Discrete Chebyshev nets and a universal permutability theorem

by

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1. History

I. Lund & Regge (1976):

- AKNS Lax pair for sine-Gordon equation $=$ parameter-dependent $SU(2)$-valued generalization of the Gauß-Weingarten equations for pseudospherical surfaces.
- Relativistic motion of a string $\longrightarrow$ Lund-Regge equation

$$\boxed{\boldsymbol{r}_{xy} = \boldsymbol{r}_x \times \boldsymbol{r}_y}$$

II. Pohlmeyer (1976, 1977):

- Lagrangian field theories $\longrightarrow$ Pohlmeyer-Lund-Regge system

$$\boxed{\begin{aligned} \theta_{xy} + \frac{\cos\theta}{\sin^3\theta}\varphi_x\varphi_y &= \sin\theta\cos\theta \\ (\cot^2\theta\,\varphi_x)_y &= (\cot^2\theta\,\varphi_y)_x \end{aligned}}$$

is an integrable generalization of the sine-Gordon equation.

- Interpretation as Gauß-Mainardi-Codazzi equations for surfaces in S^3.

III. Sym (1982): *'Soliton theory is surface theory'*

2. Chebyshev nets (1878) (‘On the cutting of our clothes’)

Theorem: Any surface $\Sigma : \boldsymbol{r} = \boldsymbol{r}(x,y)$ in $\mathbb{R}^3$ may be (locally) parametrized in such a way that

$$\boldsymbol{r}_x^2 = f(x), \quad \boldsymbol{r}_y^2 = g(y),$$

i.e. opposite sides of coordinate patches are of equal length.

First fundamental form (suitably reparametrized):

$$\mathrm{I} = d\boldsymbol{r}^2 = dx^2 + 2\cos 2\theta\, dxdy + dy^2,$$

with $2\theta =$ angle between coordinate lines.

Theorem: A surface $\Sigma : \boldsymbol{r} = \boldsymbol{r}(x,y)$ is parametrized in terms of Chebyshev coordinates if and only if $\boldsymbol{r}_{xy} \cdot \boldsymbol{r}_x = \boldsymbol{r}_{xy} \cdot \boldsymbol{r}_y = 0$ and therefore

$$\boxed{\boldsymbol{r}_{xy} = \sigma\, \boldsymbol{r}_x \times \boldsymbol{r}_y,} \qquad \sigma = \sigma(x,y)$$

or, equivalently,

$$\boxed{\boldsymbol{r}_{xy} \parallel \hat{\boldsymbol{N}},} \qquad \hat{\boldsymbol{N}} = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{|\boldsymbol{r}_x \times \boldsymbol{r}_y|}.$$

3. The Pohlmeyer-Lund-Regge system (1976)

The Lund-Regge equation corresponds to $\sigma = 1$:

$$\boxed{\boldsymbol{r}_{xy} = \boldsymbol{r}_x \times \boldsymbol{r}_y}$$

Second fundamental form:

$$\mathrm{II} = -d\boldsymbol{r} \cdot d\widehat{\mathbf{N}} = 2\cot\theta\,\varphi_x\,dx^2 + 2\sin 2\theta\,dxdy + 2\cot\theta\,\varphi_y\,dy^2$$

Associated Gauß-Mainardi-Codazzi equations:

$$\begin{aligned} \theta_{xy} + \frac{\cos\theta}{\sin^3\theta}\varphi_x\varphi_y &= \sin\theta\cos\theta \\ (\cot^2\theta\,\varphi_x)_y &= (\cot^2\theta\,\varphi_y)_x \end{aligned}$$

Invariance: $\varphi \to \tilde{\varphi} = -\varphi$

Thus: Lund-Regge surfaces come in isometric pairs $(\Sigma, \tilde{\Sigma})$!

4. The $O(4)$ nonlinear σ-model (1976)

Pohlmeyer (1976): The Pohlmeyer-Lund-Regge system is equivalent to the $O(4)$ nonlinear σ-model

$$\boxed{\mathsf{N}_{xy} + (\mathsf{N}_x \cdot \mathsf{N}_y)\mathsf{N} = 0, \quad \mathsf{N} \in S^3}$$

If $\Sigma = \tilde{\Sigma}$, that is $\varphi = 0$, then one obtains the spherical representation

$$\boxed{\hat{\boldsymbol{N}}_{xy} + (\hat{\boldsymbol{N}}_x \cdot \hat{\boldsymbol{N}}_y)\hat{\boldsymbol{N}} = \mathbf{0}, \quad \hat{\boldsymbol{N}}^2 = 1}$$

of pseudospherical surfaces.

Lelieuvre formulae:

$$\boxed{\boldsymbol{r}_x = \hat{\boldsymbol{N}} \times \hat{\boldsymbol{N}}_x, \quad \boldsymbol{r}_y = \hat{\boldsymbol{N}}_y \times \hat{\boldsymbol{N}}}$$

provide the link between a pseudospherical surface and its spherical representation.

5. Generalized Lelieuvre formulae

Given a solution N of the $O(4)$ nonlinear σ-model, the generalized Lelieuvre formulae

$$\boxed{R_x = 2(\mathsf{N}_x\mathsf{N}^\top - \mathsf{N}\mathsf{N}_x^\top), \quad R_y = 2(\mathsf{N}\mathsf{N}_y^\top - \mathsf{N}_y\mathsf{N}^\top), \quad R \in so(4)}$$

are compatible. These imply, in turn, that

$$R_{xy} = \frac{1}{2}[R_x, R_y]$$

Isomorphisms: $so(4) \cong so(3) \oplus so(3)$ and $so(3) \cong su(2) \cong \mathbb{R}^3$

$$\Rightarrow \qquad \boldsymbol{r}_{xy} = \boldsymbol{r}_x \times \boldsymbol{r}_y, \quad \tilde{\boldsymbol{r}}_{xy} = \tilde{\boldsymbol{r}}_x \times \tilde{\boldsymbol{r}}_y$$

Conclusion: N encodes the pair $(\Sigma, \tilde{\Sigma})$ of Lund-Regge surfaces. Furthermore:

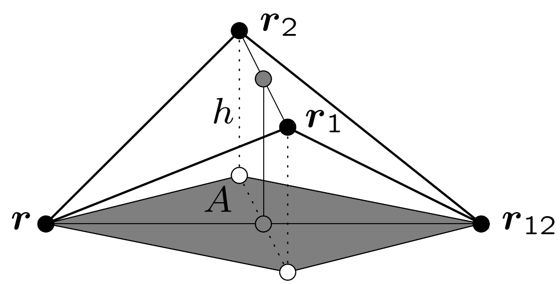
$$\bar{\boldsymbol{r}}_x = \boldsymbol{N} \times \boldsymbol{N}_x, \quad \bar{\boldsymbol{r}}_y = \boldsymbol{N}_y \times \boldsymbol{N}, \quad \bar{\boldsymbol{r}} = \frac{\boldsymbol{r} + \tilde{\boldsymbol{r}}}{2}, \quad \mathsf{N} =: (N_0, \boldsymbol{N})$$

Thus: $\boldsymbol{N}$ is a normal to the mid-surface $\bar{\Sigma}$ and x, y are asymptotic coordinates thereon.

Chiral model connection: $\boxed{(N^\dagger N_x)_y + (N^\dagger N_y)_x = 0, \quad N \in SU(2) \cong S^3 \ni \mathsf{N}}$

Definition: A 'discrete surface'

$$\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3, \quad (n_1, n_2) \mapsto \boldsymbol{r}(n_1, n_2)$$



is termed a discrete Chebyshev net if

$$(\Delta_1 \boldsymbol{r})^2 = f(n_1), \quad (\Delta_2 \boldsymbol{r})^2 = g(n_2).$$

Applications: discrete pseudospherical surfaces (1950), discrete smoke ring flows ...

Theorem: A discrete surface $\Sigma : \boldsymbol{r} = \boldsymbol{r}(x, y)$ constitutes a discrete Chebyshev net if and only if $(\Delta_{12}\boldsymbol{r}) \cdot (\boldsymbol{r}_{12} - \boldsymbol{r}) = (\Delta_{12}\boldsymbol{r}) \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1) = 0$ and therefore

$$\boxed{\Delta_{12}\boldsymbol{r} = \frac{\sigma}{2}(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1),} \qquad \sigma = \sigma(n_1, n_2)$$

or, equivalently,

$$\boxed{\Delta_{12}\boldsymbol{r} \parallel \hat{\boldsymbol{N}},} \qquad \hat{\boldsymbol{N}} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)|}.$$

Comparison:

$$\sigma = \frac{(\Delta_{12}\boldsymbol{r})\cdot\widehat{\boldsymbol{N}}}{\frac{1}{2}|(\boldsymbol{r}_{12}-\boldsymbol{r})\times(\boldsymbol{r}_2-\boldsymbol{r}_1)|} = \frac{2h}{A}, \qquad \text{(discrete)}$$

$$\sigma = \frac{\boldsymbol{r}_{xy}\cdot\widehat{\boldsymbol{N}}}{|\boldsymbol{r}_x\times\boldsymbol{r}_y|} = \frac{f}{\sqrt{\det \mathrm{I}}}, \qquad \text{(continuous)}$$

where f is the off-diagonal coefficient of the second fundamental form.

Question: Is the canonical choice $(\sigma = 1)$

$$\Delta_{12}\boldsymbol{r} = \frac{(\boldsymbol{r}_{12}-\boldsymbol{r})\times(\boldsymbol{r}_2-\boldsymbol{r}_1)}{2}$$

integrable and, if so, is it what we 'want'?

Answer: 'Yes' and 'No'!

7. Generalized discrete Lelieuvre formulae

Standard integrable discrete $O(n)$ nonlinear σ-model (Schief 2001) in the case $n = 4$:

$$\boxed{\mathsf{N}_{12} + \mathsf{N} = \frac{\mathsf{N} \cdot (\mathsf{N}_1 + \mathsf{N}_2)}{1 + \mathsf{N}_1 \cdot \mathsf{N}_2}(\mathsf{N}_1 + \mathsf{N}_2), \quad \mathsf{N} \in S^3}$$

Generalized discrete Lelieuvre formulae:

$$\boxed{\Delta_1 R = 2(\mathsf{N}_1 \mathsf{N}^\top - \mathsf{N}\mathsf{N}_1^\top), \quad \Delta_2 R = 2(\mathsf{N}\mathsf{N}_2^\top - \mathsf{N}_2 \mathsf{N}^\top), \quad R \in so(4)}$$

These imply that

$$\Delta_{12} R = \frac{[R_{12} - R, R_2 - R_1]}{2\mathsf{N} \cdot (\mathsf{N}_1 + \mathsf{N}_2)}$$

so that decomposition produces the discrete Lund-Regge equation

$$\boxed{\begin{gathered} \Delta_{12}\boldsymbol{r} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{\alpha(n_1) + \beta(n_2)} \\ \alpha = \sqrt{1 - (\Delta_1 \boldsymbol{r})^2}, \quad \beta = \sqrt{1 - (\Delta_2 \boldsymbol{r})^2} \end{gathered}}$$

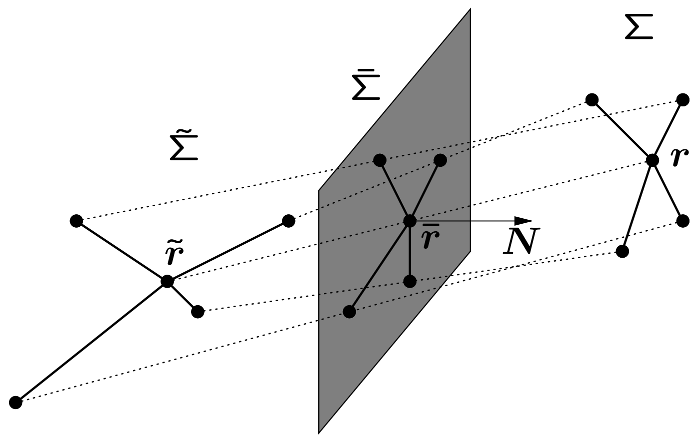
for $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$.

Remarks:

I. Discrete Lund-Regge mid-surfaces constitute discrete asymptotic nets since for

$$\bar{\boldsymbol{r}} = \frac{\boldsymbol{r} + \tilde{\boldsymbol{r}}}{2}:$$

$$\boxed{\Delta_1 \bar{\boldsymbol{r}} = \boldsymbol{N} \times \boldsymbol{N}_1, \quad \Delta_2 \bar{\boldsymbol{r}} = \boldsymbol{N}_2 \times \boldsymbol{N}}$$



II. Discrete $SU(2)$ chiral model:

$$\boxed{(N_1^\dagger + N_2^\dagger) N_{12} = N^\dagger (N_1 + N_2), \quad N \in SU(2)}$$

III. The generalized discrete Lund-Regge equation

$$\Delta_{12} \boldsymbol{r} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{\alpha(n_1) + \beta(n_2)}$$

is integrable for any prescribed choice of $\alpha(n_1)$ and $\beta(n_2)$.

8. A universal permutability theorem

$$\begin{tikzcd}
& \Phi_{(2)} \arrow[dr, "\lambda_{[1]}"] & \\
\Phi \arrow[ur, "\lambda_{[2]}"] \arrow[dr, "\lambda_{[1]}"'] & & \Phi_{(12)} \\
& \Phi_{(1)} \arrow[ur, "\lambda_{[2]}"'] &
\end{tikzcd}$$

Theorem: The four-point relation of 'Lund-Regge type'

$$\boxed{\boldsymbol{r}_{(12)} - \boldsymbol{r}_{(1)} - \boldsymbol{r}_{(2)} + \boldsymbol{r} = \frac{(\boldsymbol{r}_{(12)} - \boldsymbol{r}) \times (\boldsymbol{r}_{(2)} - \boldsymbol{r}_{(1)})}{a - b}}$$

may be interpreted as a superposition principle for any integrable system which admits commuting matrix Darboux transformations acting on an $su(2)$ linear representation.

Interpretation 1 (Example): NLS equation + conservation law (cf. Boiti *et al.* 1981)

$$\begin{aligned} iq_t + q_{xx} + \tfrac{1}{2}|q|^2 q &= 0, \\ p_x &= |q|^2, \\ p_t &= i(q_x\bar{q} - q\bar{q}_x), \end{aligned} \qquad \boldsymbol{r} = \begin{pmatrix} -\frac{1}{2}\Re(q) \\ \frac{1}{2}\Im(q) \\ -\frac{1}{4}p \end{pmatrix}$$

Interpretation 2 (Example): Heisenberg spin equation + Sym-Tafel formula

$$\boldsymbol{r}_t = \boldsymbol{r}_x \times \boldsymbol{r}_{xx}, \quad \boldsymbol{r}_x^2 = 1, \quad r = \Phi^{-1}\Phi_\lambda, \quad r \in su(2) \cong \mathbb{R}^3 \ni \boldsymbol{r}$$

Theorem ('Consistency around the cube'): The three copies

$$\Delta_{12}\boldsymbol{r} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{a(n_1) - b(n_2)}$$

$$\Delta_{23}\boldsymbol{r} = \frac{(\boldsymbol{r}_{23} - \boldsymbol{r}) \times (\boldsymbol{r}_3 - \boldsymbol{r}_2)}{b(n_2) - c(n_3)}$$

$$\Delta_{31}\boldsymbol{r} = \frac{(\boldsymbol{r}_{31} - \boldsymbol{r}) \times (\boldsymbol{r}_1 - \boldsymbol{r}_3)}{c(n_3) - a(n_1)}$$

of the generalized discrete Lund-Regge equation and their formal continuuum limit

$$\boldsymbol{r}_{xy} = 2\frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{a(x) - b(y)}$$

$$\boldsymbol{r}_{yz} = 2\frac{\boldsymbol{r}_y \times \boldsymbol{r}_z}{b(y) - c(z)}$$

$$\boldsymbol{r}_{zx} = 2\frac{\boldsymbol{r}_z \times \boldsymbol{r}_x}{c(z) - a(x)}$$

are compatible. [$\longrightarrow$ (discrete) Zakharov-Manakov connection!]