\langle t,abcdt^{2(m-r)}\rangle_{r-l}}\,E_l(x).$$

Koornwinder polynomials attached to single rows

The Koornwinder polynomial $P_{(r)}(x) = P_{(r)}(x_1, \ldots, x_m)$ attached a single row (r) $(r = 0, 1, 2, \ldots)$ is given by

$$\frac{\langle t\rangle_r}{\langle q\rangle_r} P_{(r)}(x) = \frac{\langle t^m, t^{m-1}ab, t^{m-1}ac, t^{m-1}ad\rangle_r}{\langle q, t^{2(m-r)}abcdq^{r-1}\rangle_r} \cdot \sum_{l=0}^{r} \frac{(-1)^l \langle q^{-r}, t^{2(m-1)}abcdq^{r-1}\rangle_l}{\langle t^m, t^{m-1}ab, t^{m-1}ac, t^{m-1}ad\rangle_l} H_l(x),$$

where

$$H_l(x) = \sum_{|\nu|=l} \frac{\langle t\rangle_{\nu_1} \cdots \langle t\rangle_{\nu_m}}{\langle q\rangle_{\nu_1} \cdots \langle q\rangle_{\nu_m}} \langle x_1; a\rangle_{\nu_1} \langle x_2; tq^{\nu_1}a\rangle_{\nu_2} \cdots \langle x_m; t^{m-1}q^{\nu_1+\cdots+\nu_{m-1}}a\rangle_{\nu_m}.$$

$$\langle a\rangle = a^{\frac{1}{2}} - a^{-\frac{1}{2}}, \qquad \langle x; a\rangle = \langle xa\rangle\langle x/a\rangle = x + x^{-1} - a - a^{-1},$$
$$\langle a\rangle_n = \langle a\rangle\langle qa\rangle \cdots \langle q^{n-1}a\rangle, \quad \langle x; a\rangle_n = \langle x; a\rangle\langle x; qa\rangle \cdots \langle x; q^{n-1}a\rangle.$$

Compare this formula with

Askey-Wilson case (BC_1):

$$p_r(x) = \frac{(ab, ac, ad\,; q)_r}{a^r (abcdq^{r-1}; q)_r} \, {}_4\phi_3\left(\begin{matrix} q^{-r}, abcdq^{r-1}, ax, a/x \\ ab,\ ac,\ ad \end{matrix}; q, q\right)$$
$$= \frac{\langle ab, ac, ad\rangle_r}{\langle abcdq^{r-1}\rangle_r} \sum_{l=0}^{r} \frac{(-1)^l \langle q^{-r}, abcdq^{r-1}\rangle_l}{\langle q, ab, ac, ad\rangle_l} \langle x; a\rangle_l,$$

Macdonald case (A_{m-1}):

$$\frac{(t;q)_l}{(q;q)_l} P^A_{(l)}(x) = Q^A_{(l)}(x) = \sum_{|\nu|=l} \frac{(t;q)_{\nu_1} \cdots (t;q)_{\nu_m}}{(q;q)_{\nu_1} \cdots (q;q)_{\nu_m}} x_1^{\nu_1} \cdots x_m^{\nu_m}.$$

The explicit formula above is derived from our

Kernel function of type I

for $x = (x_1, \ldots, x_m)$ and y $(n = 1)$ with $t = q^{-k}$ $(k = 0, 1, 2, \ldots)$:

$$\Phi^{(-k)}(x; y) = \prod_{j=1}^{m} x_j^{-k} \big(q^{\frac{1}{2}(1-k)} x_j y; q\big)_k \big(q^{\frac{1}{2}(1-k)} x_j / y; q\big)_k = \prod_{j=1}^{m} \langle y; q^{\frac{1}{2}(1-k)} x_j\rangle_k.$$

Kernel fuction of type I $\Longrightarrow$ $P_{(r)}(x)$ (single rows)

Expand the kernel function in the form

$$\Phi^{(-k)}(x;y)=\prod_{j=1}^{m}\langle y;q^{\frac{1}{2}(1-k)}x_j\rangle_k=\sum_{r=0}^{km}G_r(x)\,\widetilde{p}_{km-r}(y)$$

in terms of Askey-Wilson polynomials

$$\widetilde{p}_r(y)=p_r(y;\sqrt{qt}/a,\sqrt{qt}/b,\sqrt{qt}/c,\sqrt{qt}/d|q).$$

From the property of the kernel function, it follows that the coefficients $G_r(x)$ are eigenfunctions of Koornwinder's q-difference operator $\mathcal{D}_x$ with $t=q^{-k}$:

$$\mathcal{D}_x\,G_r(x)=d_{(r)}G_r(x),\quad d_{(r)}=q^{-r}(1-q^r)(1-abcdq^{r-1}t^{2(m-1)}).$$

If one can find an explicit formula for $G_r(x)$ such that the coefficients are *rational* in $t=q^{-k}$ and that is valid for all $k=0,1,2,\ldots$, then it gives an explicit formula for a multiple of $P_{(r)}(x)$.

Expand the kernel function in the form

$$\prod_{j=1}^{m}\langle y;q^{\frac{1}{2}(1-k)}x_j\rangle_k=\sum_{l=0}^{km}H_l(x)\,\langle y;\sqrt{qt}/a\rangle_{km-l}$$

with base point $\sqrt{qt}/a$. Then the coefficients are given by

$$H_l(x)=\sum_{|\nu|=l}\frac{\langle t\rangle_{\nu_1}\cdots\langle t\rangle_{\nu_m}}{\langle q\rangle_{\nu_1}\cdots\langle q\rangle_{\nu_m}}\langle x_1;a\rangle_{\nu_1}\langle x_2;tq^{\nu_1}a\rangle_{\nu_2}\cdots\langle x_m;t^{m-1}q^{\nu_1+\cdots+\nu_{m-1}}a\rangle_{\nu_m}.$$

After that, expand $\langle y;\sqrt{qt}/a\rangle_{km-l}$ in terms of the Askey-Wilson polynomials $\widetilde{p}_{km-r}(y)$ as in the case of single columns, so as to obtain explicit formulas for $G_r(x)$.

Elliptic difference operators of Ruijsenaars

A_{m-1} (GL_m form)

The Ruijsenaars operator in the variables $x = (x_1, \ldots, x_m)$ of type A_{m-1} is defined by

$$D_x^{(\delta,\kappa)} = \sum_{i=1}^{m} \prod_{j \neq i} \frac{[x_i - x_j + \kappa]}{[x_i - x_j]} T_{x_i}^{\delta},$$

where $[z] = \sigma(z; \Omega)$ denotes the Weierstrass sigma function with the period lattice $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, and $T_{x_i}^{\delta}$ stands for the shift operator $x_i \to x_i + \delta$. We fix an elliptic gamma function $G(z; \delta)$ such that

$$G(z + \delta; \delta) = [z]\, G(z; \delta).$$

We consider two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$.

Kernel function of type I (Ruijsenaars):
When $m = n$, the function

$$\Phi(x; y) = \prod_{j=1}^{m} \prod_{l=1}^{n} \frac{G(x_j + y_l + u - \kappa; \delta)}{G(x_j + y_l + u; \delta)}$$

(or its multiple by any δ-periodic fuction) satisfies the kernel relation

$$D_x^{(\delta,\kappa)} \Phi(x; y) = D_y^{(\delta,\kappa)} \Phi(x; y).$$

Kernel function of type II:
Under the *balancing condition* $m\kappa + n\delta = 0$, the function

$$\Psi(x; y) = \prod_{j=1}^{m} \prod_{l=1}^{n} [y_l - x_j]$$

satisfiles the kernel relation

$$[\kappa]\, D_x^{(\delta,\kappa)} \Psi(x; y) + [\delta] D_y^{(\kappa,\delta)} \Psi(x; y) = 0.$$

Remark: In the trigonometric and the rational cases, one need not impose the balancing condition. When $[z] = z$ or $[z] = \sin(\pi z/\omega)$, one can prove

$$D_x^{(\delta,\kappa)} \Phi(x; y) - D_y^{(\delta,\kappa)} \Phi(x; y) = [(m - n)\kappa] \Phi(x; y)$$

for the kernel of type I, and

$$[\kappa]\, D_x^{(\delta,\kappa)} \Psi(x; y) + [\delta] D_y^{(\kappa,\delta)} \Psi(x; y) = [m\kappa + n\delta] \Psi(x; y)$$

for the kernel of type II.

Ruijsenaars operator of type BC_m

We consider the Ruijsenaars operator $D_x^{(\delta,\kappa;\alpha,\beta)}$ of type BC_m in the variables $x=(x_1,\ldots,x_m)$ with 2+8 parameters

$$(\delta,\kappa;\alpha,\beta)=(\delta,\kappa;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4).$$

The parameters α_r, β_r correspond to the half periods $\frac{1}{2}\omega_r$ $(r=1,2,3,4)$, where $\Omega=\mathbb{Z}\omega_1\oplus\mathbb{Z}\omega_2$, $\omega_3=-\omega_1-\omega_2$, $\omega_4=0$. For each $r=1,2,3,4$, we define the constant λ_r by

$$\sigma(z+\omega_r;\Omega)=\pm\, e(\lambda_r(z+\frac{\omega_r}{2}))\,\sigma(z;\Omega),\quad e(z)=\exp(2\pi\sqrt{-1}z).$$

With the abbreviated notation $[z]=\sigma(z;\Omega)$, set

$$\begin{aligned}
&D_x^{(\delta,\kappa;\alpha,\beta)}\\
&=\sum_{i=1}^{m}\frac{[2x_i+\kappa]}{[2x_i]}\prod_{r=1}^{4}\frac{[x_i+\frac{1}{2}(\delta-\omega_r)+\alpha_r]}{[x_i+\frac{1}{2}(\delta-\omega_r)]}\frac{[x_i+\frac{1}{2}(\kappa-\omega_r)+\beta_r]}{[x_i+\frac{1}{2}(\kappa-\omega_r)]}\prod_{j\neq i}\frac{[x_i\pm x_j+\kappa]}{[x_i\pm x_j]}\,T_{x_i}^{\delta}\\
&\ +\sum_{i=1}^{m}\frac{[2x_i-\kappa]}{[2x_i]}\prod_{r=1}^{4}\frac{[x_i-\frac{1}{2}(\delta-\omega_r)-\alpha_r]}{[x_i-\frac{1}{2}(\delta-\omega_r)]}\frac{[x_i-\frac{1}{2}(a-\omega_r)-\beta_r]}{[x_i-\frac{1}{2}(\kappa-\omega_r)]}\prod_{j\neq i}\frac{[x_i\pm x_j-\kappa]}{[x_i\pm x_j]}\,T_{x_i}^{-\delta}\\
&\ +\sum_{r=1}^{4}e(-\tfrac{1}{2}c_m\lambda_r)\frac{[\alpha_r]}{[\kappa]}\prod_{s\neq r}\frac{[\frac{1}{2}\omega_{rs}+\alpha_s]}{[\frac{1}{2}\omega_{rs}]}\prod_{s=1}^{4}\frac{[\frac{1}{2}(\omega_{rs}+\kappa-\delta)+\beta_s]}{[\frac{1}{2}(\omega_{rs}+\kappa-\delta)]}\prod_{j=1}^{m}\frac{[\frac{1}{2}(\omega_r-\delta)\pm x_j+\kappa]}{[\frac{1}{2}(\omega_r-\delta)\pm x_j]},
\end{aligned}$$

where $\omega_{rs}=\omega_r-\omega_s$ and $c_m=2m\kappa+\sum_{s=1}^{4}(\alpha_s+\beta_s)$. This operator is symmetric with respect to the 8 parameters

$$\{\gamma_1,\ldots,\gamma_8\}=\{\tfrac{1}{2}(\delta-\omega_r)+\alpha_r,\ \ \tfrac{1}{2}(\kappa-\omega_r)+\beta_r\ \ (r=1,\ldots,4)\}.$$

Kernel functions for the BC_m Ruijsenaars operator

We consider two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$.

Kernel function of type I (Ruijsenaars, $m = n$):
Under the *balancing condition*

$$2(m-n)\kappa + \textstyle\sum_{s=1}^{4}(\alpha_s + \beta_s) = 0,$$

The function

$$\Phi(x;y) = \prod_{j=1}^{m}\prod_{l=1}^{n}\frac{G(x_j + y_l + \frac{1}{2}(\delta-\kappa);\delta)\, G(x_j - y_l + \frac{1}{2}(\delta-\kappa);\delta)}{G(x_j + y_l + \frac{1}{2}(\delta+\kappa);\delta)\, G(x_j - y_l + \frac{1}{2}(\delta+\kappa);\delta)}$$

(or its multiple by any δ-periodic factor) satisfies the kernel relation

$$D_x^{(\delta,\kappa;\alpha,\beta)}\Phi(x;y) = D_y^{(-\delta,-\kappa;\beta,\alpha)}\Phi(x;y).$$

Kernel function of type II:
Under the *balancing condition*

$$2m\kappa + 2n\delta + \textstyle\sum_{s=1}^{4}(\alpha_s + \beta_s) = 0,$$

the function

$$\Psi(x;y) = \prod_{j=1}^{m}\prod_{l=1}^{n}\,[x_j + y_l][x_j - y_l]$$

satisfiles the kernel relation

$$[\kappa]\, D_x^{(\delta,\kappa;\alpha,\beta)}\Psi(x;y) + [\delta] D_y^{(\kappa,\delta;\beta,\alpha)}\Psi(x;y) = 0.$$

It would be an important problem to explore how to use these kernel functions for the study of eigenfunctions of the Ruijsenaars operators.