Integrable lattice maps: Q_5 a rational version of Q_4

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Parametrization of integrable of lattice maps

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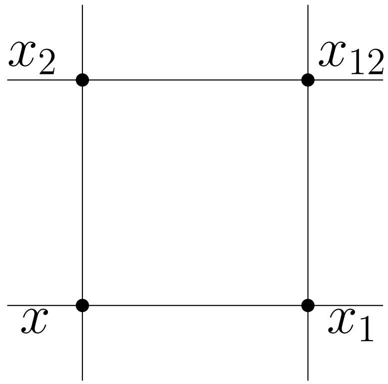
The setting

On a lattice, we define a relation which will induce an evolution. It relates the values of some field around an elementary cell. One of the simplest cases is obtained with a two-dimensional square lattice and a multilinear relation.



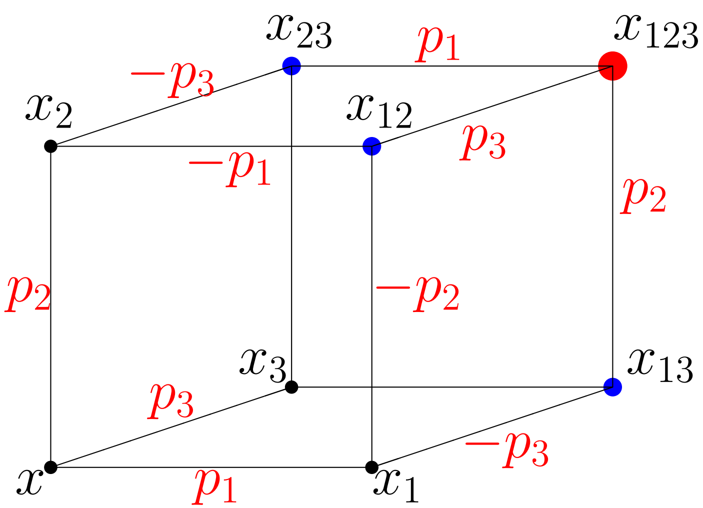
The typical relation is multilinear:

$$
\begin{gathered}
Q = q_1 \cdot x\, x_1\, x_2\, x_{12} + q_2 \cdot x\, x_1\, x_2 + q_3 \cdot x\, x_1 x_{12} + q_4 \cdot x_1\, x_2\, x_{12} + q_5 \cdot x\, x_2\, x_{12} \\
+ q_6 \cdot x\, x_2 + q_7 \cdot x_1\, x_2 + q_8 \cdot x_2\, x_{12} + q_9 \cdot x\, x_1 + p_{10} \cdot x\, x_{12} + q_{11} \cdot x_1\, x_{12} \\
+ q_{12} \cdot x_2 + q_{13} \cdot x + q_{14} \cdot x_1 + q_{15} \cdot x_{12} + q_{16} = 0
\end{gathered}
$$

so that any of the four corner values can be rationally expressed in terms of the three others. We will be interested in global properties of the evolutions defined by this infinitesimal relation, as well as local constraints (like consistency around the cube or factorization properties). $\hookleftarrow$

Integrability: Lax pair and consistency around the cube (CAC)

Consider the archetypal case of discrete mKdV: $p_1\ (x\ x_1 - x_2\ x_{12}) + p_2\ (x\ x_2 - x_1\ x_{12}) = 0$
It is possible to embed the two-dimensional cell into a three-dimensional one:



where one imposes a similar relation to all faces (the same for opposite faces).

$$p_i\ (x\ x_i - x_j\ x_{ij}) + p_j\ (x\ x_j - x_i\ x_{ij}) = 0, \qquad i, j = 1, 2, 3$$

The higher dimensional system is compatible, i.e. the value of x_{123} is independent of the way it is calculated. This is called consistency around the cube (CAC).

The major output of CAC is that it ensures the existence of a Lax pair, which is accepted as a proof of integrability[1]. $\hookleftarrow$

[1] This however relies on the specific form of relation $Q = 0$

Consistency around the cube: Q4

While the defining plaquette relation is written on one cell, and is thus infinitesimal, the CAC relation is written on a loop of cells, and is a local relation.

The way it is written associates the parameters to bonds rather than to faces (this implies to spot the spectral parameters, which may not be easy).

It is a very constraining equation, and is not easy to manipulate: if one takes the most general form of the defining relation $Q = 0$, the expressions of x_{123} get quite difficult to handle, they are big.

We will be interested in the generic solution of CAC, i.e. the Adler solution. Its form has been improved by Nijhoff, and by Hietarinta. It was shown to be the generic solution of CAC by Adler-Bobenko-Suris.

The solution is called Q_4. There are different avatars of it, linked in particular to different parametrizations.

$$
\begin{gathered}
k_0\, x\, x_1\, x_2\, x_{12} - k_1(x\, x_1\, x_2 + x_1\, x_2\, x_{12} + x\, x_2\, x_{12} + x\, x_1\, x_{12}) + k_2(x\, x_{12} + x_1\, x_2) \\
-k_3(x\, x_1 + x_2\, x_{12}) - k_4(x\, x_2 + x_1\, x_{12}) + k_5(x + x_1 + x_2 + x_{12}) + k_6 = 0
\end{gathered}
$$

with $k_0 = \alpha + \beta, \quad k_1 = \alpha\nu + \beta\mu, \quad k_2 = \alpha\nu^2 + \beta\mu^2, \quad k_5 = \frac{g_3}{2}k_0 + \frac{g_2}{4}k_1, \quad k_6 = \frac{g_2^2}{16}k_0 + g_3 k_1,$

$$
k_3 = \tfrac{\alpha\beta(\alpha+\beta)}{2(\nu-\mu)} - \alpha\nu^2 + \beta(2\mu^2 - \tfrac{g_2}{4}), \qquad k_4 = \tfrac{\alpha\beta(\alpha+\beta)}{2(\mu-\nu)} - \beta\mu^2 + \alpha(2\nu^2 - \tfrac{g_2}{4}).
$$

and $\alpha^2 = r(\mu), \qquad \beta^2 = r(\nu), \qquad r(z) = 4\, z^3 - g_2\, z - g_3$

$$
\begin{gathered}
A\,((x-b)\,(x_2-b)-d)\,((x_1-b)\,(x_{12}-b)-d) \\
+B\,((x-a)\,(x_1-a)-e)\,((x_2-a)\,(x_{12}-a)-e) = f
\end{gathered}
$$

where (a, A), (b, B), $(c, C) = (b, B) - (a, A)$ on the curve $Z^2 = r(z)$,

and $\qquad d = (a-b)\,(c-b) \qquad e = (b-a)\,(c-a), \qquad f = A\;B\;C\,(a-b)$

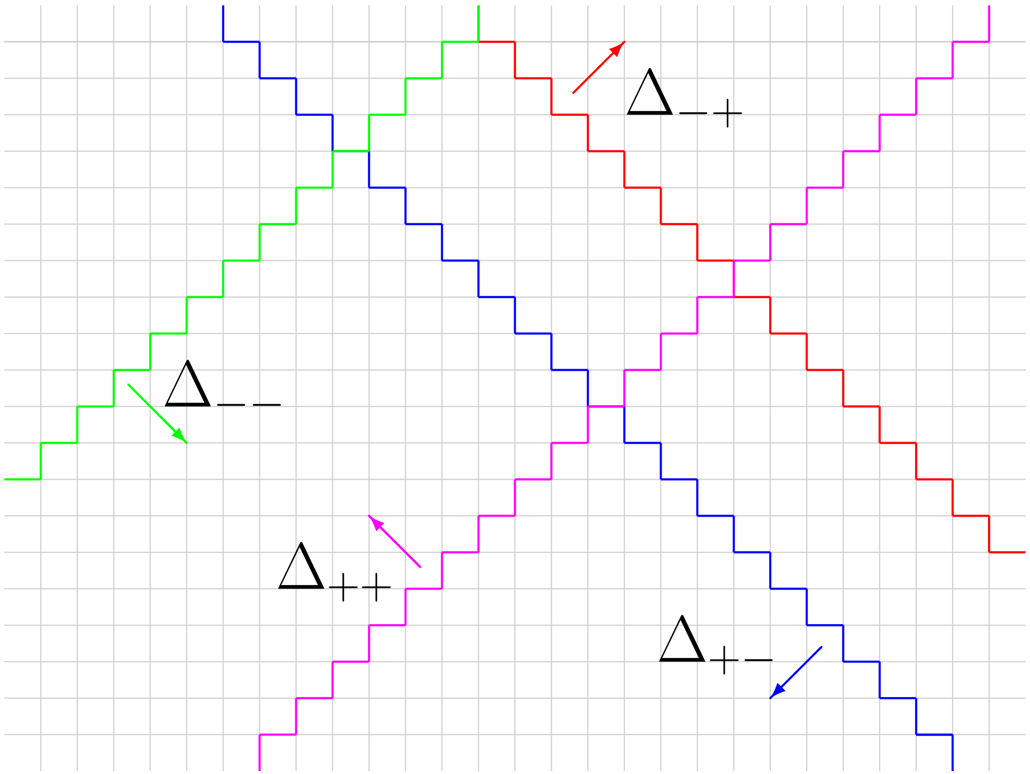
$$
\begin{gathered}
sn(\alpha)\; sn(\beta)\; sn(\alpha+\beta)(k^2\, x\, x_1\, x_2\, x_{12} + 1) + sn(\alpha+\beta)(x\, x_{12} + x_1\, x_2) \\
-sn(\alpha)(x\, x_1 + x_2\, x_{12}) - sn(\beta)(x\, x_2 + x_1\, x_{12}) = 0
\end{gathered}
$$

What we will see is that there is another interesting one. To see that, we will use the notion of algebraic entropy.

$\hookleftarrow$

Integrability: Algebraic entropy

Given a lattice map on a plane square lattice, we may define four fundamental evolutions, corresponding to initial data given on diagonals with slope $+1$ or -1, and evolutions towards the four corners of the lattice:



Fundamental evolutions on a square lattice

$\hookleftarrow$

Algebraic entropy of the lattice map

The space of initial data is infinite dimensional. Initial data are given on a line which must allow the determination of the values at all points of the lattice. The simplest possible choice is to take a regular diagonal staircase going diagonally.

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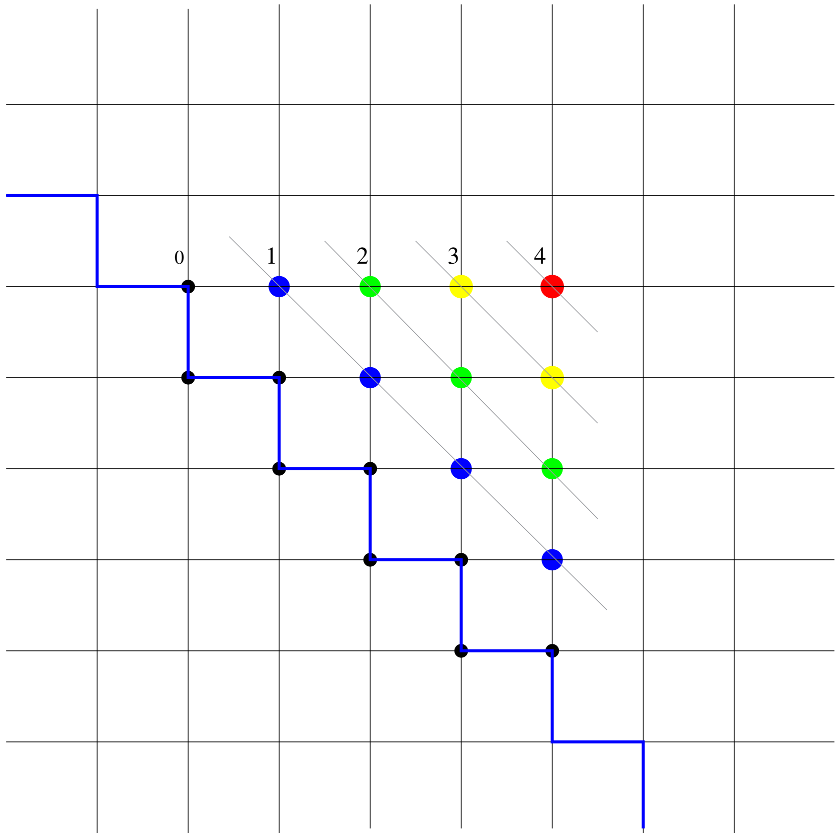
We may iterate the lattice map by calculating the values on diagonals moving away from the initial staircase, and define a sequence of degrees d_n in terms of the initial data.



Definition

We define the entropy by

$$\epsilon = \lim_{n\to\infty} \frac{1}{n} \, log(d_n).$$

Theorem: The limit defined above exists.

The reason is the same as for maps (subadditivity of the $log(d_n)$).

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When the entropy vanishes, the growth of the degree is polynomial, and the degree of that polynomial is a secondary characterization of the complexity.

The outcome of our numerous experiments, as well as what we know for maps leads to the claim that integrability of the lattice map is equivalent to the vanishing of its entropy. (see arXiv:math-ph/060943)

$\hookleftarrow$

Q_5

Apply this calculation to Q_4. The most general form having the same symmetries:

$$a_1\, x\, x_1\, x_2\, x_{12} + a_2\, (x\, x_2\, x_{12} + x_1\, x_2\, x_{12} + x\, x_1\, x_{12} + x\, x_1\, x_2)$$
$$+ a_3\, (x\, x_1 + x_2\, x_{12}) + a_4\, (x\, x_{12} + x_1\, x_2) + a_5\, (x_1\, x_{12} + x\, x_2)$$
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For our entropy analysis, is much better to have integer coefficients. We can find integer coefficient verifying the conditions fulfilled by $\{a_1, \ldots, a_7\}$ (of the type we saw before : they are constructed from points on an elliptic curve).

We can take the curve

$$Z^2 = 4\, z^3 - 32\, z + 4$$

and the points $(a, A) = (0, 2), \quad (c, C) = (3, 4), \quad (b, B) = (a, A) \oplus (c, C) = (-26/9, -2/27)$

We get the sequence $\{d_n\} = \{1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, \ldots\}$, that is to say the quadratic growth

$$d_n = 1 + n\, (n - 1)$$

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With arbitrary values of the parameters, we get the same quadratic growth as with constrained values:

$$\{d_n\} = \{1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, \ldots\}$$

fitted with

$$g(s) = \sum_{n=0}^{\infty} d_n \, s^n = \frac{1+s^2}{(1-s)^3}, \qquad \text{and } d_n = 1 + n\,(n-1)$$

This indicates integrability of the unconstrained form, with 7 free homogeneous parameters (intersection of hyperplanes in the space of multilinear relations).

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Remark: the sequence of degrees verifies a finite recursion relation

$$d_n = 3\, d_{n-1} - 3\, d_{n-2} + d_{n-3}$$

This means that the global behaviour of the sequence degrees is dictated by a local condition.

Q_4 vs Q_5

We may wonder about the number of parameters, but it seems the count is right.

$$\begin{aligned} 16 \;-\; 1 - 4 \times 3 &= \;3 \\ 7 \;-\; 1 - \;3 &= \;3 \\ 1 \;+\; 1 \;+\; 1 &= \;3 \end{aligned}$$

The general multilinear relations has 16 homogeneous parameters.

The group of homographies is $SL(2)$ and has 3 parameters.

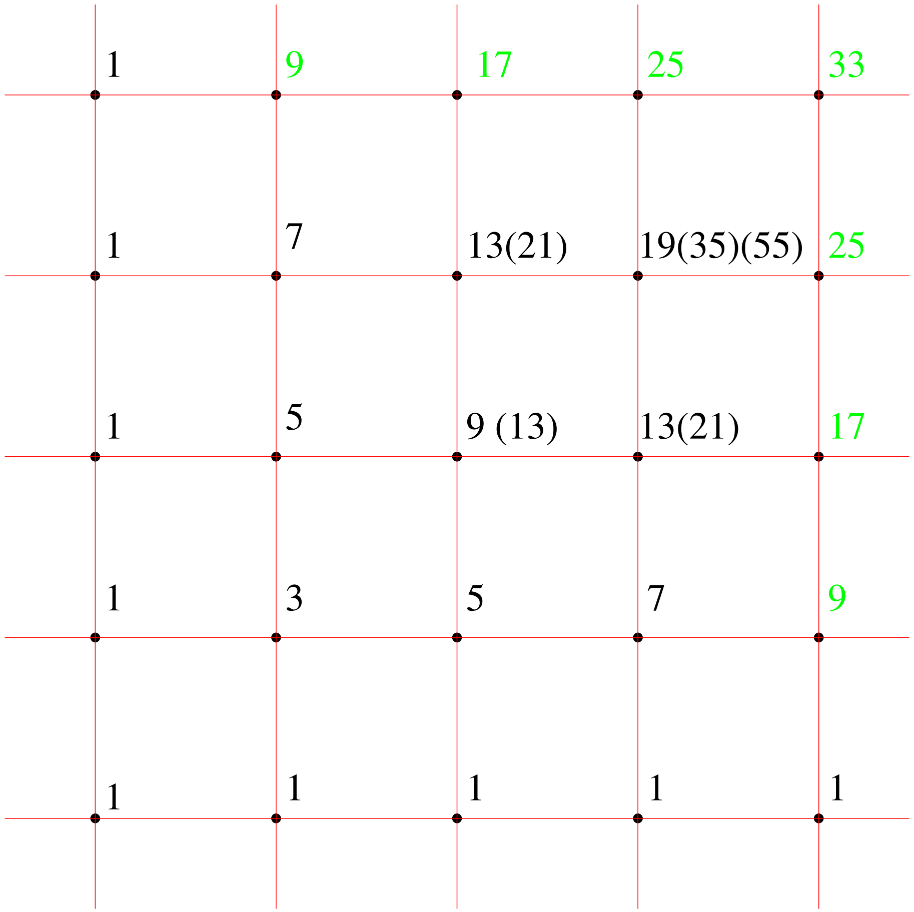
The CAC relation, seen as a condition on one face is acted upon by 4 copies of $SL(2)$.

Integrability is preserved only if the four corners of a plaquette are acted upon with the same element of $SL(2)$.

$\hookleftarrow$

Factorization

Another configuration is interesting: the first quadrant of the lattice

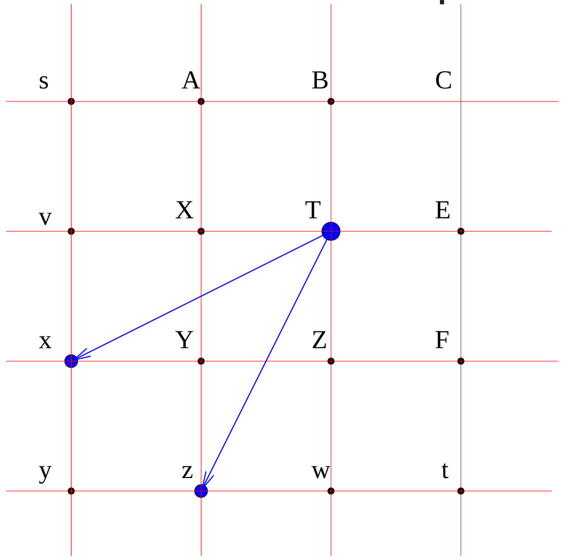


$$d_{ij} = 1 + 2\; i\; j$$

The diagonal degree growth is quadratic ($d_n = 1 + 2\; n^2$) = integrability

$\hookleftarrow$

What makes the degree drop is the factorization process. Consider the corner



and calculate X, Y, Z, T for generic Q.

$$\deg(Y) = 1 + 1 + 1 = 3, \qquad \deg(X) = \deg(Z) = \deg(Y) + 1 + 1 = 5$$
$$\deg(T) = \deg(X) + \deg(Y) + \deg(Z) = 13$$

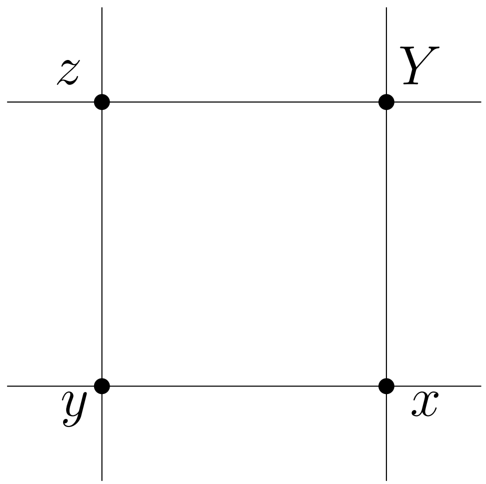
but for Q_5 there is a factorization

$$T = \frac{H(x,z)\; P(x,y,z,u,v)}{H(x,z)\; Q(x,y,z,u,v)} \simeq \frac{P}{Q}$$

$$\deg(T) = \deg(X) + \deg(Y) + \deg(Z) - \deg(H) = 13 - 4 = 9$$

The factor $H(x,z)$ is the bi-quadratic which appears in the singularity analysis.

Suppose we look at the elementary plaquette



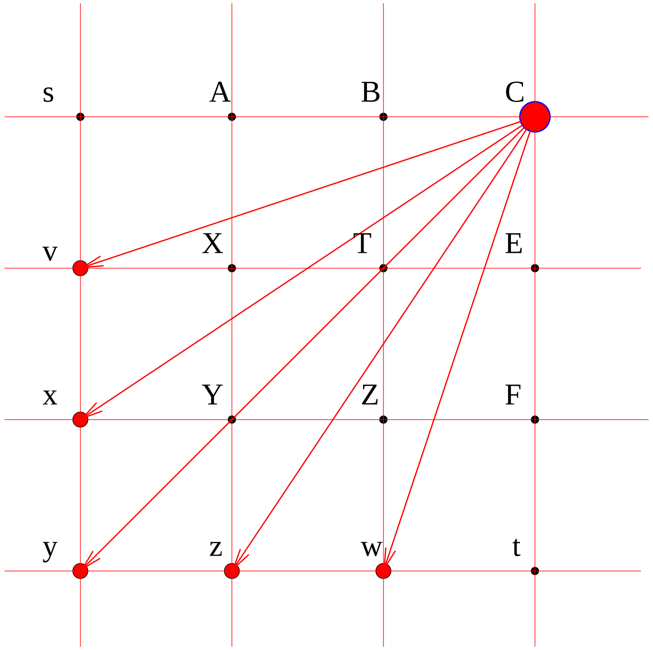
The relation Q give a projective linear map $\varphi_{xz} : y \longrightarrow Y$, whose inverse φ^{-1} is projective linear. The composed map $\varphi \cdot \varphi^{-1}$ is proportional to the biquadratic $H(x, z)$.

$$\begin{gathered} H(x,z) = (q_{16}q_{10} - q_{15}q_{13}) + (-q_8q_6 + q_{12}q_5)\,x^2 + (q_7q_3 - q_2q_{11} - q_9q_4 + q_{14}q_1)\,z^2x \\ + (-q_6q_4 - q_2q_8 + q_7q_5 + q_{12}q_1)\,x^2z + (-q_4q_2 + q_7q_1)\,x^2z^2 + (-q_{11}q_9 + q_{14}q_3)\,z^2 \\ + (-q_2q_{15} - q_6q_{11} + q_7q_{10} - q_9q_8 + q_{12}q_3 + q_{16}q_1 - q_{13}q_4 + q_{14}q_5)\,xz \\ + (q_{16}q_3 - q_{13}q_{11} + q_{14}q_{10} - q_9q_{15})\,z + (q_{16}q_5 + q_{12}q_{10} - q_{13}q_8 - q_6q_{15})\,x \end{gathered}$$

In the case of Q_5 the drop at d_{22} is $13 - 9 = 4$. What factorizes from the iterate is precisely equation of the bi-quadratic $H(x, z)$.

The elliptic curve of the known forms of Q_4 is lurking there.

But this does not account for the whole process, and higher degree curves appear at later steps (total degree 16, degree 4 in x, y, z, and bi-quadratic in v, w).



What may however happen is that, due to the specific form of the relation Q, it sufficient to ensure that the first factorization happens to have them all.

$\hookleftarrow$

This is spirit of the systematic analysis performed by JH last year, for quadratic relations, and with the additional hypothesis that factors are made out of linear pieces (we know we will not find Q_4 this way).

This produced 80 a priori different models. We have run an algebraic entropy test over those, and finally came out with a short list of integrable cases,not all in the Adler-Bobenko-Suris list, and a list of models with non-vanishing entropy.

The non-vanishing values of the entropy we got range from $\log((1+\sqrt{2})/2)$ to $\log(1+\sqrt{2})$. When the entropy vanishes, the growth was either linear either quadratic (J Hietarinta + CMV, arXiv:0705.1903).

Again some local structure (extending over a finite range of elementary cells) ensures a global property (integrability), as may be seen form the existence of a finite recurrence relation on the degrees.

$\hookleftarrow$

Conclusion

- The three levels infinitesimal/local/global appear in the discrete world. In the setting we use, which is strongly constrained (multilinearity of the elementary relation, birationality of the evolution), a local property is good enough to ensure integrability.
- One should clarify what is the group of coordinate transformations we want to take into account.
- About the rationality vs elliptic nature of the parametrization, the phenomenon is apparently the same as the one we saw (J Hietarinta+CMV, q-alg/9504028) for the celebrated Baxters solution of the Yang-Baxter equations. There exists a rational form of Baxter's R-matrix. It is gauge equivalent to the usual elliptic form, which reappears when one request a symmetric form of the solution .
- This phenomenon invites us to examine again the “Yang-Baxter maps” constructed from lattice maps.
- Finally Q_5 will be useful if one wants to look at the possible “de-autonomisations” of Q_4.