The B. and M. Shapiro Conjecture in real algebraic geometry and the Bethe ansatz

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Shapiro's Conjecture in Real Algebraic Geometry†

(Joint work with Mukhin and Tarasov)

§1 Statement of the result

Fix $r \geq 2$.

Let $V \subset \mathbb{C}[x]$ be a complex vector space, $\dim V = r$.

Def: V is real if it has a basis consisting of polynomials in $\mathbb{R}[x]$

Let $f_1, \ldots, f_r \in V$ be a basis

Def: Wronskian: $W(f_1, \ldots, f_r) = \begin{vmatrix} f_1^{(r-1)} & \cdots & f_1 \\ \vdots & \ddots & \vdots \\ f_r^{(r-1)} & \cdots & f_r \end{vmatrix}$

The Wronskian does not depend on the choice of basis, up to mult. by a constant. The monic representative will be called the Wronskian of V, denoted Wr_V.

If V is real, then Wr_V has real coefficients

Question Is converse true?

† Notes taken by I. Strachan, who takes responsibility for any errors!

Counterexample: $Wr(x^3+3ix^2, x-i) = \underbrace{2x(x^2+3)}_{\text{real coefficients (but complex roots)}}$

B. and M. Shapiro Conjecture: If all roots of Wr_V are real, then V is real.

(formulated in early 90's. Proved, for 2-dim. spaces V by Eremenko and Gabrielov, using delicate tools from classical complex analysis. Ann. of Math. 2002)

Varchenko, Mukhin, Tarasov: arXiv: math/0512299. Proof for spaces of arbitrary dim. using methods from quantum integrable systems.

§2 Parametrized rational curves with real ramification points

Consider $\mathbb{C}P^{r-1}$ with projective coordinates $(v_1 : \cdots : v_r)$.

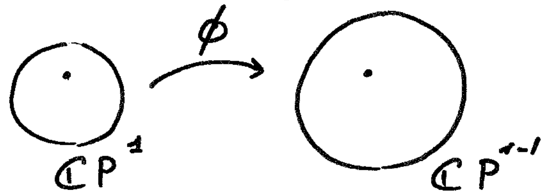
Then $\mathbb{R}P^{r-1} = \{(v_1 : \cdots : v_r) \mid v_i \in \mathbb{R}\}$ is called the real projective space

The real projective space depends on the choice of the projective coordinates

(E.g. $\mathbb{C}P^1 \supset \mathbb{R}P^1$ is a circle, S^1 in S^2: lots of such subspaces)

A parametrized rational curve in $\mathbb{C}P^{r-1}$ is a polynomial map

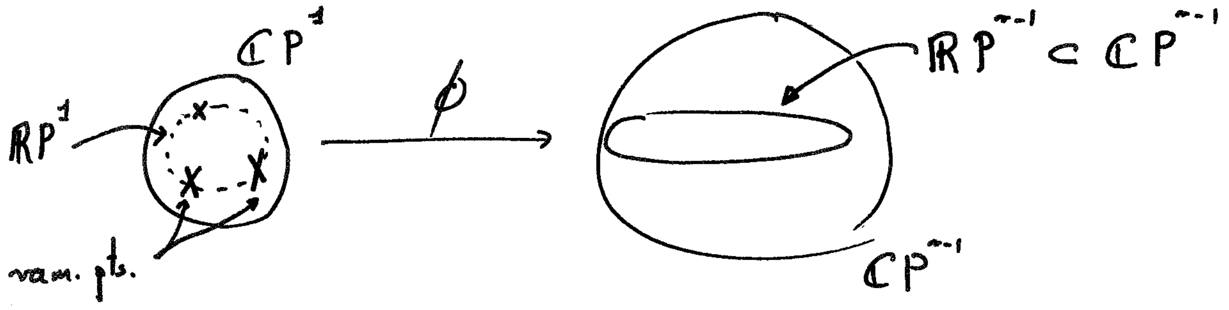
$$\phi : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^{r-1}$$



locally: $\quad u \longmapsto (g_1(u), \dots, g_{r-1}(u)) = g(u)$

<u>Def:</u> A point in $\mathbb{C}P^1$ is a <u>ramification point</u> of ϕ if the vectors $g'(u), g''(u), \dots, g^{(r-1)}(u)$ are linearly dependent at this point.

<u>Cor. of Shapiro's conjecture</u> If all ramification points of ϕ lie on a circle in $\mathbb{C}P^1$, then ϕ maps the circle into a suitable real subspace $\mathbb{R}P^{r-1} \subset \mathbb{C}P^{r-1}$.



§3 <u>Proof</u>

It is enough to prove the special case of Shapiro's conjecture:

<u>Thm 1</u> Assume that all roots of Wr_V are real and <u>simple</u>.

Then V is real.

§4 The number of (complex) vector spaces of polynomials with given Wronskian

Let $z_1, \dots, z_n \in \mathbb{C}$ be distinct

$$T(x) = \prod_{s=1}^{n} (x - z_s)$$

<u>Question</u> What is the number of r-dim. subspaces $V \subset \mathbb{C}[x]$ with $Wr_V = T$?

The answer is given by Schubert calculus.

The Lie algebra $\mathfrak{gl}_r$ acts naturally on $\mathbb{C}^r$ and hence on

$$(\mathbb{C}^r)^{\otimes n} = \underbrace{\mathbb{C}^r \otimes \cdots \otimes \mathbb{C}^r}_{n\text{-factors}} \quad ; \quad \dim (\mathbb{C}^r)^{\otimes n} = r^n .$$

Decompose $(\mathbb{C}^r)^{\otimes n}$ into irreps of $\mathfrak{gl}_r$:

$$(\mathbb{C}^r)^{\otimes n} = \bigoplus_i Q_i \qquad (*)$$

Let $\boxed{N_n = \# \text{ of irreps in } (*)}$ ⟵ important number.

<u>Claim</u> For all distinct $z_1, \dots, z_n$:

$$\# \{ V \subset \mathbb{C}[x] : \dim V = r, \ Wr_V = T \} \leq N_n$$

Moreover, for generic $z_1, \dots, z_n$, $\# \{ \dots \} = N_n$

To prove Th. 1 it is enough:

For generic real $z_1, \dots, z_n$, to construct exactly N_n real spaces $V \subset \mathbb{C}[x]$ with $\dim V = r$, $Wr_V = T$.

To do this we will consider the Gaudin model on $(\mathbb{C}^r)^{\otimes n}$ and the Bethe ansatz for that model. Before this, encode real spaces of polynomials in terms of o.d.e.'s.

§5 Spaces of polynomials and differential equations

<u>Lemma</u> Let $V \subset \mathbb{C}[x]$ be a vector space of dim r. Then

a) $\exists!$ linear operator

$$D = \left(\frac{d}{dx}\right)^r + \lambda_1(x)\left(\frac{d}{dx}\right)^{r-1} + \dots + \lambda_r(x) \qquad (\dagger)$$

s.t. $\ker D = V$. Moreover $\lambda_1(x), \dots, \lambda_r(x)$ are rational functions.

b) V is real iff $\lambda_1(x), \dots, \lambda_r(x)$ are <u>real</u> rational functions.

<u>Proof</u> a) Let $f_1, \dots, f_r \in V$ be a basis. Consider the diff. eq$^{\underline{n}}$. w.r.t. the unknown function $u(x)$:

$$\begin{vmatrix} u^{(r)} & u^{(r-1)} & \cdots & u \\ f_1^{(r)} & f_1^{(r-1)} & \cdots & f_1 \\ \vdots & & \ddots & \vdots \\ f_r^{(r)} & \cdots & & f_r \end{vmatrix} = 0$$

Clearly, any linear combination of $f_1 \cdots f_r$ is a solution of this diff. eq$^{\underline{n}}$. Expanding gives:

$$W_r(f_1, \ldots, f_r)\, u^{(r)} + \underline{\quad}\, u^{(r-1)} + \cdots + \underline{\quad}\, u = 0,$$

where all coefficients are polynomials in x. Dividing by W we get eq$^{\underline{n}}$ $(\dagger)$.

b) If $f_1, \ldots, f_r$ are real then $\lambda_1(x), \ldots, \lambda_r(x)$ are real.

It is easy to see that if we know that all solutions of the differential equation are polynomial and all coefficients $\lambda_i(x)$ are real rational functions then there is a basis of solutions consisting of real polynomials.

<u>Cor</u>. To determine if V is real or not, it is enough to check if $\lambda_i(x)$ are real or not.

§6 Generators of the Lie algebra gl_r.

$$E_{ij} = \begin{array}{cc} \left(\begin{array}{ccc} & | & \\ \text{---} & \boxed{1} & \text{---} \\ & | & \end{array} \right) & i \\ j & \end{array} \qquad ; \; i,j = 1, \ldots, r.$$

These matrices act on $\mathbb{C}^r$ in the standard way.

For $s = 1, \ldots, n$, define

$$E_{ij}^{(s)} : (\mathbb{C}^r)^{\otimes n} \longrightarrow (\mathbb{C}^r)^{\otimes n} \quad \text{by}$$

$$E_{ij}^{(s)} = 1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ s^{th} \text{ place}}}{E_{ij}} \otimes 1 \cdots \otimes 1$$

§7 The Gaudin model on $(\mathbb{C}^r)^{\otimes n}$

This is a family of commuting linear operators on $(\mathbb{C}^r)^{\otimes n}$.

To construct the family we do the following:

Let $z_1, \ldots z_n \in \mathbb{C}$.

Define $$X_{ij} = \delta_{ij} \frac{d}{dx} - \sum_{s=1}^{n} \frac{E_{ji}^{(s)}}{x - z_s} \quad ; \; i,j = 1, \ldots, r.$$

This is a diff. op. of order 1 $(i=j)$ or order 0 $(i \neq j)$ acting on $(\mathbb{C}^r)^{\otimes n}$-valued functions of x.

Define
$$K = \sum_{\sigma \in S_r} (-1)^{|\sigma|} X_{1\sigma_1} \cdots X_{r\sigma_r}$$

This is a differential operator of order r.

<u>Example</u>: $r=2$

$$K = \left(\frac{d}{dx} - \sum_{s=1}^{n} \frac{E_{11}^{(s)}}{x - z_s}\right)\left(\frac{d}{dx} - \sum_{s=1}^{n} \frac{E_{22}^{(s)}}{x - z_s}\right) - \left(\sum_{s=1}^{n} \frac{E_{21}^{(s)}}{x - z_s}\right)\left(\sum_{s=1}^{n} \frac{E_{12}^{(s)}}{x - z_s}\right)$$

Write
$$K = \left(\frac{d}{dx}\right)^{r} + K_1(x)\left(\frac{d}{dx}\right)^{r-1} + \cdots + K_{r-1}(x)\left(\frac{d}{dx}\right) + K_r(x)$$

where $K_i(x) : (\mathbb{C}^r)^{\otimes n} \longrightarrow (\mathbb{C}^r)^{\otimes n}$ are linear operators, dependent on x, called the <u>Gaudin Hamiltonians</u> or <u>transfer matrices</u>

<u>Properties</u> 1) $[K_i(u), K_j(v)] = 0 \quad \forall u, v; i, j.$

2) $K_i(x)$, $i = 1, \ldots, r$ commute with the $\mathfrak{gl}_r$-action on $(\mathbb{C}^r)^{\otimes n}$

Thus to $z_1, \ldots, z_n$ we assign a commutative subalgebra of $\mathrm{End}(\mathbb{C}^r)^{\otimes n}$ generated by the linear operators

$$\{ K_1(x), \ldots, K_r(x) \mid x \in \mathbb{C} \}$$

Problem: Find common eigenvectors and eigenvalues of the Hamiltonians

The Bethe ansatz method is a method to construct eigenvectors of the Hamiltonians.

While constructing eigenvectors by the Bethe ansatz method the following was obtained:

Thm [MTV] All solutions of the dif. eq: $KF = 0$ are $(\mathbb{C}^r)^{\otimes n}$ - valued polynomials.

- this important result indicates a connection with algebraic geometry
- the question is how to get from this theorem the r-dimensional spaces of scalar polynomials which we are interested in.

Remark Assume that $v \in (\mathbb{C}^r)^{\otimes n}$ is an eigenvector,

$$K_i(x)\, v = \lambda_i(x)\, v \qquad ; \quad i = 1, \dots, r,$$

for suitable scalar functions $\lambda_1(x), \dots, \lambda_r(x)$. Let us look for a solution of $KF(x) = 0$ in the form $F = f(x)\, v$, where f is a scalar function. Then f must lie in the kernel of the scalar diff. operator

$$D_v = \left(\frac{d}{dx}\right)^r + \lambda_1(x)\left(\frac{d}{dx}\right)^{r-1} + \cdots + \lambda_r(x)$$

Thus with every eigenvector v we associate a scalar diff. operator D_v. By Thm 2 [MTV] its kernel is an r-dim. space of polynomials.

<u>Main Results</u>

<u>Thm 3</u> [MTV] For generic $z_1, \ldots, z_n \in \mathbb{C}$:

- $\exists$ N_n eigenvectors of the Gaudin Hamiltonians $v_1, \ldots, v_{N_n}$ s.t. $\forall i \neq j$ $\exists$ l s.t. $H_l(x)$ has different eigenvalues on v_i and v_j;

- For any i, let V_i be the kernel of D_{v_i}. Then V_i is an r-dim space of polynomials with $Wr_{V_i} = \prod_{s=1}^{n}(x - z_s)$

<u>Cor</u> $V_1, \ldots, V_{N_n}$ are all possible distinct spaces of polynomials with Wronskian $\prod_{s=1}^{n}(x - z_s)$.

<u>Question</u> How to get reality?

Th$\underline{m}$ 4 [MTV]

If $z_1, \dots, z_n \in \mathbb{R}$, and are generic, then all $V_1, \dots, V_{N_n}$ are real spaces.

Proof If $z_1, \dots, z_n \in \mathbb{R}$, then all Hamiltonians $K_i(x)$ are real linear operators on $(\mathbb{R}^r)^{\otimes n} \subset (\mathbb{C}^r)^{\otimes n}$.

They are symmetric w.r.t. the standard scalar product $\langle , \rangle$ on this space, i.e. $\langle K_i(x)\, v, w \rangle = \langle v, K_i(x)\, w \rangle$.

Hence they have real eigenvalues, hence D_{V_i} have real coefficients and hence V_i are real spaces.

The correspondence between the eigenvectors v_i of the Hamiltonians and the differential operators D_{V_i} with polynomial kernel is in the spirit of the Geometric Langlands correspondence.

§ 8 Another form of Shapiro Conjecture

Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be distinct numbers.
Let $\mu_1, \dots \mu_n \in \mathbb{C}$
Consider the $n \times n$ - matrix $M = (m_{ij})$, where $m_{ij} = \begin{cases} \dfrac{1}{\lambda_i - \lambda_j} & , i \neq j, \\ \mu_i & , i = j. \end{cases}$

<u>Thm [MTV]</u> If M is nilpotent, $M^n = 0$, then $\mu_1, \dots \mu_n \in \mathbb{R}$.

<u>Proof</u> $(n=2)$. $M = \begin{pmatrix} \mu_1 & \frac{1}{\lambda_1 - \lambda_2} \\ \frac{1}{\lambda_2 - \lambda_1} & \mu_2 \end{pmatrix}$.

$$M^2 = 0 \quad \text{iff} \quad \begin{cases} \mu_1 + \mu_2 = 0 \\ \mu_1 \mu_2 + \dfrac{1}{(\lambda - \lambda_2)^2} = 0 \end{cases}$$

$$\Rightarrow \quad \mu_1^2 = \frac{1}{(\lambda_1 - \lambda_2)^2} > 0.$$

In general, proof uses ideas from integrable systems.