Linear deformations of the matrix product and integrable ODEs.

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Consider the following class of matrix ODEs

$$\frac{dx}{dt} = [x, Q(x)],$$

where x is $n \times n$ matrix, Q is a constant linear operator $Q : Mat_n \to Mat_n$.

For instance, the matrix equation

$$\frac{dx}{dt} = [x, xc + cx] = x^2 c - c x^2. \tag{1}$$

is integrable for any constant matrix c and any n. This equation possesses an infinite set of homogeneous integrals $H_{i,j}$, where i and j are degrees with respect to x and c.

For example,

$$H_{1,1} = \text{trace}\,(xc)\,, \qquad H_{2,1} = \text{trace}\,(x^2c).$$

Equation (1) is Hamiltonian one with respect to the standard matrix linear Poisson bracket, given by the Hamiltonian operator ad_x, and Hamiltonian function $H_{2,1}$.

Matrix equations of arbitrary size like (1) are important because of a possibility to make different reductions.

For the most trivial reduction one may regard x as a block-matrix. In this case (1) becomes a system of several matrix equations for the block entries of x.

Under reduction $x^T = -x, c^T = c$ equation (1) is a commuting flow for the n-dimensional Euler equation.

Let x and c in the equation $\frac{dx}{dt} = x^2\,c - c\,x^2$ be represented by matrices of the form

$$x = \begin{pmatrix} 0 & u_1 & 0 & 0 & \cdot & 0 \\ 0 & 0 & u_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & u_{N-1} \\ u_N & 0 & 0 & 0 & \cdot & 0 \end{pmatrix}, \; c = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 & J_N \\ J_1 & 0 & 0 & \cdot & 0 & 0 \\ 0 & J_2 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & J_{N-1} & 0 \end{pmatrix},$$

where u_k and J_k are block matrices (of any dimension). It follows from equation (1) that u_k satisfy the non-abelian Volterra equation

$$\frac{d}{dt}u_k = u_k \circ u_{k+1} \circ J_{k+1} - J_{k-1} \circ u_{k-1} \circ u_k.$$

A multiplication $\circ$ defined on the vector space Mat_n of all $n \times n$ matrices is said to be compatible with the matrix multiplication if the product

$$X \bullet Y = XY + \lambda\, X \circ Y \tag{2}$$

is associative for any constant λ.

Since the matrix algebra is rigid, the multiplication (2) is isomorphic to the matrix multiplication for almost all values of the parameter λ. This means that there exists a formal series of the form

$$S_\lambda = \mathbf{1} + R\ \lambda + O(\lambda^2), \tag{3}$$

with the coefficients being linear operators on Mat_n, such that

$$S_\lambda(X)\, S_\lambda(Y) = S_\lambda(XY + \lambda\ X \circ Y). \tag{4}$$

It follows from this formula that the multiplication $\circ$ is given by

$$X \circ Y = R(X)Y + XR(Y) - R(XY). \tag{5}$$

where $R: Mat_n \to Mat_n$ is a linear operator.

Example 1. Let c be an element of Mat_n and $R: X \to cX$ be the operator of left multiplication by c. Then the corresponding multiplication $X \circ Y = X\,c\,Y$ is associative and compatible with matrix multiplication.

Example 2. Suppose $a, b \in Mat_2$; then the product

$$X \circ Y = (aX - Xa)\,(bY - Yb)$$

is compatible with the standard product in Mat_2. The corresponding operator R is given by $R(X) = aXb - abX$.

Classical associative Yang-Baxter equation

Let $\lambda \to S_\lambda$ be a meromorphic function with values in $End(Mat_n)$ such that $S_0 = Id$ and

$$S_\lambda(X)\, S_\lambda(Y) = S_\lambda(XY + \lambda\ X \circ Y),$$

where $\circ$ is an associative multiplication compatible with the matrix one.

Theorem. The formula

$$r(u,v) = \frac{1}{u-v} S_u S_v^{-1}$$

defines a solution to the associative Yang-Baxter equation

$$(r(u,w)x)(r(u,v)y) - r(u,v)((r(v,w)x)y) =$$

$$r(u,w)(x(r(w,v)y)).$$

Integrable matrix ODEs.

It is convenient to write down the operator R in the form

$$R(x) = a_1\, x\, b^1 + ... + a_p\, x\, b^p + c\, x, \tag{6}$$

where $a_i, b^i, c \in Mat_n$, with p being smallest possible.

Let $\circ$ be a multiplication (5) compatible with the matrix product. Consider the following matrix differential equation

$$\frac{dx}{dt} = [R(x) + R^*(x), x], \tag{7}$$

where

$$R^*(x) = b^1\, x\, a_1 + ... + b^p\, x\, a_p + x\, c.$$

It turns out that equation (7) possesses the Lax representation

$$\frac{dL}{dt} = [A,\, L],$$

where

$$L = \left(S_\lambda^{-1}\right)^* (x), \qquad A = \frac{1}{\lambda} S_\lambda(x). \tag{8}$$

As usual, the integrals of motion for (7) are given by coefficients of different powers of λ in trace$\,(L^k)$, $k = 1, 2...$

To make these formulas constructive, we should find S_λ and S_λ^{-1} in a closed form i.e. as analytic operator-valued functions.

In the case of Example 1, we have

$$R(x) = cx, \qquad S_\lambda(x) = (1 + \lambda c)\, x,$$

$$L = x(1 + \lambda c)^{-1}.$$

The Lax equation is equivalent to the well known integrable matrix equation

$$\frac{dx}{dt} = [x,\, xc + cx] = x^2\, c - c\, x^2.$$

The L-operator produces an infinite set of homogeneous integrals $H_{i,j}$, where i and j are degrees with respect to x and c. The simplest are

$$H_{1,1} = \mathrm{trace}\,(xc)\,, \quad H_{2,1} = \mathrm{trace}\,(x^2 c).$$

A generalization of Example 2 is defined by two arbitrary constant matrices A and B such that

$$A^2 = B^2 = \mathbf{1}. \tag{9}$$

The corresponding associative multiplication is given by (5), where

$$R(x) = AxB + BAx. \tag{10}$$

This leads to the following integrable matrix equation

$$x_t = [x,\ BxA + AxB + xBA + BAx]. \tag{11}$$

The Lax representation (8) for (11) is given by the following explicit formulas for S_λ and S_λ^{-1}:

$$S_\lambda(x) = \frac{1-q}{2}\,BxB + \frac{1+q}{2}\,x + \lambda(AxB + BAx),$$

$$S_\lambda^{-1}(x) = \frac{1}{q}(\mathbf{1}+\lambda K)^{-1}\left(\frac{q-1}{2}\,BxB + \frac{1+q}{2}\,x + \lambda(ABx - AxB)\right),$$

where $q = \sqrt{1-4\lambda^2}$, $\;K = AB + BA$.

The simplest linear and quadratic first integrals for (11) generated by the L-operator are given by

$$H_{1,1} = \operatorname{trace}\,[x(AB+BA)], \qquad H_{2,1} = \operatorname{trace}\,[ABx^2 + AxBx].$$

Equation (11) admits the following skew- symmetric reduction

$$x^T = -x, \qquad B = A^T. \tag{12}$$

Different integrable $so(n)$-models provided by reduction (12) are in one-to-one correspondence with equivalence classes with respect to the $SO(n)$ gauge action of $n \times n$ matrices A such that $A^2 = \mathbf{1}$.

For the real matrix A, a canonical form for such equivalence class can be chosen as

$$A = \begin{pmatrix} \mathbf{1}_p & T \\ 0 & \mathbf{1}_{n-p} \end{pmatrix}. \tag{13}$$

Here $\mathbf{1}_s$ stands for the unity $s \times s$ matrix and $T = \{t_{ij}\}$, where $t_{ij} = \delta_{ij}\alpha_i$.

This canonical form is defined by the discrete natural parameter p and continuous parameters $\alpha_1, \ldots, \alpha_r$, where $r = \min(p, n-p)$.

For example, in the case $n = 4$ the equivalence classes with $p = 2$ and $p = 1$ give rise to the Steklov and the Poincare integrable models, correspondingly.

Thus, whereas (1) is a matrix version of the $so(4)$ Schottky-Manakov top, equation (11)-(13) with $p = [n/2]$ and $p = 1$ can be regarded as new $so(n)$ generalizations for the $so(4)$ Steklov and Poincare models.

A generalization of the basic matrix integrable model (7):

$$\frac{dx}{dt} = [x, v] + x \star x,$$

$$\frac{dv}{dt} = [x, u] + x \star v,$$

$$\frac{du}{dt} = x \star u,$$

where $x, v, w \in Mat_n$ and

$$X \star Y = [R(X), Y] + [R^T(Y), X] + R^T([X, Y]).$$

If $v = u = 0$, we get

$$\frac{dx}{dt} = [R(x) + R^*(x), x].$$

Algebraic structures.

Let us write R in the form

$$R(x) = a_1 \, X \, b^1 + ... + a_p \, X \, b^p + c \, X \tag{14}$$

with minimal p. Here $a_i, b^i, c \in Mat_N$.

Lemma 1. If the multiplication defined by

$$X \circ Y = R(X)Y + X R(Y) - R(XY).$$

is associative, then

$$a_i a_j = \phi^k_{i,j} a_k + \mu_{i,j} \mathbf{1}, \qquad b^i b^j = \psi^{i,j}_k b^k + \lambda^{i,j} \mathbf{1}$$

for some tensors $\phi^k_{i,j}$, $\mu_{i,j}$, $\psi^{i,j}_k$, $\lambda^{i,j}$.

This means that the vector spaces spanned by $1, a_1, \ldots a_p$ and $1, b_1, \ldots b_p$ are associative algebras. We denote them by $\mathcal{A}$ and $\mathcal{B}$. The algebras $\mathcal{A}$ and $\mathcal{B}$ have to be related by certain consistency conditions.

The simplest example of a similar structure can be described as follows.

Associative bi-algebras.

Let $\mathcal{A}$ and $\mathcal{B}$ be associative algebras with basis $A_1, \dots, A_p$ and $B^1, \dots, B^p$ and structural constants $\phi^i_{j,k}$ and $\psi^{\alpha,\beta}_{\gamma}$. Suppose that the structural constants satisfy the following identities:

$$\phi^s_{j,k}\psi^{l,i}_s = \phi^l_{s,k}\psi^{s,i}_j + \phi^i_{j,s}\psi^{l,s}_k, \qquad 1 \le i,j,k,l \le p.$$

Then the algebra $\mathcal{M}$ of dimension $2p+p^2$ with the basis A_i, B^j, A_iB^j and relations

$$B^iA_j = \psi^{k,i}_j A_k + \phi^i_{j,k}B^k$$

is associative. Note that A_iB^j form an associative subalgebra of dimension p^2.

Consider the vector space $\mathcal{L}$ spanned by A_i, B^j. It is clear that $\mathcal{A}$ and $\mathcal{B}$ act on $\mathcal{L}$ by right and left multiplication, correspondingly.

Let $(\cdot,\cdot)$ be a non-degenerate symmetric scalar product on the space $\mathcal{L}$ such that

$$(A_i, A_j) = (B^i, B^j) = 0, \qquad (A_i, B^j) = \delta_i^j.$$

Then the consistency condition means that

$$(b_1 b_2, v) = (b_1, b_2 v), \quad (v, a_1 a_2) = (v a_1, a_2)$$

for any $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $v \in \mathcal{L}$.

General algebraic structure.

Definition. By week $\mathcal{M}$-structure on a linear space $\mathcal{L}$ we mean a collection of the following data:

- Two subspaces $\mathcal{A}$ and $\mathcal{B}$ and distinguished element $1 \in \mathcal{A} \cap \mathcal{B} \subset \mathcal{L}$.
- Associative products $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$ with unity 1.
- Left action $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$ of the algebra $\mathcal{B}$ and right action $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ of the algebra $\mathcal{A}$ on the space $\mathcal{L}$, which commute to each other.
- A non-degenerate symmetric scalar product $(\cdot, \cdot)$ on the space $\mathcal{L}$.

These data should satisfy the following properties:

1. $\dim \mathcal{A} \cap \mathcal{B} = \dim \mathcal{L}/(\mathcal{A} + \mathcal{B}) = 1$. Intersection of $\mathcal{A}$ and $\mathcal{B}$ is a one dimensional space spanned by the unity 1.

2. Restriction of the action $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$ to subspace $\mathcal{B} \subset \mathcal{L}$ is the product in $\mathcal{B}$. Restriction of the action $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ to subspace $\mathcal{A} \subset \mathcal{L}$ is the product in $\mathcal{A}$.

3. $(a_1, a_2) = (b_1, b_2) = 0$ and

$$(b_1 b_2, v) = (b_1, b_2 v), \quad (v, a_1 a_2) = (v a_1, a_2)$$

for any $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $v \in \mathcal{L}$.

It follows from these properties that $(\cdot, \cdot)$ gives a non-degenerate pairing between $\mathcal{A}/\mathbb{C}1$ and $\mathcal{B}/\mathbb{C}1$, so $\dim \mathcal{A} = \dim \mathcal{B}$ and $\dim \mathcal{L} = 2 \dim \mathcal{A}$.

For given week $\mathcal{M}$-structure we can define an universal associative algebra generated by $\mathcal{L}$.

Definition. By week $\mathcal{M}$-algebra associated with week $\mathcal{M}$-structure on $\mathcal{L}$ we mean an associative algebra $U(\mathcal{L})$ with the following properties:

1. $\mathcal{L} \subset \mathcal{U}(\mathcal{L})$ and the actions $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$, $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ are restrictions of the product in $U(\mathcal{L})$.

2. For any algebra X with the property **1** there exists a unique homomorphism of algebras $X \to U(\mathcal{L})$, which is identical on $\mathcal{L}$.

It is easy to see that if $U(\mathcal{L})$ exists, then it is unique for given $\mathcal{L}$.

Let us describe the structure of $U(\mathcal{L})$ explicitly.

Let $\{1, A_1, ..., A_p\}$ be a basis of $\mathcal{A}$ and $\{1, B^1, ..., B^p\}$ be a dual basis of $\mathcal{B}$ (which means that $(A_i, B^j) = \delta_i^j$). Let $C \in \mathcal{L}$ doesn't belong to the sum of $\mathcal{A}$ and $\mathcal{B}$. Since $(\cdot, \cdot)$ is non- degenerate, we have $(1, C) \neq 0$. Multiplying C by constant, we may assume that $(1, C) = 1$. Adding linear combination of $1, A_1, ..., A_p, B^1, ..., B^p$ to C, we can assume that $(C, C) = (C, A_i) = (C, B^j) = 0$. Such element C is uniquely determined by choosing basis in $\mathcal{A}$ and $\mathcal{B}$.

Let us define an element $K \in U(\mathcal{L})$ by the formula $K = A_i B^i + C$.

Definition. A week $\mathcal{M}$-structure on $\mathcal{L}$ is called $\mathcal{M}$-structure if $K \in U(\mathcal{L})$ is a central element of the algebra $U(\mathcal{L})$.

Theorem 1. The algebra $U(\mathcal{L})$ is spanned by the elements K^s, $A_i K^s$, $B_j K^s$, $A_i B^j K^s$, where $i, j = 1, ..., p$, and $s = 0, 1, 2, ...$

Theorem 2. For any representation $U(\mathcal{L}) \to Mat_n$ given by

$$A_1 \to a_1, ..., A_p \to a_p, B^1 \to b^1, ..., B^p \to b^p, C \to c$$

formula (5) defines associative product on Mat_n compatible with the usual product.

Example 3. Suppose $\mathcal{A}$ and $\mathcal{B}$ are generated by elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a^{p+1} = b^{p+1} = 1$. Assume that $(b^i, a^{-i}) = \epsilon^i - 1$, $(1, c) = 1$ and other scalar products are equal to zero. Here ϵ is a primitive root of unity of order p. Let

$$b^i a^j = \frac{\epsilon^{-j} - 1}{\epsilon^{-i-j} - 1} a^{i+j} + \frac{\epsilon^i - 1}{\epsilon^{i+j} - 1} b^{i+j}$$

for $i + j \neq 0$ modulo p and

$$b^i a^{-i} = 1 + (\epsilon^i - 1)c, \quad ca^i = \frac{1}{1 - \epsilon^i} a^i + \frac{1}{\epsilon^i - 1} b^i,$$

$$b^i c = \frac{1}{\epsilon^{-i} - 1} a^i + \frac{1}{1 - \epsilon^{-i}} b^i$$

for $i \neq 0$ modulo p. These formulas define an $\mathcal{M}$-structure.

The central element has the following form

$$K = c + \sum_{0<i<p} \frac{1}{\epsilon^i - 1} a^{-i} b^i.$$

Let a, t be linear operators in some vector space. Assume that $a^{p+1} = 1$, $at = \epsilon ta$ and the operator $t - 1$ is invertible. It is easy to check that the formulas

$$A \to a, \qquad B \to \frac{\epsilon t - 1}{t - 1} a, \qquad C \to \frac{t}{t - 1}$$

define a representation of the algebra $U(\mathcal{L})$.

Case of semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$

Suppose a vector space $\mathcal{L}$ is equipped with a weak $\mathcal{M}$-structure such that

$$\mathcal{A} = \oplus_{1\le i\le r} End(V_i), \qquad \mathcal{B} = \oplus_{1\le j\le s} End(W_j),$$

$$\dim V_i = m_i, \qquad \dim W_j = n_j.$$

Then $\mathcal{L}$ as $\mathcal{A}\otimes\mathcal{B}$-module is isomorphic to $\oplus_{1\le i\le r, 1\le j\le s}(V_i^\star \otimes W_j)^{a_{i,j}}$, for some $a_{i,j} \ge 0$

Theorem. For any i, j

$$\sum_{j=1}^{s} a_{i,j} n_j = 2m_i, \qquad \sum_{i=1}^{r} a_{i,j} m_i = 2n_j. \tag{15}$$

The matrix of linear system (15) has the form

$$Q = \begin{pmatrix} 2 & -A \\ -A^t & 2 \end{pmatrix}.$$

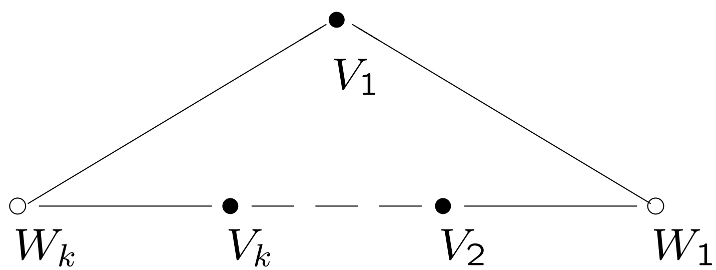
According to the result by E. Vinberg, if the kernel of indecomposable matrix Q contains an integer positive vector, them Q is the Cartan matrix of an affine Dynkin diagram.

Moreover, it follows from the structure of Q that this diagram is a simple laced affine Dynkin diagram with a partition of the set of vertices into two subsets such that vertices of the same subset are not connected.

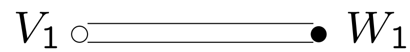
Theorem. Let A be an $r \times s$ matrix of multiplicities for an indecomposable weak $\mathcal{M}$ -structure. Then, after a possible permutation of rows and columns and the transposition, a matrix A is equal to one in the following list:

1. $A = (2)$. Here $r = s = 1$, $n_1 = m_1 = m$. The corresponding Dynkin diagram is of the type $\tilde{A}_1$.

2. $a_{i,i} = a_{i,i+1} = 1$ and $a_{i,j} = 0$ for other pairs i, j. Here $r = s = k \geq 2$, the indexes are taken modulo k, and $n_i = m_i = m$. The corresponding Dynkin diagram is $\tilde{A}_{2k-1}$.

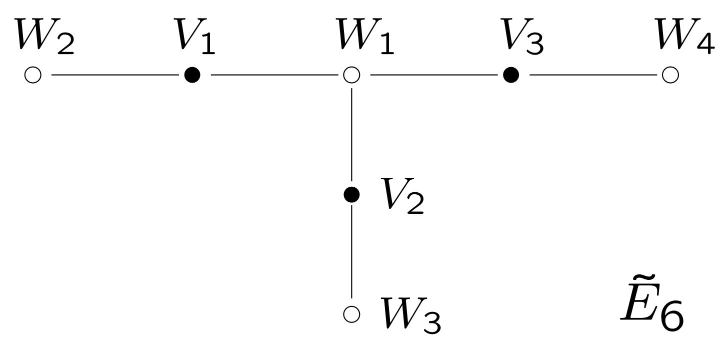


$\tilde{A}_{2k-1}$



$\tilde{A}_1$

3. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Here $r = 3$, $s = 4$ and $n_1 = 3m$, $n_2 = n_3 = n_4 = m$, $m_1 = m_2 = m_3 = 2m$. The Dynkin diagram is $\tilde{E}_6$:



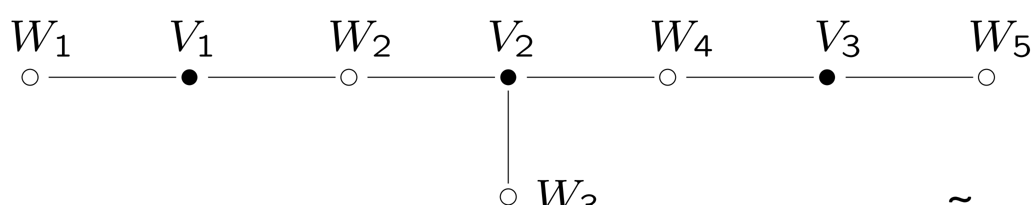
$\tilde{E}_6$

4. $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$

Here $r = 3$, $s = 5$ and

$$n_1 = m, \quad n_2 = 3m, \quad n_3 = 2m, \quad n_4 = 3m,$$
$$n_5 = m, \qquad m_1 = 2m,\ m_2 = 4m,\ m_3 = 2m.$$

The Dynkin diagram is $\tilde{E}_7$:

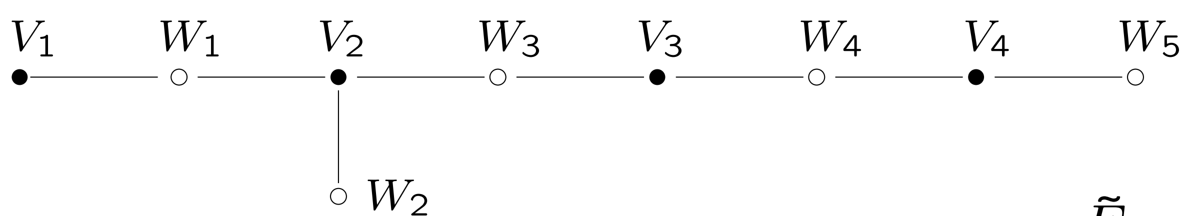


$\tilde{E}_7$

5. $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

Here $n_1 = 4m$, $n_2 = 3m$, $n_3 = 5m$, $n_4 = 3m$, $n_5 = m$, $m_1 = 2m$, $m_2 = 6m$, $m_3 = 4m$, $m_4 = 2m$.

The Dynkin diagram is $\tilde{E}_8$:

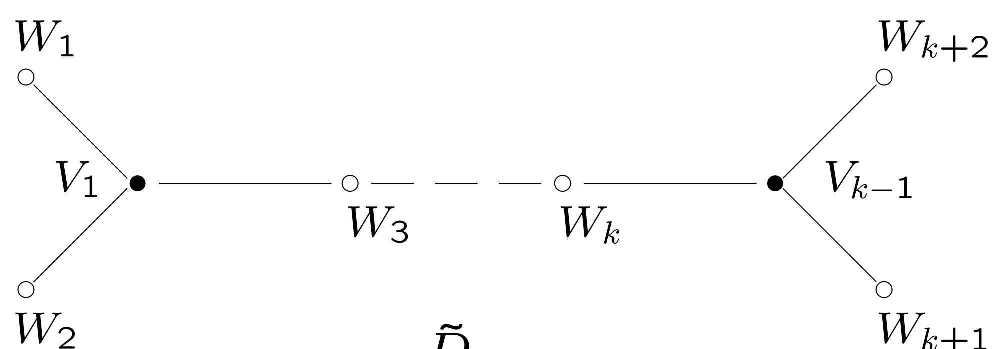


$\tilde{E}_8$

6. $A = (1,1,1,1)$. Here $r = 1, s = 4$ and $n_1 = n_2 = n_3 = n_4 = m$, $m_1 = 2m$. The corresponding Dynkin diagram is $\tilde{D}_4$.

7. $a_{1,1} = a_{1,2} = a_{1,3} = 1$, $a_{2,3} = a_{2,4} = a_{3,4} = a_{3,5} = \cdots = a_{k-2,k-1} = a_{k-2,k} = 1$, $a_{k-1,k} = a_{k-1,k+1} = a_{k-1,k+2} = 1$, and $a_{i,j} = 0$ for other (i,j).

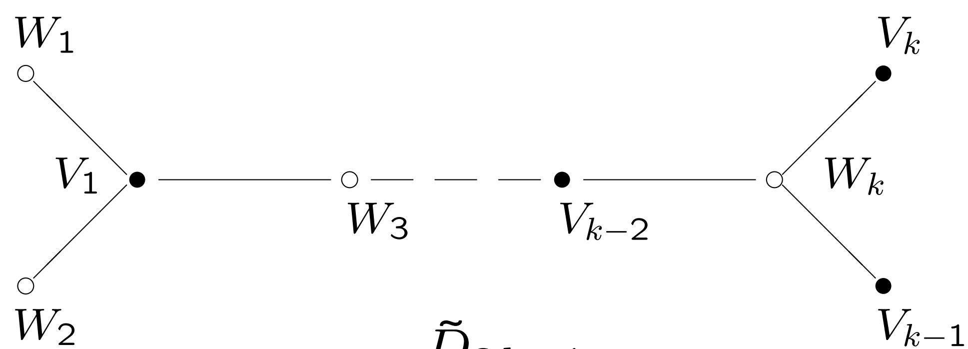
Here we have $r = k-1$, $s = k+2$ and $n_1 = n_2 = n_{k+1} = n_{k+2} = m$, $n_3 = \cdots = n_k = 2m$, $m_1 = \cdots = m_l = 2m$. The corresponding Dynkin diagram is $\tilde{D}_{2k}$, where $k \geq 3$.



$\tilde{D}_{2k}$

8. $a_{1,1} = a_{1,2} = a_{1,3} = 1$, $a_{2,3} = a_{2,4} = a_{3,4} = a_{3,5} = \cdots = a_{k-2,k-1} = a_{k-2,k} = 1$, $a_{k-1,k} = a_{k,k} = 1$, and $a_{i,j} = 0$ for other (i,j).

Here we have $r = s = k \geq 3$, $n_1 = n_2 = m$, $n_3 = \cdots = n_k = 2m$, $m_1 = \cdots = m_{k-2} = 2m$, $m_{k-1} = m_k = m$. The corresponding Dynkin diagram is $\tilde{D}_{2k-1}$:



$\tilde{D}_{2k-1}$

Note that if $k = 3$, then $a_{1,1} = a_{1,2} = a_{1,3} = 1$, $a_{2,3} = a_{3,3} = 1$.

Resume: Suppose $\mathcal{L}$ is an indecomposable $\mathcal{M}$-structure with semi-simple algebras $\mathcal{A} = \oplus_{1\leq i\leq r} End(V_i)$, $\mathcal{B} = \oplus_{1\leq j\leq s} End(W_j)$; then there exists an affine Dynkin diagram of the type A, D, or E such that:

1. There is a one-to-one correspondence between the set of vertices and the set of vector spaces

$$\{V_1, ..., V_r, W_1, ..., W_s\}$$

. **2.** For any i, j the spaces V_i, V_j are not connected by edges as well as W_i, W_j.

3. The vector

$$(\dim V_1, ..., \dim V_r, \dim W_1, ..., \dim W_s)$$

is equal to mJ, where J is the minimal imaginary positive root of the Dynkin diagram.

Remark. It can be proved that for indecomposable $\mathcal{M}$-structures $m = 1$.

To describe the $\mathcal{M}$-structure it remains to construct an embedding $\mathcal{A} \to \mathcal{L}$, $\mathcal{B} \to \mathcal{L}$ and a scalar product $(\cdot, \cdot)$ on the space $\mathcal{L}$.

If we fix an element $1 \in \mathcal{L}$, then we can define the embedding $\mathcal{A} \to \mathcal{L}$, $\mathcal{B} \to \mathcal{L}$ by the formula $a \to 1a$, $b \to b1$ for $a \in \mathcal{A}$, $b \in \mathcal{B}$. We may assume that 1 is a generic element of $\mathcal{L}$.

Thus to study $\mathcal{M}$-structures corresponding to a Dynkin diagram, one should take a generic element in $\mathcal{L} = \oplus_{1\leq i\leq r, 1\leq j\leq s}(V_i^\star \otimes W_j)^{a_{i,j}}$, find its simplest canonical form by choosing bases in the vector spaces

$$V_1, ..., V_r, W_1, ..., W_s,$$

calculate the embedding $\mathcal{A} \to \mathcal{L}$, $\mathcal{B} \to \mathcal{L}$ and the scalar product $(\cdot,\cdot)$ on the space $\mathcal{L}$.

The classification of generic elements $1 \in \mathcal{L}$ up to choice of bases in the vector spaces $V_1, ..., V_r, W_1, ..., W_s$ is equivalent to classification of irreducible representations of the quivers corresponding to our affine Dynkin diagrams and we can apply known results about these representations.

Semi-simple case.

Consider an associative algebra

$$M = \oplus_{1\le\alpha\le m} M_\alpha,$$

where M_α is isomorphic to Mat_{n_α}. We are going to study associative products in this algebra compatible with the initial one.

The main algebraic object is the weak $\mathcal{PM}$-structure (of size m) on $\mathcal{L}$. All properties coincide with the properties of weak $\mathcal{M}$-structure except for

1. $\dim \mathcal{A}\cap\mathcal{B} = \dim \mathcal{L}/(\mathcal{A}+\mathcal{B}) = m$. The intersection of $\mathcal{A}$ and $\mathcal{B}$ is a m-dimensional algebra isomorphic to $\mathbb{C}\oplus ... \oplus\mathbb{C}$.

The scalar product $(\cdot,\cdot)$ defines a non- degenerate pairing between $\mathcal{A}/\mathcal{A}\cap\mathcal{B}$ and $\mathcal{B}/\mathcal{A}\cap\mathcal{B}$, so $\dim\mathcal{A} = \dim\mathcal{B}$ and $\dim\mathcal{L} = 2\dim\mathcal{A}$.

Let $\{e_\alpha; 1 \leq \alpha \leq m\}$ be a basis of the space $\mathcal{A}\cap\mathcal{B}$ such that

$$e_\alpha e_\beta = \delta_{\alpha,\beta} e_\alpha.$$

Denote by $\mathcal{L}_{\alpha,\beta}$ the vector space consisting of elements $v_{\alpha,\beta} \in \mathcal{L}$ with the property

$$e_\alpha v_{\alpha,\beta} = v_{\alpha,\beta} e_\beta = v_{\alpha,\beta}.$$

Let $\mathcal{A}_{\alpha,\beta} = \mathcal{A}\cap\mathcal{L}_{\alpha,\beta}$ and $\mathcal{B}_{\alpha,\beta} = \mathcal{B}\cap\mathcal{L}_{\alpha,\beta}$.

We sum by repeated Latin indexes and do not sum by repeated Greek indexes

The following properties hold:

- $\mathcal{L} = \oplus_{1\le\alpha,\beta\le m}\mathcal{L}_{\alpha,\beta}$, $\quad \mathcal{A} = \oplus_{1\le\alpha,\beta\le m}\mathcal{A}_{\alpha,\beta}$ and $\mathcal{B} = \oplus_{1\le\alpha,\beta\le m}\mathcal{B}_{\alpha,\beta}$.

- $\dim \mathcal{A}_{\alpha,\beta} \cap \mathcal{B}_{\alpha,\beta} = \dim \mathcal{L}/(\mathcal{A}_{\alpha,\beta} + \mathcal{B}_{\alpha,\beta}) = \delta_{\alpha,\beta}$. The intersection of $\mathcal{A}_{\alpha,\alpha}$ and $\mathcal{B}_{\alpha,\alpha}$ is an one-dimensional space spanned by e_α.

- $\mathcal{B}_{\alpha,\beta}\mathcal{L}_{\beta',\gamma} = 0$ for $\beta \ne \beta'$ and $\mathcal{B}_{\alpha,\beta}\mathcal{L}_{\beta,\gamma} \subset \mathcal{L}_{\alpha,\gamma}$. Similarly $\mathcal{L}_{\alpha,\beta}\mathcal{A}_{\beta',\gamma} = 0$ for $\beta \ne \beta'$ and $\mathcal{L}_{\alpha,\beta}\mathcal{A}_{\beta,\gamma} \subset \mathcal{L}_{\alpha,\gamma}$. In particular, $\mathcal{A}_{\alpha,\beta}\mathcal{A}_{\beta',\gamma} = \mathcal{B}_{\alpha,\beta}\mathcal{B}_{\beta',\gamma} = 0$ for $\beta \ne \beta'$ and $\mathcal{A}_{\alpha,\beta}\mathcal{A}_{\beta,\gamma} \subset \mathcal{A}_{\alpha,\gamma}$, $\mathcal{B}_{\alpha,\beta}\mathcal{B}_{\beta,\gamma} \subset \mathcal{B}_{\alpha,\gamma}$.

- $\mathcal{L}_{\alpha,\beta} \perp \mathcal{L}_{\beta',\alpha'}$ if $\alpha \neq \alpha'$ or $\beta \neq \beta'$.

It follows from these properties that $(\cdot,\cdot)$ gives non-degenerate pairing between $\mathcal{A}_{\alpha,\beta}$ and $\mathcal{B}_{\beta,\alpha}$ for $\alpha \neq \beta$ and between $\mathcal{A}_{\alpha,\alpha}/\mathbb{C}e_\alpha$ and $\mathcal{B}_{\alpha,\alpha}/\mathbb{C}e_\alpha$. Therefore $\dim \mathcal{A}_{\alpha,\beta} = \dim \mathcal{B}_{\beta,\alpha}$.

Definition. By weak $\mathcal{PM}$-algebra associated with a weak $\mathcal{PM}$-structure $\mathcal{L}$ we mean an associative algebra $U(\mathcal{L})$ possessing the following properties:

1. $\mathcal{L} \subset U(\mathcal{L})$ and the actions $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$, $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ are the restrictions of the product in $U(\mathcal{L})$.

2. For any algebra X with the property **1** there exists a unique homomorphism of algebras $X \to U(\mathcal{L})$ identical on $\mathcal{L}$.

It is clear that $U(\mathcal{L}) = \oplus_{1\leq\alpha,\beta\leq m} U(\mathcal{L})_{\alpha,\beta}$, where $U(\mathcal{L})_{\alpha,\beta} = \{v \in U(\mathcal{L}); e_\alpha v = v e_\beta = v\}$. We have $U(\mathcal{L})_{\alpha,\beta} U(\mathcal{L})_{\beta',\gamma} = 0$ for $\beta \neq \beta'$ and $U(\mathcal{L})_{\alpha,\beta} U(\mathcal{L})_{\beta,\gamma} \subset U(\mathcal{L})_{\alpha,\gamma}$.

Let $\{e_\alpha, A_{i,\alpha,\alpha}; 1 \leq i \leq p_{\alpha,\alpha}\}$ be a basis of $\mathcal{A}_{\alpha,\alpha}$ and $\{e_\alpha, B^i_{\alpha,\alpha}; 1 \leq i \leq p_{\alpha,\alpha}\}$ be the dual basis of $\mathcal{B}_{\alpha,\alpha}$. Let $\{A_{i,\alpha,\beta}; 1 \leq i \leq p_{\beta,\alpha}\}$ be a basis of $A_{\alpha,\beta}$ for $\alpha \neq \beta$ and $\{B^i_{\beta,\alpha}; 1 \leq i \leq p_{\beta,\alpha}\}$ be the dual basis of $B_{\beta,\alpha}$.

Take $C_\alpha \in \mathcal{L}_{\alpha,\alpha}$ that does not belong to the sum of $\mathcal{A}_{\alpha,\alpha}$ and $\mathcal{B}_{\alpha,\alpha}$. Without loss of generality we can assume that $(e_\alpha, C_\alpha) = 1$, $(C_\alpha, C_\alpha) = (C_\alpha, A_{i,\alpha,\alpha}) = (C_\alpha, B^j_{\alpha,\alpha}) = 0$. Such element C_α is uniquely determined by choosing of basis in $\mathcal{A}_{\alpha,\alpha}$.

Let us define the element $K_\alpha \in U(\mathcal{L})$ by the formula $K_\alpha = C_\alpha + \sum_{1 \leq \nu \leq m} A_{i,\alpha,\nu} B^i_{\nu,\alpha}$.

Definition. A weak $\mathcal{PM}$-structure $\mathcal{L}$ is called $\mathcal{PM}$-structure if $K = \sum_{1 \leq \alpha \leq m} K_\alpha \in U(\mathcal{L})$ is a central element of the algebra $U(\mathcal{L})$.

Theorem 3.4. A basis of $U(\mathcal{L})_{\alpha,\beta}$ for $\alpha \neq \beta$ consists of the elements

$$\{A_{i,\alpha,\beta}K_{\beta}^{s},\ \ B_{\alpha,\beta}^{j}K_{\beta}^{s},\ \ A_{i_1,\alpha,\nu}B_{\nu,\beta}^{j_1}K_{\beta}^{s}\},$$

where $1 \leq i \leq p_{\beta,\alpha},\ 1 \leq j \leq p_{\alpha,\beta},\ 1 \leq \alpha,\beta,\nu \leq m, 1 \leq i_1 \leq p_{\nu,\alpha},\ 1 \leq j_1 \leq p_{\nu,\beta},\ s = 0,1,2,\ldots$.

A basis of $U(\mathcal{L})_{\alpha,\alpha}$ consists of the elements

$$\{e_{\alpha},\ \ A_{i,\alpha,\alpha}K_{\alpha}^{s},\ \ B_{\alpha,\alpha}^{j}K_{\alpha}^{s},\ \ A_{i_1,\alpha,\nu}B_{\nu,\alpha}^{j_1}K_{\alpha}^{s}\},$$

where $1 \leq i,j \leq p_{\alpha,\alpha},\ 1 \leq \nu \leq m,\ 1 \leq i_1, j_1 \leq p_{\nu,\alpha},\ s = 0,1,2,\ldots$.

Theorem. Let $\mathcal{L}$ be a $\mathcal{PM}$-structure. Then for any representation of $U(\mathcal{L})$ given by $A_{i,\beta,\alpha} \quad \rightarrow \quad a_{i,\beta,\alpha}, \quad B^i_{\alpha,\beta} \quad \rightarrow \quad b^i_{\alpha,\beta}, \quad C_\alpha \quad \rightarrow \quad c_\alpha;$ $1 \le i \le p_{\alpha,\beta}, \;\; 1 \le \alpha, \beta \le m$ the formula

$$x_\alpha \circ y_\beta = a_{i,\beta,\alpha} x_\alpha b^i_{\alpha,\beta} y_\beta + x_\alpha a_{i,\alpha,\beta} y_\beta b^i_{\beta,\alpha}, \quad \alpha \neq \beta,$$

$$x_\alpha \circ y_\alpha = a_{i,\alpha,\alpha} x_\alpha b^i_{\alpha,\alpha} y_\alpha + x_\alpha a_{i,\alpha,\alpha} y_\alpha b^i_{\alpha,\alpha} -$$

$$a_{i,\alpha,\alpha} x_\alpha y_\alpha b^i_{\alpha,\alpha} + x_\alpha c_\alpha y_\alpha.$$

defines a product on $M = \oplus_{1 \le \alpha \le m} Mat_{n_\alpha}$ compatible with the usual one.

Lemma 2. If (5) is associative, then for some tensor t^i_j

$$\phi^s_{j,k}\psi^{l,i}_s = \phi^l_{s,k}\psi^{s,i}_j + \phi^i_{j,s}\psi^{l,s}_k + \delta^l_k t^i_j - \delta^i_j t^l_k - \delta^i_j \phi^l_{s,r}\psi^{r,s}_k,$$

and

$$b^i a_j = \psi^{k,i}_j\, a_k + \phi^i_{j,k}\, b^k + t^i_j + \delta^i_j\, c.$$

Lemma 3. If (5) is associative, then

$$b^i\, c = \lambda^{k,i} a_k - t^i_k\, b^k - \phi^i_{k,l}\psi^{l,k}_s\, b^s - \phi^i_{k,l}\lambda^{l,k},$$

$$c\, a_j = \mu_{j,k}\, b^k - t^k_j\, a_k - \phi^s_{k,l}\psi^{l,k}_j\, a_s - \mu_{k,l}\psi^{l,k}_j$$

and moreover

$$\phi^s_{j,k} t^i_s = \psi^{s,i}_j \mu_{s,k} + \phi^i_{j,s} t^s_k - \delta^i_j \psi^{s,r}_k \mu_{r,s},$$

$$\psi^{k,i}_s t^s_j = \phi^i_{j,s}\lambda^{k,s} + \psi^{s,i}_j t^k_s - \delta^i_j \phi^k_{s,r}\lambda^{r,s}.$$

Proposition 1. The algebra $U(\mathcal{L})$ is defined by the following relations

$$A_iA_j = \phi^k_{i,j}A_k + \mu_{i,j}, \qquad B^iB^j = \psi^{i,j}_k B^k + \lambda^{i,j}$$

$$B^iA_j = \psi^{k,i}_j A_k + \phi^i_{j,k}B^k + t^i_j + \delta^i_j C,$$

$$B^iC = \lambda^{k,i}A_k + u^i_k B^k + p^i, \quad CA_j = \mu_{j,k}B^k + u^k_j A_k + q_i$$

for certain tensors $\phi^k_{i,j}, \psi^{i,j}_k, \mu_{i,j}, \lambda^{i,j}, u^i_k, p^i, q_i$.

Lemma. $\mathcal{L}$ as $\mathcal{A}$-module is isomorphic to $\oplus_{1\le i\le r}(V_i^\star)^{2m_i}$.

Definition. The $r \times s$ matrix $A = (a_{i,j})$ is called matrix of multiplicities of a weak $\mathcal{M}$-structure.

Definition. The matrix A is called decomposable if there exist partitions $\{1, ..., r\} = I \sqcup I'$ and $\{1, ..., s\} = J \sqcup J'$ such that $a_{i,j} = 0$ for $(i, j) \in I \times J' \sqcup I' \times J$.

Lemma. If A is decomposable, then the corresponding $\mathcal{M}$-structure is decomposable.