Grothendieck rings of basic classical Lie superalgebras

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Simple Lie algebras

The representation ring $R(\mathfrak{g})$ of a complex simple Lie algebra $\mathfrak{g}$ is isomorphic to the ring $\mathbb{Z}[P]^W$ of W-invariants in the integral group ring $\mathbb{Z}[P]$, where P is the corresponding weight lattice and W is the Weyl group. The isomorphism is given by the character map $Ch : R(\mathfrak{g}) \to \mathbb{Z}[P]^W$.

Our goal is to generalise this result to the case of Lie superalgebras.

Basic classical Lie superalgebras and generalised root systems

Following **V. Kac (1977,1978)** we call Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *basic classical* if

a) $\mathfrak{g}$ is simple,

b) Lie algebra $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$,

c) there exists a non-degenerate invariant even bilinear form on $\mathfrak{g}$.

Kac proved that the complete list of basic classical Lie superalgebras, which are not Lie algebras, consists of Lie superalgebras of the type

$$A(m,n), B(m,n), C(n), D(m,n), D(2,1,\lambda), F(4), G(3)$$

In full analogy with the case of simple Lie algebras one can consider the decomposition of $\mathfrak{g}$ with respect to adjoint action of Cartan subalgebra $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\alpha),$$

where the sum is taken over the set R of non-zero linear forms on $\mathfrak{h}$, which are called *roots* of $\mathfrak{g}$. With the exception of the Lie superalgebra of type $A(1,1)$ the corresponding root subspaces $\mathfrak{g}_\alpha$ have dimension 1 (for $A(1,1)$ type the root subspaces corresponding to the isotropic roots have dimension 2).

It is turned out that the corresponding root systems admit the following simple geometric description found by **V.Serganova (1996)**.

Let V be a finite dimensional complex vector space with a non-degenerate bilinear form $(,)$.

Definition. The finite set $R \subset V \setminus \{0\}$ is called a *generalised root system* if the following conditions are fulfilled :

1) R spans V and $R = -R$;

2) if $\alpha, \beta \in R$ and $(\alpha, \alpha) \neq 0$ then $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ and $s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in R$;

3) if $\alpha \in R$ and $(\alpha, \alpha) = 0$ then there exists an invertible mapping $r_\alpha : R \to R$ such that $r_\alpha(\beta) = \beta$ if $(\beta, \alpha) = 0$ and $r_\alpha(\beta) \in \{\beta + \alpha, \beta - \alpha\}$ otherwise.

The roots α such that $(\alpha, \alpha) = 0$ are called *isotropic*. A generalised root system R is called *reducible* if it can be represented as a direct orthogonal sum of two non-empty generalised root systems R_1 and R_2: $V = V_1 \oplus V_2$, $R_1 \subset V_1$, $R_2 \subset V_2$, $R = R_1 \cup R_2$. Otherwise the system is called *irreducible*.

Any generalised root system has a partial symmetry described by the finite group W_0 generated by the reflections with respect to the non-isotropic roots.

A remarkable fact proved by Serganova is that classification list for the irreducible generalised root systems with isotropic roots coincides with the root systems of the basic classical Lie superalgebras from the Kac list (with the exception of $A(1,1)$) and $B(0,n)$). Note that the superalgebra $B(0,n)$ has no isotropic roots: its root system is the usual non-reduced system of $BC(n)$ type.

Main Result

Let $\mathfrak{g}$ be such Lie superalgebra different from $A(1,1)$ and $\mathfrak{h}$ be its Cartan subalgebra (which in this case is also Cartan subalgebra of the Lie algebra $\mathfrak{g}_0$). Let $P_0 \subset \mathfrak{h}^*$ be the abelian group of weights of $\mathfrak{g}_0$, W_0 be the Weyl group of $\mathfrak{g}_0$ and $\mathbb{Z}[P_0]^{W_0}$ be the ring of W_0-invariants in the integral group ring $\mathbb{Z}[P_0]$. The decomposition of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{h}$ gives the (generalised) root system R of Lie superalgebra $\mathfrak{g}$. By definition $\mathfrak{g}$ has a natural non-degenerate bilinear form on $\mathfrak{h}$ and hence on $\mathfrak{h}^*$, which will be denoted as $(\,,\,)$. In contrast to the theory of semisimple Lie algebras some of the roots $\alpha \in R$ are isotropic: $(\alpha, \alpha) = 0$. For isotropic roots one can not define the usual reflection, which explains the difficulty with the notion of Weyl group in this case.

Define the following *ring of exponential super-invariants* $J(\mathfrak{g})$, replacing the algebra of Weyl group invariants in the classical case of Lie algebras:

$$J(\mathfrak{g}) = \{f \in \mathbb{Z}[P_0]^{W_0} : D_\alpha f \in (e^\alpha - 1)$$

$$\text{for any isotropic root } \alpha\}$$

where $(e^\alpha - 1)$ denotes the principal ideal in $\mathbb{Z}[P_0]^{W_0}$ generated by $e^\alpha - 1$ and the derivative D_α is defined by the property $D_\alpha(e^\beta) = (\alpha, \beta)e^\beta$. This ring is a variation of the algebra of invariant polynomials investigated for Lie superalgebras by **F. Berezin (1977), A.Sergeev (1982,1999), V. Kac (1984).** For the special case of the Lie superalgebra $A(1,1)$ one should slightly modify the definition because of the multiplicity 2 of the isotropic roots (see section 8 below).

Our main result is the following

Theorem. *The Grothendieck ring $K(\mathfrak{g})$ of finite dimensional representations of a basic classical Lie superalgebra $\mathfrak{g}$ is isomorphic to the ring $J(\mathfrak{g})$. The isomorphism is given by the supercharacter map $Sch : K(\mathfrak{g}) \to J(\mathfrak{g})$.*

The elements of $J(\mathfrak{g})$ can be described as the invariants in the weight group rings under the action of the following groupoid $\mathfrak{W}$, which we call *super Weyl groupoid*. It is defined as a disjoint union

$$\mathfrak{W}(R) = W_0 \coprod W_0 \ltimes \mathfrak{T}_{iso},$$

where $\mathfrak{T}_{iso}$ is the groupoid, whose base is the set R_{iso} of all isotropic roots of $\mathfrak{g}$ and the set of morphisms from $\alpha \to \beta$ is non-empty if and only if $\beta = \pm\alpha$ in which case it consists of just one element τ_α. This notion was motivated by our work on deformed Calogero-Moser systems **A.Sergeev, A.Veselov (2004)**.

The group W_0 is acting on $\mathfrak{T}_{iso}$ in a natural way and thus defines a semi-direct product groupoid $W_0 \ltimes \mathfrak{T}_{iso}$ (see details in section 9). One can define a natural action of $\mathfrak{W}$ on $\mathfrak{h}$ with τ_α acting as a shift by α in the hyperplane $(\alpha, x) = 0$. If we exclude the special case of $A(1,1)$ our Theorem can now be reformulated as the following version of the classical case:

The Grothendieck ring $K(\mathfrak{g})$ of finite dimensional representations of a basic classical Lie superalgebra $\mathfrak{g}$ is isomorphic to the ring $\mathbb{Z}[P_0]^{\mathfrak{W}}$ of the invariants of the super Weyl groupoid $\mathfrak{W}$.

Concluding remarks

Thus we have now a description of the Grothendieck rings of finite-dimensional representations for all basic classical Lie superalgebras. The fact that the corresponding rings can be described by simple algebraic conditions seems to be remarkable. We believe that these rings as well as the corresponding super Weyl groupoids will play an important role in the representation theory.

An important problem is to describe "good" bases of the rings $K(\mathfrak{g})$ as modules over $\mathbb{Z}$ and transition matrices between them. For example, in the classical case of Lie algebra of type $A(n)$ we have various bases labeled by Young diagrams λ: Schur polynomials s_λ (or characters of the irreducible representations), symmetric functions h_λ and e_λ . We hope also that our result could lead to a better understanding of the algorithms of computing the characters proposed by **V. Serganova (1996) and J.Brundan (2003)**. The investigation of the deformations of the Grothendieck rings and the spectral decompositions of the corresponding analogues of the deformed Calogero-Moser and Macdonald operators which were introduced by **M. Feigin, O. Chalykh, A.Veselov (1996), A.Sergeev, A.Veselov (2004)** may help to clarify the situation.

One can define also the Grothendieck ring $P(\mathfrak{g})$ of projective finite-dimensional $\mathfrak{g}$-modules . It can be shown that $P(\mathfrak{g}) \subset K(\mathfrak{g})$ is an ideal in the Grothendieck ring $K(\mathfrak{g})$. The problem is to describe the structure of $P(\mathfrak{g})$ as a $K(\mathfrak{g})$-module.