AKNS system by operator methods

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1 Basic Strategy (Marchenko, Carl)

AKNS
$q = q(x,t;\, \mathbf{a},\mathbf{b})$ $r = r(x,t;\, \mathbf{a},\mathbf{b})$

$\xrightarrow{\text{"translate"}}$

non-commutative AKNS
$Q = Q(x,t;\, \mathbf{A},\mathbf{B})$ $R = R(x,t;\, \mathbf{A},\mathbf{B})$

$\xrightarrow{\text{"projection"}}$

solution formula for AKNS with **operator-valued** parameters
$\hat{q} = \hat{q}(x,t;\, \mathbf{A},\mathbf{B})$ $\hat{r} = \hat{r}(x,t;\, \mathbf{A},\mathbf{B})$

Advantage: A, B arbitrary operators!

2 Non-commutative AKNS

f, g polynomials

The operator functions R, Q (R with values in $\mathcal{L}(F,E)$, Q with values in $\mathcal{L}(E,F)$, E, F Banach spaces) solve the non-commutative AKNS cf

$$g(T_{R,Q})\begin{pmatrix} R_t \\ Q_t \end{pmatrix} = f(T_{R,Q})\begin{pmatrix} R \\ -Q \end{pmatrix}$$

where

$$T_{R,Q}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U_x - \left[R\int_{-\infty}^{x}(QU+VR)\,dx' + \int_{-\infty}^{x}(UQ+RV)\,dx' \cdot R \right] \\ -V_x + \left[Q\int_{-\infty}^{x}(UQ+RV)\,dx' + \int_{-\infty}^{x}(QU+VR)\,dx'\, Q \right] \end{pmatrix}$$

3 Operator solutions

$f_0 = f/g$

Theorem (Sch.'05) E, F Banach spaces; $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(F)$.

Let $L = L(x,t)$, $M = M(x,t)$ be families of operators with values in $\mathcal{L}(F,E)$, $\mathcal{L}(E,F)$ respectively, which satisfy

$$L_x = AL \qquad M_x = BM$$
$$L_t = f_0(A)L \qquad M_t = -f_0(-B)M$$

Then

$$Q = (I - LM)^{-1}(BM + MA)$$
$$R = (I - ML)^{-1}(AL + LB)$$

solves the non-commutative AKNS.

Assumptions:

a) $\operatorname{spec}(A) \cup \operatorname{spec}(-B)$ is contained in the domain where f_0 is holomorphic

b) L, M are sufficiently smooth and behave sufficiently well for $x \to -\infty$

c) $(I - LM)$, $(I - ML)$ are invertible.

Rem Partial results by Bauhardt/Pöppe.

4 "Projection" to scalar AKNS

Use:
- Traces and determinants on quasi-Banach ideals (Pietsch, Carl)
- Theory of elementary operators (Eschmeier, Dash/Schechter)
- Factorization theorems (Grothendieck, Ransford/Taylor/White)

Result: Solution formulas depending on operator parameters A, B.

Rem Also possible: "projection" to matrix equations

5 Applications

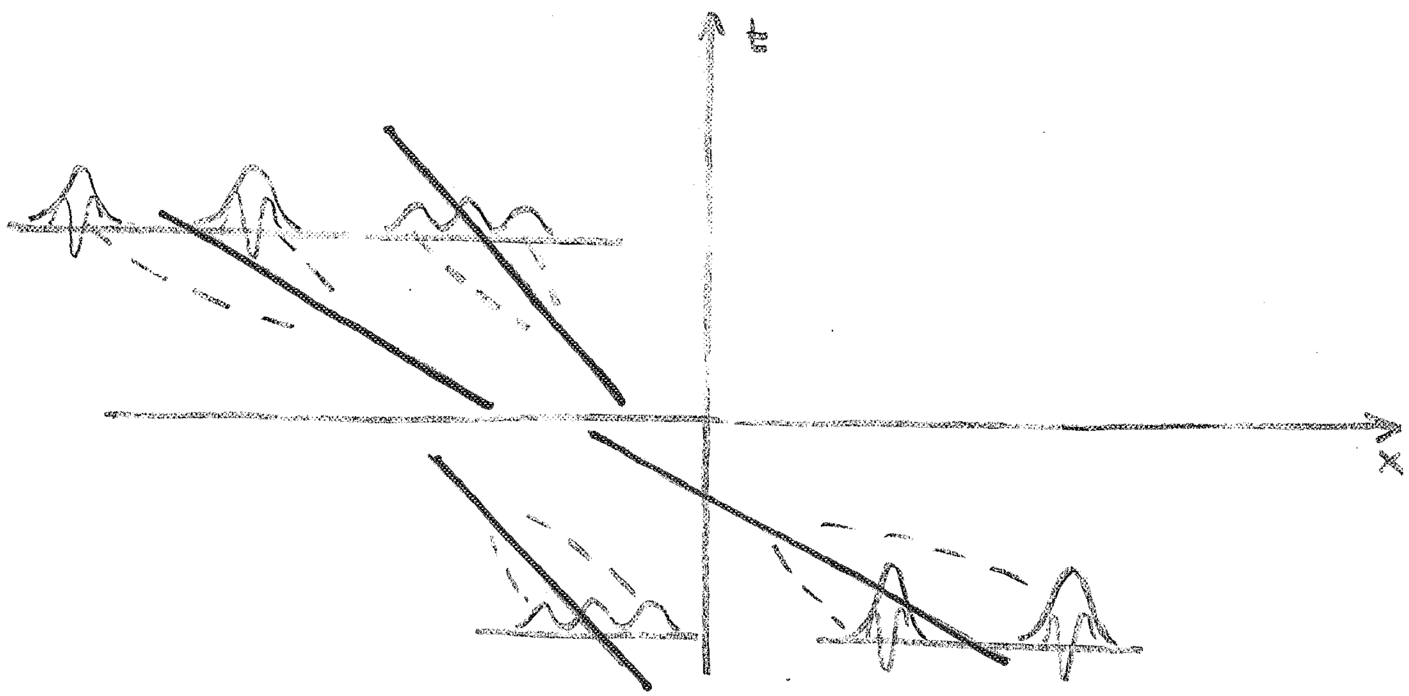
- closed formulas for "N-solitons" of the nonreduced AKNS
- long time existence and regularity of solutions of reduced equations
- countable superposition by limit construction (complementing results by Gesztesy et al.)
- Complete description of multipole solutions

  partial results: Wadati/Ohkuma, Tsuru/Wadati, Olmedilla, Fuchssteiner

Systematic approach for the related class of positons: V. Matveev.

A Jordan matrix with N blocks (eigenvalues α_j, sizes n_j)

Result (Sch'05) for $r = -\overline{q}$



Ass: a) $v_j = -\mathrm{Re}\, f_0(\alpha_j)/\mathrm{Re}(\alpha_j)$ pairwise different

b) $v_j + f_0'(\alpha_j) \neq 0$

Asymptotic form

$$u \approx \sum_{j=1}^{N} \sum_{k=1}^{n_j} u_{jk} \quad \text{for } t \approx \pm\infty$$

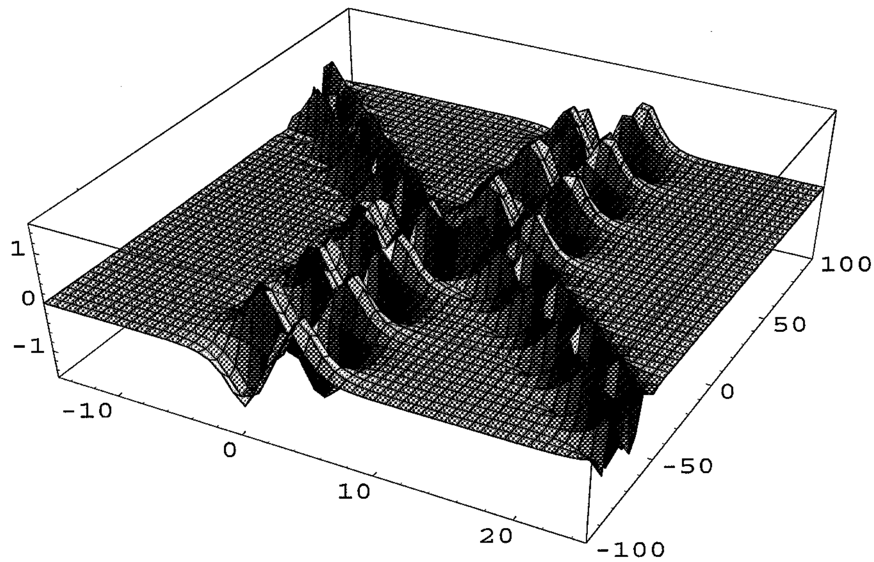
Here the u_{jk} are solitons characterized by α_j moving along the asymptotic curves

$$\alpha_j x + f_0(\alpha_j)t + \mathcal{O}(\log|t|) + \varphi_j + \varphi_j^{\pm} + \varphi_{jk}^{\pm} = 0$$

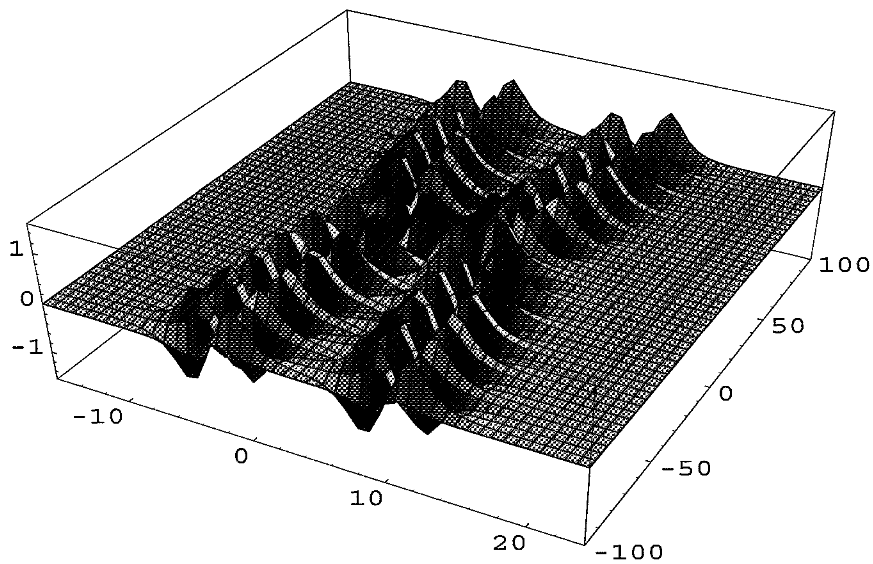
Formula for the phaseshift

$$e^{\varphi_j^{\pm}} = \prod_{k:\, v_k \lessgtr v_j} \left(\frac{\alpha_j - \alpha_k}{\alpha_j + \overline{\alpha_k}}\right)^{2n_k}$$

SG

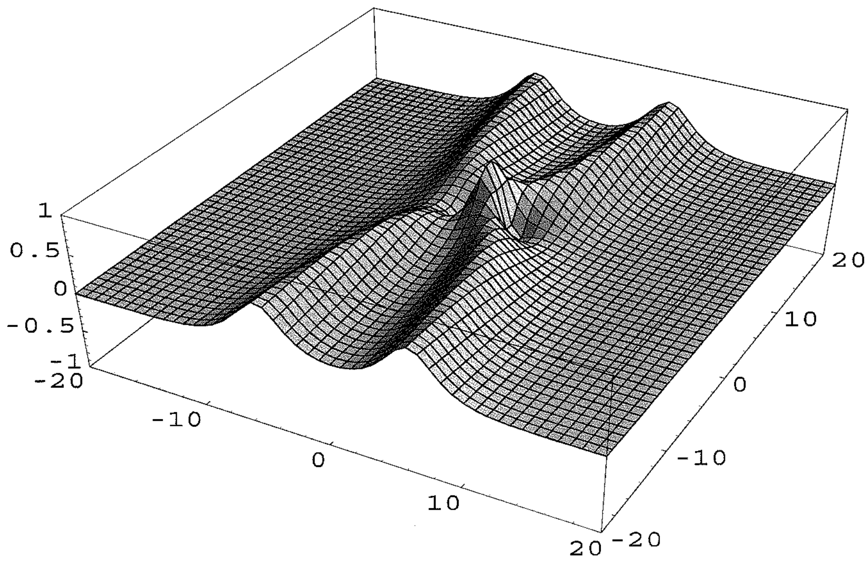


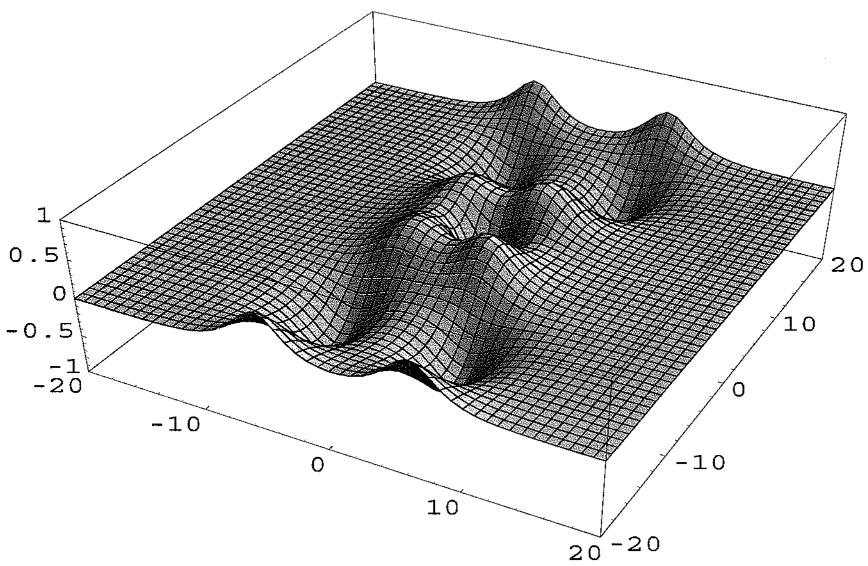
two breathers ($a_1 = 0.8(\sqrt{1-0.4^2}+0.4\mathrm{i})$, $a_2 = \sqrt{1-0.2^2}+0.2\mathrm{i}$) meet



negaton ($a = \sqrt{1-0.4^2}+0.4\mathrm{i}$) consisting of two breathers http://www.minet.uni-jena.de/~cornelia

NLS

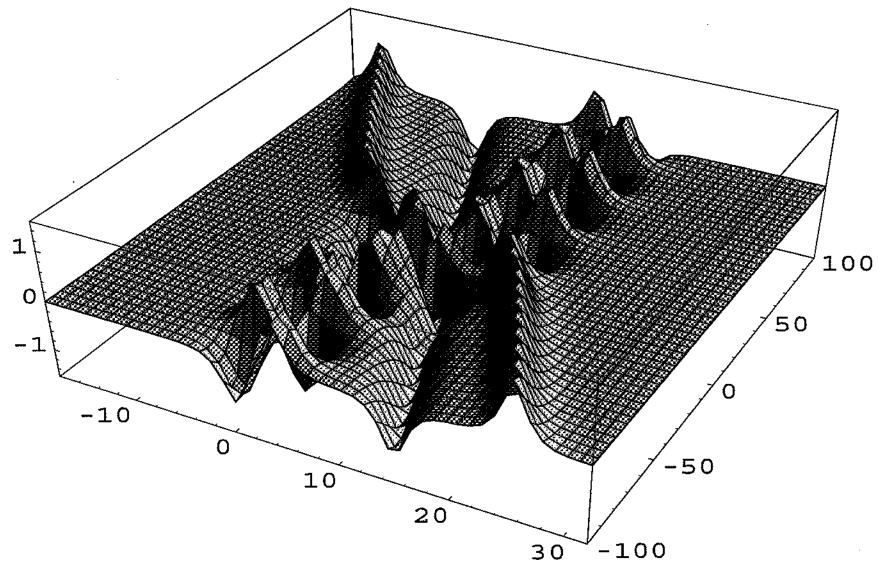




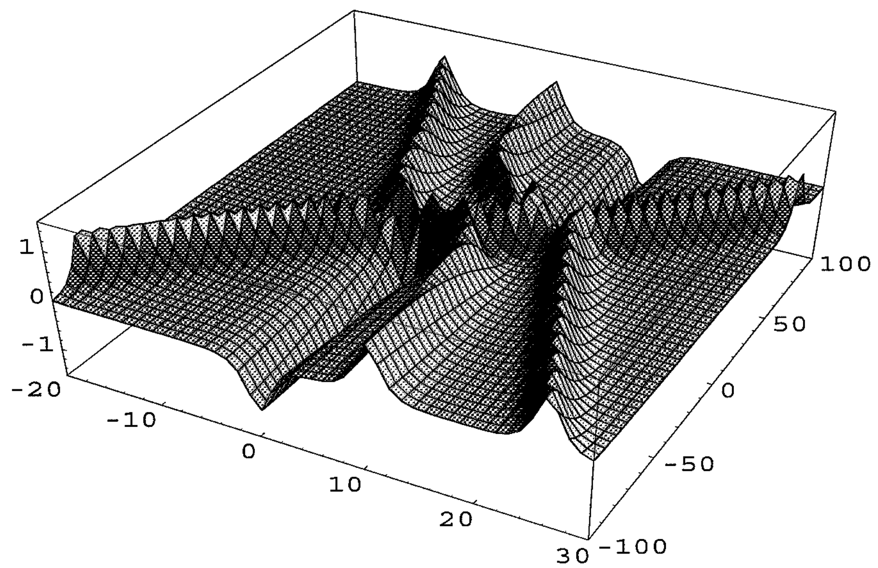
negaton ($a = 0.6$) consisting of two solitons

This solution is a stationary negaton, which is drawn in the coordinates (x, t). The plot above shows its modulus, the plot below its real part.

SG



negaton ($a_1 = 0.9$) consisting of a soliton and an antisoliton meets breather ($a_2 = \sqrt{1 - 0.2^2} + 0.2\mathrm{i}$)



negaton ($a_1 = 1$) consisting of a soliton and an antisoliton meets two solitons ($a_2 = 0.9$, $a_3 = 1.3$)

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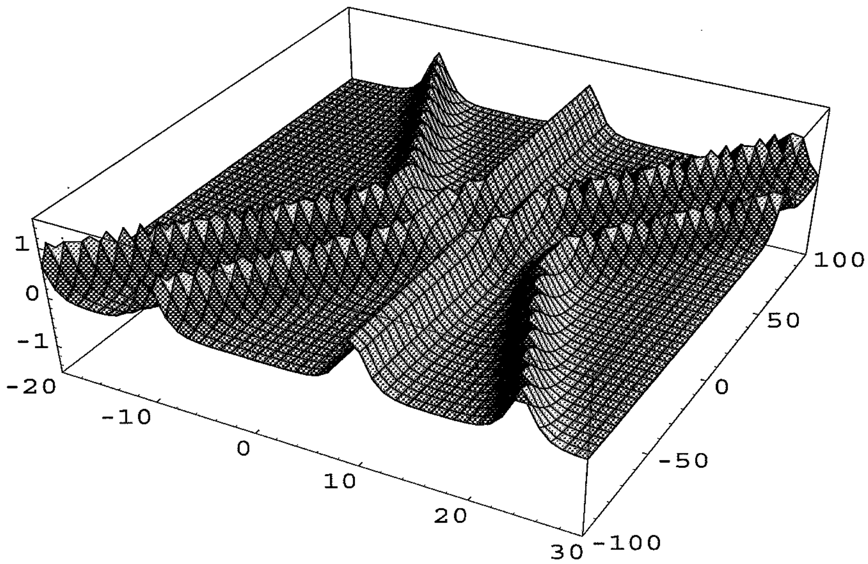
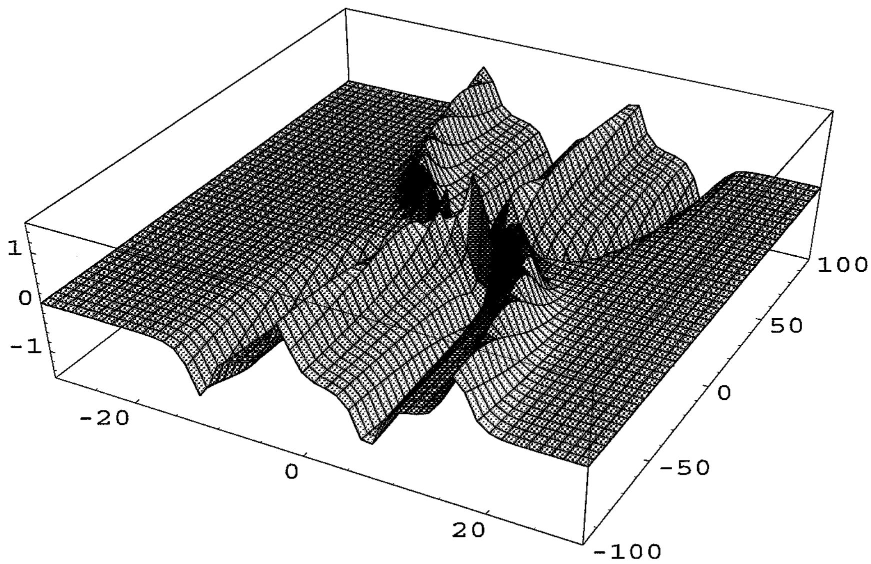
NE sG



four solitons ($a_1 = 0.9$, $a_2 = 1$, $a_3 = 1.2$, $a_4 = 1.25$)



negaton ($a = 1$) consisting of two solitons and two antisolitons