Polynomial solutions of Knizhnik-Zamolodchikov equations and Schur–Weyl duality

A.P. Veselov

Joint work with **G. Felder**

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KZ equations with values in S_N-modules

$$\partial_i \psi = m \sum_{j \neq i}^{N} \frac{s_{ij} + 1}{z_i - z_j} \psi, \quad i = 1, \dots, N$$

Here $\psi(z)$ takes values in an irreducible representation W^λ of the symmetric group S_N with Young diagram λ and s_{ij} is the action of the corresponding elementary transposition in W^λ.

We will assume that m is a positive integer, then all the solutions are polynomial (**Opdam; Felder-V.**) of degree equal to the value of the central element

$$C = m \sum_{i<j} (s_{ij} + 1) = m \sum_{i<j} s_{ij} + m \frac{N(N-1)}{2}$$

in the irreducible representation W^λ. Our aim is an explicit integral formula for these solutions.

Schur-Weyl duality

Let V be an n-dimensional complex vector space.

The classical **Schur–Weyl theorem** states that, as a $GL(V) \times S_N$ module, $V^{\otimes N}$ has a decomposition into a direct sum

$$V^{\otimes N} \cong \oplus_\lambda M^\lambda \otimes W^\lambda$$

where M^λ are inequivalent irreducible $GL(V)$-modules and W^λ are inequivalent irreducible S_N-modules. The sum is over partitions λ of N into at most n parts, which are sequences of integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ with $\sum \lambda_i = N$. If $n \geq N$ all irreducible S_N modules appear.

From this one can realise $W^\lambda = (V^{\otimes N})_\lambda^{\mathsf{n}_+}$ as the set of primitive vectors of weight λ. This gives the following basis of W^λ labeled by standard Young tableaux.

Recall that a *standard tableau* on λ is a numbering $T\colon \lambda \to \{1, \ldots, N\}$ of the boxes of λ, which is increasing from left to right and from top to bottom.

Let λ be a Young diagram with N boxes with rows of lengths $\lambda_1, \ldots, \lambda_m$. To each numbering T we associate a vector $e_T = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_N} \in V^{\otimes N}$ so that $\alpha_k = i$ whenever $T^{-1}(k)$ is in the ith row. For example, if T is the numbering

$$\begin{array}{|c|c|c|}
\hline
3 & 4 & 1 \\
\hline
2 & 5 \\
\cline{1-2}
6 \\
\cline{1-1}
\end{array}$$

then $e_T = e_1 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_3$.

Then the claim is that the vectors

$$v_T = \sum_{\sigma \in C(T)} \mathsf{sign}(\sigma) e_{\sigma T},$$

where T runs over the set $\mathcal{T}(\lambda)$ of standard tableaux on λ, form a basis of the S_N-module $W^\lambda = (V^{\otimes N})_\lambda^{\mathfrak{n}_+}$.

Configuration space C_λ and integration cycles

For given $\lambda = (\lambda_1, \ldots, \lambda_n)$ define the integers m_i from the relation

$\lambda = (m_0 - m_1, m_1 - m_2, \ldots, m_{n-2} - m_{n-1}, m_{n-1}):$

m_s is the number of boxes in the rows of λ strictly lower than s.

Consider n sets $X_0, X_1, \ldots, X_{n-1}$ of points on the complex plane $\mathbb{C}$ consisting of $m_0, \ldots, m_{n-1}$ points respectively with the condition that X_i and X_{i+1} have no common points. Denote the elements of X_0 as $z_1, \ldots, z_N$ and fix them. The set of all admissible $\{X_1, \ldots, X_{n-1}\}$ is our *configuration space* $C_\lambda(z_1, \ldots, z_N)$. Let

$$X_s = \{t_s^b \in \mathbb{C}, b \in \lambda,\, r(b) > s\},$$

then

$$C_\lambda = \{t_s^b \in \mathbb{C}, b \in \lambda : \; t_{s+1}^b \neq t_s^{b'},\, t_1^b \neq z_k\}.$$

On C_λ we have a natural action of the group $G_\lambda = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_{n-1}}$. The *integration cycles* $\sigma_T, T \in \mathcal{T}(\lambda)$ in the top homology group $H_{top}(C_\lambda(z_1, \ldots, z_N))$ are defined as follows.

Consider first the product Γ_T of circles consecutively surrounding anti-clockwise z_k with the variables $t^b_1, \ldots, t^b_{r(b)-1}$, $b = T^{-1}(k)$ located on these circles:

$$\Gamma_T = \{t^b_s \in \mathbb{C} : |t^b_s - z_k| = \epsilon s,\, b = T^{-1}(k)\}$$

for any real positive ϵ small enough.

The cycle σ_T is the skew-symmetrisation of Γ_T by the action of G_λ:

$$\sigma_T = \sum_{g \in G_\lambda} (-1)^g g_*(\Gamma_T).$$

The formula

Define the form ω_T as

$$\omega_T = \frac{1}{(2\pi i)^{d_\lambda}} \Phi_\lambda^m \phi_T dt,$$

where

$$\Phi_\lambda = \Delta(z) \prod_{s,b\neq b'} (t_s^b - t_s^{b'})^2 \prod_{s,b,b'} \frac{1}{(t_{s+1}^b - t_s^{b'})} \prod_{k,b} \frac{1}{(t_1^b - z_k)},$$

$$\phi_T = \prod_{s,b} (t_{s+1}^b - t_s^b)^{-1} \prod_b (t_1^b - z_{T(b)})^{-1}$$

$\Delta(z) = \prod_{i<j}^N (z_i - z_j)^2$ and $dt = \prod_{s,b} dt_s^b$.

Theorem. *A fundamental set of solutions of the KZ equation with values in S_N-module W^λ has a form*

$$\psi_T(z_1, \dots, z_N) = \sum_{T' \in \mathcal{T}(\lambda)} \psi_{T,T'}(z_1, \dots, z_N) v_{T'}$$

where

$$\psi_{T,T'} = \int_{\sigma_T} \omega_{T'}.$$

This integral can be effectively computed as an iterated residue and gives a polynomial in $z_1, \ldots, z_N$ with **integer** coefficients.

The proof is based on Schur-Weyl duality and the results of **Matsuo**, who found some integral formulas for the solutions of the original $SU(n)$ KZ equation inspired by **Zamolodchikov-Fateev** and **Christe-Flume**.

In the asymptotic region $0 \ll |z_1| \ll \cdots \ll |z_N|$ we have

$$\psi_{T,T}(z) \sim C \prod_{b \in \lambda} z_{T(b)}^{m(T(b)-1+c(b)-r(b))} + \cdots$$

Corollary (Frobenius) *The value $f_2(\lambda)$ of the central element $\sum_{i<j} s_{ij}$ in the representation W^λ can be computed as*

$$f_2(\lambda) = \sum_{b \in \lambda} (c(b) - r(b)).$$

Example. In the simplest non-trivial example when $N = 3$ and $\lambda = (2, 1)$, which corresponds to the usual two-dimensional representation of S_3. In this case there are two standard tableaux:

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \qquad S = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

The corresponding primitive vectors are $v_T = \epsilon_3 - \epsilon_1$, $v_S = \epsilon_2 - \epsilon_1$.

The residues can be computed explicitly to give:

$$\psi_1 = z_{23}^{2m} \sum_{k=0}^{m} d_{m,k} \left((m-k)v_T + kv_S\right) z_{12}^{m-k} z_{13}^{k},$$

$$\psi_2 = z_{13}^{2m} \sum_{k=0}^{m} (-1)^{m-k} d_{m,k} \left((m-k)v_T - mv_S\right) z_{12}^{m-k} z_{2}^{k}$$

where $z_{ij} = z_i - z_j$ and

$$d_{m,k} = -\frac{1}{m}\binom{-m}{k}\binom{-m}{m-k}.$$

Duality $m \leftrightarrow -m$ and intersection pairing

To apply our results to negative m we can use the following isomorphism between the space of solutions

$$KZ(V,m) \approx KZ(V \otimes \mathsf{Alt}, -m)$$

of the KZ equation with values in the representations V and $V \otimes \mathsf{Alt}$, where $\mathsf{Alt} = \mathbb{C}\epsilon$ is the alternating representation:

If $\psi \in KZ(V,m)$ then $\phi = \prod_{i>j}(z_i - z_j)^{-2m}\psi \otimes \epsilon \in KZ(V \otimes \mathsf{Alt}, -m)$.

In particular it follows that for negative m all solutions are rational functions.

It is well-known that the involution $V \to V \otimes \mathsf{Alt}$ corresponds to the *transposition* of the Young diagram $\lambda \to \lambda'$, so we have shown that

$$KZ(\lambda, m) \approx KZ(\lambda', -m).$$

It turns out that there is a link between the spaces of KZ solutions with the *same* Young diagram:

$$j : KZ(\lambda, m) \approx KZ(\lambda, -m)^*.$$

More precisely, there exists a natural pairing

$$KZ(V, m) \times KZ(V^*, -m) \to \mathbb{C},$$

where V^* is the dual space to V : for any two solutions $\psi \in KZ(V, m)$ and $\phi \in KZ(V^*, -m)$ the product $\langle \psi(z_1, \ldots, z_N), \phi(z_1, \ldots, z_N) \rangle$ is independent of $z_1, \ldots, z_N$ and thus defines a pairing.

A fundamental matrix for $KZ(\lambda, -m)$ is

$$\Phi_{\lambda,-m}(z_1, \ldots, z_N) = (\Phi_{\lambda,m}(z_1, \ldots, z_N)^{-1})^T$$

and the determinant has the form

$$\det \Phi_{\lambda,m}(z_1, \ldots, z_N) = C \prod_{i<j} (z_i - z_j)^{2md_+(\lambda)},$$

where $C = C(\lambda, m)$ is a non-zero constant and $d_+(\lambda) = \dim W_+^\lambda$ is the dimension of the fixed subspace of reflection s_{ij} acting in the representation W^λ.

We now give the topological interpretation of this duality in the special case of the standard $(N-1)$-dimensional representation of S_N, corresponding to $\lambda = (N-1, 1)$.

For positive m our integral formula gives

$$\psi_a = \Delta(z)^m \,\mathrm{res}_{t=z_a} \prod_{i=1}^{N} (t - z_i)^{-m} \sum_{b=1}^{N} \frac{1}{t - z_b} \epsilon_b \, dt$$

with the relation $\psi_1 + \cdots + \psi_N = 0$.

For the space $KZ(\lambda, -m)$ with positive m there is a different integral representation (**Felder - V.**):

$$\phi_a = \Delta(z)^{-m} \int_{z_a}^{z_N} \prod_{i=1}^{N} (t - z_i)^{m} \sum_{b=1}^{N} \frac{1}{t - z_b} \epsilon_b \, dt$$

give a basis in $KZ(\lambda, -m)$.

Thus we have two maps

$$H_1(\mathbb{C} \setminus \{z_1, \ldots, z_N\}) \to W^\lambda,$$

$$H_1(\mathbb{C}, \{z_1, \ldots, z_N\}) \to W^\lambda,$$

sending horizontal sections for the Gauss–Manin connection to solutions in $KZ(\lambda, m)$ and $KZ(\lambda, -m)$, respectively.

Theorem. *The intersection pairing*

$$H_1(\mathbb{C} \setminus \{z_1, \ldots, z_N\}) \times H_1(\mathbb{C}, \{z_1, \ldots, z_N\}) \to \mathbb{Z},$$

is proportional to the pairing

$$KZ(V, m) \times KZ(V^*, -m) \to \mathbb{C}.$$

More precisely,

$$\langle \psi_\sigma(z_1, \ldots, z_N), \phi_\tau(z_1, \ldots, z_N) \rangle = C_N \frac{1}{m} (\sigma \cdot \tau),$$

$$\sigma \in H_1(\mathbb{C} \setminus \{z_1, \ldots, z_N\}), \quad \tau \in H_1(\mathbb{C}, \{z_1, \ldots, z_N\}),$$

for some constant $C_N \neq 0$ depending on the normalization of the isomorphism $(W^\lambda)^* \to W^\lambda$.

Some open problems

Intersection pairing interpretation of duality for an arbitrary representation W^λ

Generalisation to the quantum KZ equation and possible combinatorial links (**Razumov and Stroganov, Di Francesco and P. Zinn-Justin**)

Large m limit and new approach to representation theory of symmetric group (**Vershik and Okounkov**)

Relations with representation theory of Cherednik algebras (**Berest and Chalykh**)

Reference

G. Felder and A.P. Veselov, math.RT/0610383
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