# Symmetry Structures of integrable hierarchies

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**Def.** A partial differential equation is called integrable if it possess an infinite hierarchy of symmetries.

**Ultimate goal:** *To give a description of all systems of the form*

$$\mathbf{u}_t = L(D_x)\mathbf{u} + F_2[\mathbf{u}] + F_3[\mathbf{u}] + \cdots ,$$

*which possess infinite hierarchy of symmetries.*

What is known:

- Mikhailov, Shabat and Yamilov ('85–'87): second order systems of the form

$$\mathbf{u}_t = \mathbf{A}(\mathbf{u})\mathbf{u}_{xx} + F(\mathbf{u}, \mathbf{u}_x), \quad A(\mathbf{u}) = \mathrm{diag}(a,\ b), \quad a = -1, \quad b = 1.$$

- Sokolov and Svinolupov ('89): studied the Burgers and KdV type equations, i.e. the case $a = b = 1$.

- Mikhailov, Shabat and Sokolov ('91): the symmetry approach

* Sokolov and Wolf ('99): systems of second order

* Foursov and Olver ('00): symmetrically-coupled evolutionary equations

* Tsuchida and Wolf ('05): systems of mixed scalar and vector dependent variables

# Beukers, Sanders and Wang ('98,'01): systems of Bakirov-type

# Sanders and Wang ('01): systems of two equations of second order

# Van der Kamp ('03): Systems of generalised Bakirov-type

$$\mathbf{u}_t = \Lambda \mathbf{u}_n + \mathbf{F}(\mathbf{u}_{n-1}, \ldots, \mathbf{u})$$

Our aims are

- Formulate simple and effective test for integrability
- To derive necessary conditions of existence of infinite hierarchies of higher symmetries
- To determine admissible $\Lambda$ and admissible dispersion relations for higher symmetries
- Classification of integrable systems of fixed order
- Global classification (i.e. in all orders)

**Two component systems**

We consider systems of two evolutionary equations of the form

$$\left\{ \begin{array}{l} u_t = \lambda_1 u_n + K_1(u_{n-1}, v_{n-1}, \ldots, u, v) \\ v_t = \lambda_2 v_n + K_2(u_{n-1}, v_{n-1}, \ldots, u, v) \end{array} \right. , \quad n \in 2\mathbb{N}+1,\ \lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}.$$

We assume that $\lambda_1 \neq \lambda_2$. We also assume that $K_1, K_2 = o(\mathcal{R}^1)$.

In the symbolic representation the system takes the form

$$\left\{ \begin{array}{l} u_t = \lambda_1 u \xi_1^n + f_1 \\ v_t = \lambda_2 v \zeta_1^n + f_2, \end{array} \right.$$

where

$$f_i = \sum_{s \geq 2} \sum_{j+k=s} u^j v^k a_{j,k}^i(\xi_1, \ldots, \xi_j, \zeta_1, \ldots, \zeta_k), \quad i = 1, 2$$

## Symmetries and approximate symmetries

Since $\lambda_1 \neq \lambda_2$, we shall seek the higher symmetries of the system without loss of generality in the form

$$\begin{cases} u_\tau = \mu_1 u_m + R_1(u_{m-1}, v_{m-1}, \ldots, u, v) \\ v_\tau = \mu_2 v_m + R_2(u_{m-1}, v_{m-1}, \ldots, u, v). \end{cases}$$

We shall assume that $m \in 2\mathbb{N} + 1$.

In the symbolic representation the symmetry takes the form

$$\begin{cases} u_\tau = \mu_1 u \xi_1^m + e_1, \\ v_\tau = \mu_2 v \zeta_1^m + e_2, \end{cases}$$

where

$$e_i = \sum_{s \geq 2} \sum_{j+k=s} u^j v^k A^i_{j,k}(\xi_1, \ldots, \xi_j, \zeta_1, \ldots, \zeta_k), \quad i = 1, 2$$

**Prop.** The above expression is an approximate symmetry of degree 2 of the system only if functions $A^i_{jk}$, $i = 1, 2$, $j + k = 2$, determined as follows, are polynomials in their arguments:

(Group 1):

$$A^1_{20} = \frac{\mu_1}{\lambda_1} \frac{(\xi_1 + \xi_2)^m - \xi_1^m - \xi_2^m}{(\xi_1 + \xi_2)^n - \xi_1^n - \xi_2^n} a^1_{20}, \quad A^2_{02} = \frac{\mu_2}{\lambda_2} \frac{(\xi_1 + \xi_2)^m - \xi_1^m - \xi_2^m}{(\xi_1 + \xi_2)^n - \xi_1^n - \xi_2^n} a^2_{02},$$

(Group 2):

$$A^1_{11} = \frac{\mu_1(\xi_1 + \xi_2)^m - \mu_1\xi_1^m - \mu_2\xi_2^m}{\lambda_1(\xi_1 + \xi_2)^n - \lambda_1\xi_1^n - \lambda_2\xi_2^n} a^1_{11}, \quad A^2_{20} = \frac{\mu_2(\xi_1 + \xi_2)^m - \mu_1\xi_1^m - \mu_1\xi_2^m}{\lambda_2(\xi_1 + \xi_2)^n - \lambda_1\xi_1^n - \lambda_1\xi_2^n} a^2_{02},$$

(Group 3):

$$A^1_{02} = \frac{\mu_1(\xi_1 + \xi_2)^m - \mu_2(\xi_1^m + \xi_2^m)}{\lambda_1(\xi_1 + \xi_2)^n - \lambda_2(\xi_1^n + \xi_2^n)} a^1_{02}, \quad A^2_{11} = \frac{\mu_2(\xi_1 + \xi_2)^m - \mu_1\xi_1^m - \mu_2\xi_2^m}{\lambda_2(\xi_1 + \xi_2)^n - \lambda_1\xi_1^n - \lambda_2\xi_2^n} a^2_{11}$$

**Prop.** Let $G(m,\mu_1,\mu_2) = \mu_1(x+y)^m - \mu_2(x^m + y^m)$. If there exists infinite many pairs of $(m,\mu_1,\mu_2)$ such that the defined functions have nontrivial common divisor, the greatest common divisors are

**Case I**. $(x+y)(x-qy)(y-qx)$ if $m = 2k+1, k \in \mathbb{N}$, and

$$\frac{\mu_2}{\mu_1} = \frac{(1+q)^m}{1+q^m}, \quad q \in \mathbb{C} \setminus \{0,-1\}.$$

**Case II**. $(x+y)(x-qy)^2(y-qx)^2$ if $q$ is a primitive root of unity of order $l$ and $m = 1 \bmod l$ if $l$ is even and $m = 1 \bmod 2l$ if $l$ is odd and

$$\frac{\mu_2}{\mu_1} = (1+q)^{m-1}.$$

**Case III**. $(x+y)(x-qy)(y-qx)(x-sy)(y-sx)$ if

$$q = \alpha\frac{\beta-1}{\alpha-1}, \;\; s = \beta^{-1}\frac{\beta-1}{\alpha-1}$$

$\alpha, \beta$ are primitive roots of unity of orders $l_1, l_2 \in 2\mathbb{N}+1$, such that $\alpha \neq \beta, \beta^{-1}$ and $m = (1+2k)\mathrm{lcm}(l_1,l_2)$ and $\frac{\mu_2}{\mu_1} = \frac{(\alpha\beta-1)^m}{(\alpha-1)^m+(\beta-1)^m}$.

## Degenerate dispersion relations

Suppose that there are infinitely many both $G(2k+1, \mu_1, \mu_2)$ and $G(2k+1, \mu_2, \mu_1)$ with nontrivial common factor (not $(x+y)$), where $\mu_1 \neq \mu_2$. Then $\frac{\mu_2}{\mu_1} = \frac{(1+q)^m}{(1+q^m)}$, where

- $q = e^{\frac{2\pi i}{5}}$ and $m = 1, 3, 7, 9 \mod 10$, that is, $\frac{\mu_2}{\mu_1} = -\frac{7+3\sqrt{5}}{2}$; $-\frac{47+21\sqrt{5}}{2}$; $\frac{47+21\sqrt{5}}{2}$; $\frac{123+55\sqrt{5}}{2}$....

- $q = e^{\frac{\pi i}{6}}$ and $m = 1, 5, 7, 11 \mod 12$, that is, $\frac{\mu_2}{\mu_1} = 26 + 15\sqrt{3}$; $-97 - 56\sqrt{3}$; $-362 - 209\sqrt{3}$; $-1351 - 780\sqrt{3}$.....

**Example** 3rd order systems of the Korteweg-de Vries type.

Consider

$$\begin{cases} u_t = \lambda_1 u_3 + 2d_1 uu_1 + d_2 vu_1 + d_3 uv_1 + 2d_4 vv_1, \\ v_t = \lambda_2 v_3 + 2d_5 uu_1 + d_6 vu_1 + d_7 uv_1 + 2d_8 vv_1. \end{cases}$$

Here $d_i \in \mathbb{C}$, $i = 1, \ldots, 8$ and at least one of these constants does not vanish.

**Prop.** If the system possesses an infinite hierarchy of approximate symmetries of degree 2 of odd order then up to re-scaling and up to change $u \leftrightarrow v$ it is one of the following:

$$\begin{cases} u_t = (5 - 3\sqrt{5})u_3 + d_1 uu_1 + d_2 vu_1 + d_3 uv_1 + d_4 vv_1, \\ v_t = (5 + 3\sqrt{5})v_3 + d_5 uu_1 + d_6 vu_1 + d_7 uv_1 + d_8 vv_1, \end{cases}$$

$$\begin{cases} u_t = u_3 + 2d_1 uu_1 + 2d_2 vv_1, \\ v_t = \lambda v_3 + d_3 vu_1 + d_4 uv_1 + 2d_5 vv_1, \qquad \lambda = \frac{(1+q)^3}{1+q^3},\ q \in \mathbb{C} \setminus \{0, -1\} \end{cases}$$

The first system of the list corresponds to the case $\mu_2(m)/\mu_1(m) = (1 + e^{\frac{2\pi i}{5}})^m/(1 + e^{\frac{2m\pi i}{5}}),\ m = 1, 3, 7, 9 \bmod 10$ and $\lambda_{1,2} = \mu_{1,2}(3)$, so we can choose $\lambda_{1,2} = 5 \mp 3\sqrt{5}$.

**Prop.** If the system possess an infinite hierarchy of approximate symmetries of degree 3 of odd order and at least one of the constants $d_1, \ldots, d_8$ does not vanish then up to re-scaling and up to change $u \leftrightarrow v$ it is one of the following:

$$\begin{cases} u_t = (5-3\sqrt{5})u_3 - 2uu_1 + (3-\sqrt{5})vu_1 + 2uv_1 + (1+\sqrt{5})vv_1, \\ v_t = (5+3\sqrt{5})v_3 - 2vv_1 + (3+\sqrt{5})uv_1 + 2vu_1 + (1-\sqrt{5})uu_1, \end{cases}$$

$$\begin{cases} u_t = u_3 + \alpha uu_1 + \beta vv_1, \\ v_t = -2v_3 - \alpha uv_1, \end{cases}$$

$$\begin{cases} u_t = u_3 + 2uu_1, \\ v_t = 4v_3 + vu_1 + 2uv_1, \end{cases}$$

$$\begin{cases} u_t = u_3 + vv_1, \\ v_t = \lambda v_3, \qquad \lambda \in \mathbb{C} \setminus \{0,1\}, \end{cases}$$

$$\begin{cases} u_t = u_3 + \alpha uu_1, \\ v_t = \lambda v_3 + \beta vv_1, \qquad \lambda \in \mathbb{C} \setminus \{0,1\}, \end{cases}$$

$$\begin{cases} u_t = u_3 + \alpha uu_1 + \beta vv_1, \\ v_t = -2v_3 - \alpha(uv_1 + vu_1), \end{cases}$$

**Prop.** If the system possess an infinite hierarchy of symmetries of odd orders and at least one of the constants $d_1, \dots, d_8$ does not vanish then up to re-scaling and up to change $u \leftrightarrow v$ it is one of the following:

$$\begin{cases} u_t = (5-3\sqrt{5})u_3 - 2uu_1 + (3-\sqrt{5})vu_1 + 2uv_1 + (1+\sqrt{5})vv_1, \\ v_t = (5+3\sqrt{5})v_3 - 2vv_1 + (3+\sqrt{5})uv_1 + 2vu_1 + (1-\sqrt{5})uu_1, \end{cases}$$

$$\begin{cases} u_t = u_3 + \alpha uu_1 + \beta vv_1, \\ v_t = -2v_3 - \alpha uv_1, \end{cases}$$

$$\begin{cases} u_t = u_3 + 2uu_1, \\ v_t = 4v_3 + vu_1 + 2uv_1, \end{cases}$$

$$\begin{cases} u_t = u_3 + vv_1, \\ v_t = \lambda v_3, \qquad \lambda \in \mathbb{C} \setminus \{0, 1\}, \end{cases}$$

$$\begin{cases} u_t = u_3 + \alpha uu_1, \\ v_t = \lambda v_3 + \beta vv_1, \qquad \lambda \in \mathbb{C} \setminus \{0, 1\}, \end{cases}$$

$$\begin{cases} u_t = u_3 + uu_1, \\ v_t = -2v_3 - uv_1 - vu_1, \end{cases}$$

## 5th order systems

The following systems possess infinite hierarchies of higher symmetries

$$
\begin{aligned}
u_t &= (9-5\sqrt{3})u_5 + D\left[2(9-5\sqrt{3})uu_2 + (-12+7\sqrt{3})u_1^2\right] \\
&\quad +2(3-\sqrt{3})u_3 v + 2(6-\sqrt{3})u_2 v_1 + 2(3-2\sqrt{3})u_1 v_2 - 6(1+\sqrt{3})uv_3 \\
&\quad +D\left[2(33+19\sqrt{3})vv_2 + (21+12\sqrt{3})v_1^2\right] + \frac{4}{5}(-12+7\sqrt{3})u^2 u_1 + \\
&\quad \frac{8}{5}(3-2\sqrt{3})\left[vuu_1 + u^2 v_1\right] + \frac{4}{5}(24+13\sqrt{3})v^2 u_1 + \frac{8}{5}(36+20\sqrt{3})uvv_1 \\
&\quad -\frac{8}{5}(45+26\sqrt{3})v^2 v_1, \\
v_t &= (9+5\sqrt{3})v_5 + D\left[2(33-19\sqrt{3})uu_2 + (21-12\sqrt{3})u_1^2\right] \\
&\quad -6(1-\sqrt{3})u_3 v + 2(3+2\sqrt{3})u_2 v_1 + 2(6+\sqrt{3})u_1 v_2 + 2(3+\sqrt{3})uv_3 \\
&\quad +D\left[2(9+5\sqrt{3})vv_2 - (12+7\sqrt{3})v_1^2\right] - \frac{8}{5}(45-26\sqrt{3})u^2 u_1 + \\
&\quad \frac{8}{5}(36-20\sqrt{3})vuu_1 + \frac{4}{5}(24-13\sqrt{3})u^2 v_1 + \frac{8}{5}(3+2\sqrt{3})\left[v^2 u_1 + uvv_1\right] \\
&\quad -\frac{4}{5}(12+7\sqrt{3})v^2 v_1
\end{aligned}
$$

This system corresponds to the case $\mu_2(m)/\mu_1(m) = \frac{(1+e^{\frac{\pi i}{6}})^m}{1+e^{\frac{m\pi i}{6}}}$ for $m = 1, 5, 7, 9, 11 \bmod 12$ and $\mu_{1,2}(5) = \lambda_{1,2},\ i = 1, 2$, so we can choose $\lambda_{1,2} = 9 \mp 5\sqrt{3}$.

$$
\begin{aligned}
u_t &= 15u_5 + 30u_1u_2 - 30u_3v - 45u_2v_1 - 35u_1v_2 - 10uv_3 - 6u^2u_1 \\
&\quad +6u^2v_1 + 12v^2u_1 + 12uvv_1, \\
v_t &= -\frac{5}{3}v_5 - 10uu_3 - 15u_1u_2 + 10vv_3 + 25v_1v_2 - 6u^2u_1 + 6u^2v_1 \\
&\quad +12uvu_1 - 12v^2v_1.
\end{aligned}
$$

This system possess a reduction $u = 0$ to Kaup-Kupershmidt equation.

$$
\begin{aligned}
u_t &= 15u_5 - 30u_3v - 45u_2v_1 - 35u_1v_2 - 10uv_3 + 6u^2u_1 + 12(v^2u_1 + uvv_1) \\
v_t &= -\frac{5}{3}v_5 - 10uu_3 - 15u_1u_2 + 10vv_3 + 25v_1v_2 + 6u^2v_1 + 12uvu_1 - 12v^2v
\end{aligned}
$$

This system also possess a reduction $u = 0$ to Kaup-Kupershmidt equation. It has a symmetry $u_\tau = u_3 - 2vu_1 - uv_1, \quad v_\tau = 3uu_1$. (Compare to Drinfeld-Sokolov systems and to Popowicz system).