Tzitzeica solitons vs. relativistic Calogero-Moser 3-body clusters (joint with Jon Nimmo)

1. The two relativistic dynamics at issue

- Tzitzeica equation (1910; geometry of surfaces)

$$\Psi_{uv} = e^{\Psi} - e^{-2\Psi} \qquad (T)$$

(aka Dodd-Bullough-Jiber-Shabat-Mikhailov equation)

u, v light cone coord.; with $t = u-v$, $y = u+v$, get $\Psi_{uv} \to \Psi_{yy} - \Psi_{tt}$

so (T) can be viewed as relativistic wave equation

- Relativistic Calogero-Moser system ('85; S.R.+H.Schneider)

$$S_\pm = \sum_{1\le i\le N_+} \exp(\pm p_i^+) V_i^+ + \sum_{1\le j\le N_-} \exp(\pm p_j^-) V_j^-$$

$$(V_i^+)^2 = \prod_{\substack{1\le k\le N_+ \\ k\ne i}} \left(1 + \frac{\sin^2 c}{\mathrm{sh}^2(x_i^+ - x_k^+)/2}\right) \prod_{1\le j\le N_-} \left(1 - \frac{\sin^2 c}{\mathrm{ch}^2(x_j^- - x_i^+)/2}\right)$$

$$(V_j^-)^2 = \prod_{\substack{1\le l\le N_- \\ l\ne j}} \left(1 + \frac{\sin^2 c}{\mathrm{sh}^2(x_j^- - x_l^-)/2}\right) \prod_{1\le i\le N_+} \left(1 - \frac{\sin^2 c}{\mathrm{ch}^2(x_j^- - x_i^+)/2}\right)$$

Time translation generator: $H = \frac{M}{2}(S_+ + S_-)$ $\qquad (CM)_{rel}$

Space translation generator: $P = \frac{M}{2}(S_+ - S_-)$

Boost generator: $B = \sum_{i=1}^{N_+} x_i^+ + \sum_{j=1}^{N_-} x_j^-$

2. A simpler soliton-particle correspondence: sine-Gordon

Specialize CM coupling c to $\pi/2$, and define

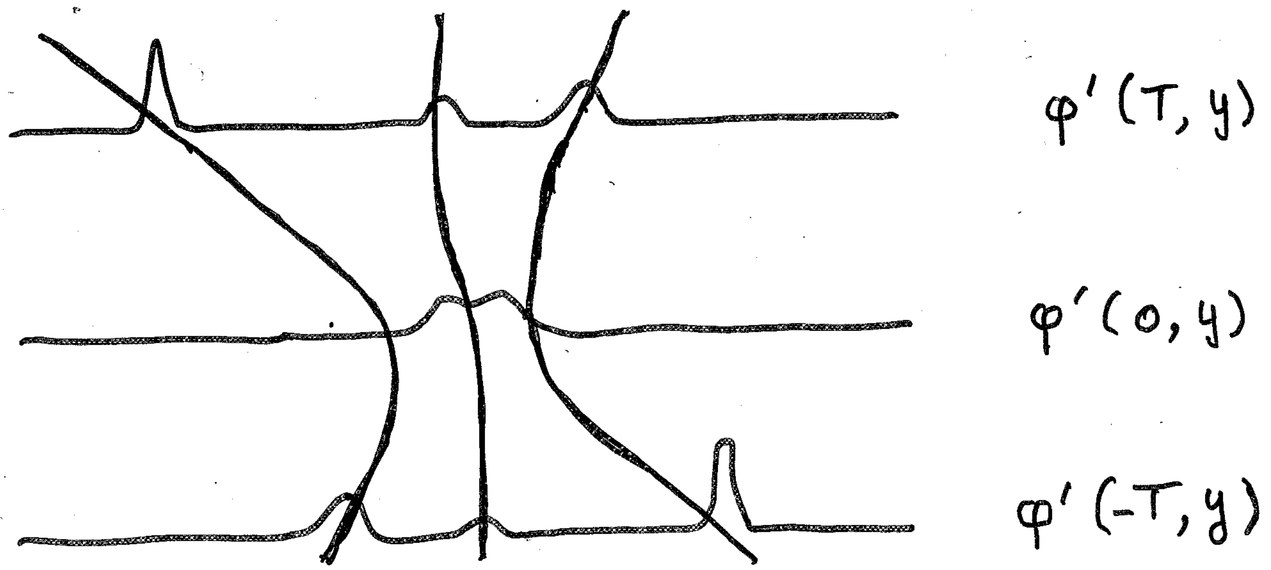
$$x_i^+(t,y) = \left(e^{tH-yP}(x,p)\right)_i \,, \quad x_j^-(t,y) = \left(e^{tH-yP}(x,p)\right)_{N_++j}$$

with $i=1,\dots,N_+$, $j=1,\dots,N_-$. Then

$$\varphi(t,y) = 4\sum_{i=1}^{N_+} \mathrm{Arctg}\left(e^{x_i^+(t,y)}\right) + 4\sum_{j=1}^{N_-} \mathrm{Arctg}\left(e^{x_j^-(t,y)}\right)$$

solves $\varphi_{yy} - \varphi_{tt} = \sin\varphi$; It is a solution with N_+ solitons and N_- antisolitons, which may occur in ℓ breather pairs, $0 \le \ell \le \min(N_+, N_-)$. Requiring $x_i^+(t,y)=0$, $x_j^-(t,y)=0$, yields space-time trajectories $y_i^+(t)$, $y_j^-(t)$.

Ex. $N_+ = 3$, $N_- = 0$



Soliton-particle correspondence equivariant under Poincaré group; it extends to the quantum level

3. Sine-Gordon and Tzitzeica as 2D Toda reductions

Recall 2D Toda equation (Mikhailov '79)

$$(\varphi_n)_{uv} = e^{\varphi_n - \varphi_{n-1}} - e^{\varphi_{n+1} - \varphi_n}, \quad n \in \mathbb{Z}, \quad \varphi_n = \ln(\tau_n / \tau_{n-1})$$

$$\Downarrow \quad (\varphi_n - \varphi_{n-1})_{uv} = 2 e^{\varphi_n - \varphi_{n-1}} - e^{\varphi_{n+1} - \varphi_n} - e^{\varphi_{n-1} - \varphi_{n-2}}$$

- Require 2-periodicity $\tau_{n+2} = \tau_n$ and put $\varphi_1 - \varphi_0 = i\psi \Rightarrow$

$$\psi_{uv} = 4 \sin\psi \qquad \text{(sine-Gordon)}$$

- Require 3-periodicity $\tau_{n+3} = \tau_n$ and put

$$h_n = e^{\varphi_n - \varphi_{n-1}} = \tau_n \tau_{n-2} / \tau_{n-1}^2 \qquad \text{(3-per.)}$$

to get

$$h_1 h_2 h_3 = 1$$

$$(\ln h_1)_{uv} = 2h_1 - h_2 - \frac{1}{h_1 h_2}, \quad (\ln h_2)_{uv} = 2h_2 - h_1 - \frac{1}{h_1 h_2}$$

Now require the extra constraint

$$h_1 = h_2 = h = e^{\Psi} \qquad \text{(extra)}$$

to get the T. eq. $\Psi_{uv} = e^{\Psi} - e^{-2\Psi}$. By (3-per.) this yields

$$(\tau_0 / \tau_1)^3 = 1 \qquad \text{(tauq)}$$

4. Reducing the 2D Toda solitons

The 2D Toda solitons are given by

$$\tau_n = \sum_{\mu_1,\dots,\mu_N=0,1} \exp\Big(\sum_{j<k} \mu_j \mu_k B_{jk} + \sum_j \mu_j \xi_{j,n} \Big), \quad n \in \mathbb{Z}$$

$$\exp(B_{jk}) = \frac{(a_j - a_k)(b_j - b_k)}{(a_j - b_k)(b_j - a_k)}$$

$$\xi_{j,n} = \xi_j^0 + n \ln(a_j/b_j) + i \sum_{\ell=1}^{\infty} \big((a_j^\ell - b_j^\ell) t_{\ell,+} + (a_j^{-\ell} - b_j^{-\ell}) t_{\ell,-} \big)$$

(Kyoto school: Date, Jimbo, Miwa, ...)

Set $t_{\ell,+} = t_{\ell,-} = 0$ for $\ell > 1$, and $t_{1,+} = -v$, $t_{1,-} = u$.

Then $\varphi_n = \ln(\tau_n / \tau_{n-1})$ solves the 2D Toda equation.

Next, rewrite τ_n as

$$\tau_n = \det(\mathbb{1}_N + C D_n), \qquad n \in \mathbb{Z}$$

$$C_{jk} = \frac{a_j - b_j}{a_j - b_k}, \qquad D_n = \mathrm{diag}(e^{\xi_{1,n}}, \dots, e^{\xi_{N,n}})$$

and reparametrize:

$$a_j = \exp(\eta_j - i c_j), \quad b_j = \exp(\eta_j + i c_j)$$

This yields

$$\xi_{j,n} = \xi_j^0 - 2\,i\,n\,c_j - 2\sin(c_j)\left(v e^{\eta_j} + u e^{-\eta_j}\right)$$

- Can obtain 2-periodicity by choosing $c_1 = \dots = c_N = \pi/2$; this reduction yields the sine-Gordon particle-like solutions.
- Can obtain 3-periodicity by choosing $c_1 = \dots = c_N = \pi/3$; but to get Tzitzeica solitons, also need $\tau_0^3 = \tau_1^3$. It appears impossible to achieve this for any choice of ξ_j^0, η_j.
- Can __also__ obtain 3-periodicity by choosing $c_j \in \{\frac{\pi}{3}, \frac{2\pi}{3}\}$.

Exploiting this, __can__ achieve $\tau_0 = \tau_1$! Specifically, choose variables in pairs, as follows:

$$c_{2k-1} = \frac{\pi}{3}, \quad c_{2k} = \frac{2\pi}{3}, \qquad k = 1, \dots, N$$

$$\xi_{2k-1}^0 = \frac{i\pi}{3} + q_k, \quad \xi_{2k}^0 = -\frac{i\pi}{3} + q_k, \quad \eta_{2k-1} = \eta_{2k} = \theta_k$$

Then get $\tau_0 = \tau_1$; letting $q, \theta \in \mathbb{R}^N$, also get $\tau_n \in \mathbb{R}$.

__Upshot__: Can obtain Tzitzeica N-soliton solutions as subset of 2D Toda 2N-soliton solutions; relation to Kaptsov-Shanko N-soliton solutions still unclear.

Ex. The Tzitzeica 1-soliton solution:

$$\tau_0 = \tau_1 = 1 + 2F + F^2, \qquad \tau_2 = 1 - 4F + F^2$$

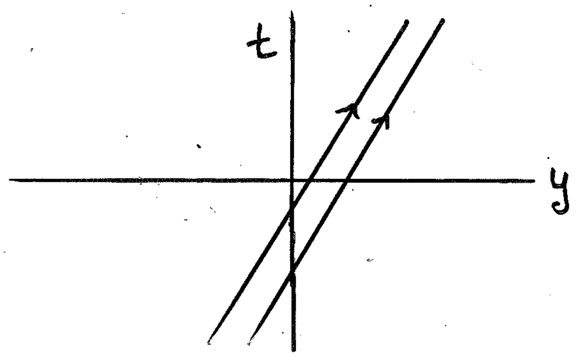
$$F = \tfrac{1}{2} \exp\left(q - \sqrt{3}\,(v e^{\theta} + u e^{-\theta})\right)$$

$$= \tfrac{1}{2} \exp\left(q + \sqrt{3}\,(t\,\mathrm{sh}\,\theta - y\,\mathrm{ch}\,\theta)\right)$$

$$\Psi = \ln(\tau_2/\tau_0) = \ln\left(\frac{1 - 4F + F^2}{(1+F)^2}\right)$$

Get singularities for $F = 2 \pm \sqrt{3} \iff$

$$y_\pm(t) = t\,\mathrm{th}\,\theta + \frac{1}{\sqrt{3}\,\mathrm{ch}\,\theta}\left(q - \ln(4 \pm 2\sqrt{3})\right):$$



5. Relativistic Calogero-Moser systems vs. 2D Toda solitons

Can tie in 2D Toda N-soliton with $c_1 = \dots = c_N = c$ with CM N-body system ($N = N_+ + N_-$) with coupling c.

Key ingredients: Lax matrix ($\simeq$ CD) and action-angle map.

- Can use fusion procedure to tie in solitons with

$c_j = n_j c$, $n_j \in \mathbb{N}^*$, $j=1,\dots,\lambda$; $c \in (0, \pi/\max(n_1,\dots,n_\lambda))$,

with CM $(n_1+\dots+n_\lambda)$-body system with coupling c.

This involves the complexification of the CM phase space, but real subvarieties and flows arise, cf.

S.R., Integrable particle systems vs. solutions to the KP and 2D Toda equations, Ann. Phys. 256 (1997) 226-301

- In particular, can take $c=\pi/3$, $\lambda = 2N$, $n_{2k-1}=1$, $n_{2k}=2$, $k=1,\dots,N$, to get above Tzitzeica N-soliton $\tau_n(q,\theta)$ via N-fold fusion from special 2D Toda 3N-soliton.

- The crux: The latter solitons correspond to a 2N-dimensional Poincaré-invariant submanifold of the 6N-dimensional phase space for 2N particles and N anti-particles. Specifically,

$$\tau_n(q,\theta) = \det\left(\mathbb{1}_{3N} + e^{-2in\pi/3} e^{i\pi/3} \mathcal{L}(q,\theta)\right),$$

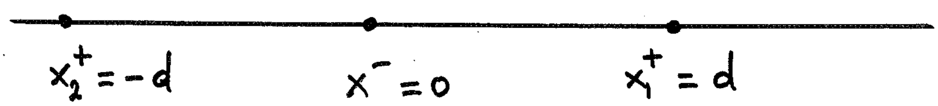
with $\mathcal{L}$ the dual Lax matrix of the $c=\pi/3$ system.

6. The physical picture

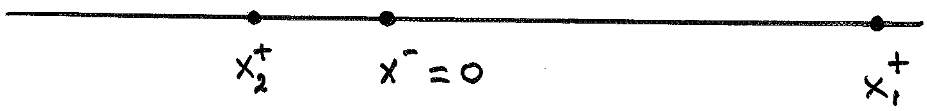
To appreciate the correspondence, consider the $N=1$ case, starting with the nonrelativistic Hamiltonian

$$H = \frac{1}{2}\left((p_1^+)^2 + (p_2^+)^2 + (p^-)^2\right) + \frac{g^2}{4}\left(\frac{1}{\mathrm{sh}^2(x_1^+ - x_2^+)/2} - \frac{1}{\mathrm{ch}^2(x^- - x_1^+)/2} - \frac{1}{\mathrm{ch}^2(x^- - x_2^+)/2}\right)$$

It has an 'obvious' equilibrium:



Since H commutes with $P = p_1^+ + p_2^+ + p^-$, also translates are equilibria. But there exist 'non-obvious' equilibria as well, obtained by acting with 3^{d} Poisson commuting Hamiltonian:



For the relativistic case there is again an 'obvious' equilibrium, and since H_{rel} commutes with $p_1^+ + p_2^+ + p^-$, also 'translates' are equilibria. But H_{rel} also commutes with P_{rel}, which yields a 1-dimensional space of 'non-obvious' equilibria. Acting with boosts (generated by $x_1^+ + x_2^+ + x^-$), we get a

2-dimensional Poincaré-invariant submanifold of the 6-dimensional phase space; these are the points yielding the Tzitzeica 1-soliton solution via the dual Lax matrix.

General case:

$$
\begin{tikzcd}
\text{relativ. CM system with } c=\pi/3, \text{ 2N particles, N antiparticles} \arrow[r, "\text{dual Lax m.}"'] \arrow[d, Rightarrow, "\text{restriction}"] & \text{special class of 2D Toda 3N-solitons} \arrow[d, Rightarrow, "\text{fusion + param. specialization}"] \\
\text{submanifold of N clusters (dead breather + particle)} \arrow[r, "\text{dual Lax m.}"] \arrow[d, "\text{Poincaré transf. on phase space}"] & \text{special 2D Toda 2N-solitons} \simeq \text{Tzitzeica N-solitons} \arrow[d, "\text{Poincaré transf. on space-time}"] \\
{} \arrow[r, "\text{dual Lax m.}"] & {}
\end{tikzcd}
$$

(The diagram commutes: Poincaré equivariance)

Upshot: A Tzitzeica soliton may be viewed as a bound state of three Calogero-Moser 'quarks'.