Extended affine Weyl groups and Frobenius manifolds-II

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Main References

1. B. Dubrovin, Geometry of 2D topological field theories

Lecture Notes in Math. 1620 (Springer, Berlin, 1996) 120–384

2. Extended affine Weyl groups and Frobenius manifolds

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3. Extended affine Weyl groups and Frobenius manifolds-II

B.Dubrovin and Youjin Zhang and Dafeng Zuo

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Outline of this talk

§1. Some definitions [Ref.1]

§2. Frobenius manifolds and Extended affine Weyl groups [Ref.2]

§3. Our question and result [Ref.3]

§1. Some definitions

Definition 1.1 A Frobenius structure of charge d on M is the data $(M, \bullet, \langle , \rangle, e, E)$ satisfying

(i) $\eta := \langle\ ,\ \rangle$ is a flat pseudo-Riemannian metric;

(ii) $\bullet$ is $\mathbb{C}$-linear, associative, commutative product on T_mM which depends smoothly on m;

(iii) e is the unity vector field for the product $\bullet$ and $\nabla e = 0$;

(iv) $(\nabla_w c)(x, y, z)$ is symmetric, where $c(x, y, z) := \langle x \bullet y, z\rangle$;

(v) A linear vector field $E \in Vect(M)$ must be fixed on M, i.e. $\nabla\nabla E = 0$ such that

$$\mathcal{L}_E\langle\ ,\ \rangle = (2-d)\langle\ ,\ \rangle, \quad \mathcal{L}_E \bullet = \bullet, \quad \mathcal{L}_E\ e = -e.$$

Theorem.[B.Dubrovin 1992] There is a one to one correspondence between a Frobenius manifold and the solution $F(\mathbf{t})$ of WDVV equations of associativity

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\delta \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\alpha \partial t^\gamma},$$

with a quasihomogeneity condition

$$\mathcal{L}_E F = (3-d)F + \text{quadratic polynomial in t.}$$

Definition 1.2 A Frobenius manifold is called semisimple if the algebra $(T_m M, \bullet)$ are semisimple at generic m.

Definition 1.3 An Intersection form of Frobenius manifold is a symmetric bilinear form on the cotangent bundle T^*M defined by

$$(\omega_1, \omega_2)^* = i_E(\omega_1 \cdot \omega_2), \quad \omega_1, \omega_2 \in T^*M.$$

Here the multiplication law on the cotangent planes is defined using the isomorphism

$$\langle\ ,\ \rangle : TM \to T^*M.$$

The discriminant Σ is defined by

$$\Sigma = \{t |\det(\ ,\ )|_{T_t^*M} = 0\} \subset M.$$

Theorem.[B.Dubrovin 1992]

The metrics $\eta := \langle\ ,\ \rangle$ and $g := (\ ,\ )^*$ form a flat pencil on $M \setminus \Sigma$, i.e.,

1. *The metric $h^{\alpha\beta} = \eta^{\alpha\beta} + \lambda g^{\alpha\beta}$ is flat for arbitrary λ and*
2. *The Levi-Civita connection for the metric $h^{\alpha\beta}$ has the form*

$$\Gamma^{\alpha\beta}_{\delta_{(h)}} = \Gamma^{\alpha\beta}_{k_{(\eta)}} + \lambda \Gamma^{\alpha\beta}_{k_{(g)}},$$

where $\Gamma^{\alpha\beta}_{\delta_{(h)}} = -h^{\alpha\gamma}\Gamma^{\beta}_{\delta\gamma_{(h)}}$, $\Gamma^{\alpha\beta}_{\delta_{(g)}} = -g^{\alpha\gamma}\Gamma^{\beta}_{\delta\gamma_{(g)}}$, $\Gamma^{\alpha\beta}_{\delta_{(\eta)}} = -\eta^{\alpha\gamma}\Gamma^{\beta}_{\delta\gamma_{(\eta)}}$.

The holonomy of the local Euclidean structure defined on $M \setminus \Sigma$ by the intersection form $(\ ,\ )^*$ gives a representation

$$\mu : \pi_1(M \setminus \Sigma) \to \mathit{Isometries}(\mathbb{C}^n).$$

Definition 1.4 The group

$$W(M) := \mu(\pi_1(M \setminus \Sigma)) \subset \mathit{Isometries}(\mathbb{C}^n)$$

is called a monodromy group of Frobenius manifold.

$$\Longrightarrow \qquad M \setminus \Sigma = \Omega / W(M), \qquad \Omega \subset \mathbb{C}^n.$$

§2. Frobenius manifolds and Extended affine Weyl groups

Motivation. Quantum cohomology of $\mathbb{P}^1$:

$$F = \frac{1}{2}t_1^2 t_2 + e^{t_2}, E = t_1\partial_1 + 2\partial_2, e = \partial_1, W(M) = \widetilde{W}(A_1)$$

Question: How to construct this kind of Frobenius manifolds? That is,

$$F = F(t_1, \cdots, t_n, t_{n+1}, e^{t_{n+1}})$$

$$E = \sum_{\alpha=1}^{n} d_\alpha t_\alpha \partial_\alpha + d_{n+1}\partial_{n+1}$$

$$W(M) = \widetilde{W}^{(k)}(R)$$

Notations

Let R be an irreducible reduced root system defined on $(V, (\ ,\ ))$.

$\{\alpha_j\}$: a basis of simple roots, $\{\alpha_j^\vee\}$: the corresponding coroots.

W Weyl group, $W_a(R)$ affine Weyl group (the semi-direct product of W by the lattice of coroots)

$W_a(R) \curvearrowright V$: affine transformations

$$\mathbf{x} \mapsto w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee, \quad w \in W,\ m_j \in \mathbb{Z}.$$

ω_j: the fundamental weights, $(\omega_i, \alpha_j^\vee) = \delta_{ij}$

Definition.[B.Dubrovin, Y.Zhang 1998]

The extended affine Weyl group $\widetilde{W} = \widetilde{W}^{(k)}(R)$ acts on the extended space

$$\widetilde{V} = V \oplus \mathbb{R}$$

and is generated by the transformations

$$x = (\mathbf{x}, x_{l+1}) \mapsto (w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^{\vee},\ x_{l+1}), \quad w \in W,\ m_j \in \mathbb{Z},$$

and

$$x = (\mathbf{x}, x_{l+1}) \mapsto (\mathbf{x} + \gamma\, \omega_k,\ x_{l+1} - \gamma).$$

Here $\gamma = 1$ except for the cases when $R = B_l, k = l$ and $R = F_4, k = 3$ or $k = 4$, in these three cases $\gamma = 2$.

Definition.[B.Dubrovin, Y.Zhang 1998]

$\mathcal{A} = \mathcal{A}^{(k)}(R)$ is the ring of all $\widetilde{W}$-invariant Fourier polynomials of the form

$$\sum_{m_1,\dots,m_{l+1}\in\mathbb{Z}} a_{m_1,\dots,m_{l+1}} e^{2\pi i(m_1 x_1+\dots+m_l x_l+\frac{1}{f}\, m_{l+1}x_{l+1})}$$

that are bounded in the limit

$$\mathbf{x} = \mathbf{x}^0 - i\,\omega_k\tau, \quad x_{l+1} = x^0_{l+1} + i\,\tau, \quad \tau \to +\infty$$

for any $x^0 = (\mathbf{x}^0, x^0_{l+1})$, where f is the determinant of the Cartan matrix of the root system R.

We introduce a set of numbers

$$d_j = (\omega_j, \omega_k), \quad j = 1, \dots, l$$

and define the following Fourier polynomials

$$\tilde{y}_j(x) = e^{2\pi i d_j x_{l+1}}\, y_j(\mathbf{x}), \quad j = 1, \dots, l,$$

$$\tilde{y}_{l+1}(x) = e^{\frac{2\pi i}{\gamma} x_{l+1}}.$$

Here

$$y_j(\mathbf{x}) = \frac{1}{n_j} \sum_{w \in W} e^{2\pi i(\omega_j, w(\mathbf{x}))},$$

$$n_j = \#\{w \in W | e^{2\pi i(\omega_j, w(\mathbf{x}))} = e^{2\pi i(\omega_j, \mathbf{x})}\}.$$

B.Dubrovin and Y.Zhang considered a particular choice of α_k based on the following observations

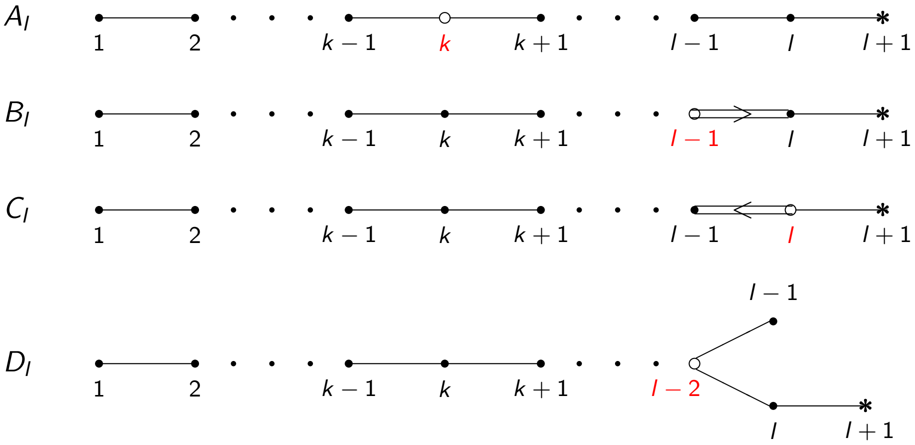
1. The Dynkin graph of $R_k := \{\alpha_1, \cdots, \hat{\alpha_k}, \cdot, \alpha_l\}$ (α_k is omitted) consists of 1, 2 or 3 branches of A_r type for some r.

2. $d_k > d_s, s \neq k$.

Chevalley-Type Theorem [B.Dubrovin, Y.Zhang 1998]

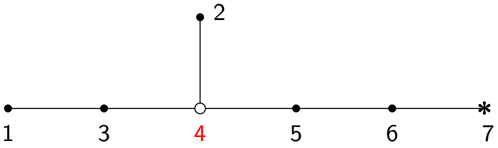
For the above particluar choice of α_k*,*

$$\mathcal{A}^{(k)}(R) \simeq \mathbb{C}[\tilde{y}_1, \cdots, \tilde{y}_{l+1}].$$

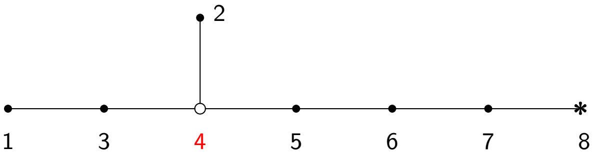


A_l
1 2 $k-1$ k $k+1$ $l-1$ l $l+1$
B_l
1 2 $k-1$ k $k+1$ $l-1$ l $l+1$
C_l
1 2 $k-1$ k $k+1$ $l-1$ l $l+1$
D_l
$l-1$
1 2 $k-1$ k $k+1$ $l-2$
l $l+1$

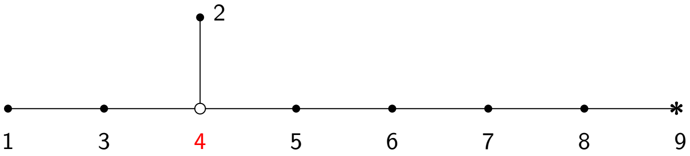
E_6



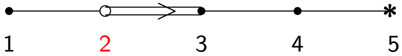
E_7



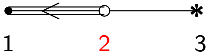
E_8



F_4



G_2



$\mathcal{M} = \mathrm{Spec}\mathcal{A}$: the orbit space of $\widetilde{W}^{(k)}(R)$

global coordinates on $\mathcal{M}$: $\{\tilde{y}_1(x), \cdots, \tilde{y}_{l+1}(x)\}$

local coordinates on $\mathcal{M}$:

$$y^1 = \tilde{y}_1, \ldots, y^l = \tilde{y}_l, \;\; y^{l+1} = \log \tilde{y}_{l+1} = 2\pi i \, x_{l+1}.$$

the metric $(\; , \;)^{\sim}$ on $\widetilde{V} = V \oplus \mathbb{C}$

$$(dx_a, dx_b)^{\sim} = \frac{1}{4\pi^2}(\omega_a, \omega_b),$$

$$(dx_{l+1}, dx_a)^{\sim} = 0, \qquad 1 \le a, \; b \le l,$$

$$(dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4\pi^2(\omega_k, \omega_k)} = -\frac{1}{4\pi^2 d_k}$$

$\rightsquigarrow (\mathcal{M} \setminus \Sigma, g^{ij}(y))$,

$$g^{ij}(y) := (dy^i, dy^j)^{\sim} = \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} (dx^a, dx^b)^{\sim}. \tag{0.1}$$

Claim: *$g^{ij}(y)$ is flat. Moreover for the particular choice, $g^{ij}(y)$ are at most linear w.r.t y^k.*

$$\rightsquigarrow \eta^{ij}(y) = \mathcal{L}_e g^{ij}(y) = \frac{\partial g^{ij}(y)}{\partial y^k}, \quad e := \frac{\partial}{\partial y^k}.$$

Theorem. [B.Dubrovin, Y.Zhang 1998]

For the particular choice of α_k, $\eta^{ij}(y)$ *and* $g^{ij}(y)$ *form a flat pencil. Moreover there exists a unique Frobenius structure on the orbit space* $\mathcal{M} = \mathcal{M}(R, k)$ *polynomial in* $t^1, \ldots, t^l, e^{t^{l+1}}$ *such that*

1. *the unity vector field coincides with* $\frac{\partial}{\partial y^k} = \frac{\partial}{\partial t^k}$;

2. *the Euler vector field has the form*

$$E = \frac{1}{2\pi i d_k} \frac{\partial}{\partial x_{l+1}} = \sum_{\alpha=1}^{l} \frac{d_\alpha}{d_k}\ t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{1}{d_k} \frac{\partial}{\partial t^{l+1}}.$$

3. *The intersection form of the Frobenius structure coincides with the metric* $(\ ,\ )^{\sim}$ *on* $\mathcal{M}$.

Theorem.[P.Slodowy 1998,Preprint but unpublished]

The ring $\mathcal{A}^{(k)}(R)$ is isomorphic to the ring of polynomials of $\tilde{y}_1(x), \cdots, \tilde{y}_{l+1}(x)$ for arbitrary choice.

Another proof

We give an alternative proof of Chevelly-Type theorem associated to the root systems $B_l, C_l, D_l, (F_4, G_2)$.

§3. Our question and result

An natural question:[P.Slodowy, B.Dubrovin and Y.Zhang 1998]

Is whether the geometric structures that were revealed in the above for particular choice also exist on the orbit spaces of the extended affine Weyl groups for an arbitrary choice of α_k?

Difficulty: d_k will be not the maximal number except the particular choice.

1. Note that the $g^{ij}(y)$ may be not linear with respect to y^k. Thus if we define $\eta^{ij}(y) = \dfrac{\partial g^{ij}(y)}{\partial y^k}$ as before, we can not obtain the flat pencil.

2. If we can obtain a flat pencil, how to find flat coordinates and construct Frobenius manifolds?

For the question 1, our strategy is to change the unity vector field.

Main theorem 1. *For any fixed integer $0 \leq m \leq l - k$ there is a flat pencil of metrics $(g^{ij}(y)), (\eta^{ij}(y))$ (bilinear forms on T^*M) with $(g^{ij}(y))$ given by (0.1) and $\eta^{ij}(y) = \mathcal{L}_e g^{ij}(y)$ on the orbit space $\mathcal{M}$ of $\widetilde{W}^{(k)}(C_l)$. Here the unity vector field*

$$e := \sum_{j=k}^{l} a_j \frac{\partial}{\partial y^j}$$

is defined by the generating function

$$\sum_{j=k}^{l} a_j u^{l-j} = (u+2)^m (u-2)^{l-k-m}$$

for the constants $a_k, \ldots, a_l$.

For the question 2, it is very technical.

Main theorem 2. *In the flat coordinates* $t^1, \ldots, t^{l+1}$, *the nonzero entries of the matrix* (η^{ij}) *are given by*

$$\eta^{ij} = \begin{cases} k, & j = k - i, \quad 1 \le i \le k-1, \\ 1, & i = l+1, j = k \quad \text{or } i = k, \ j = l+1, \\ C, & j = l - m + k - i + 1, \quad k+2 \le i \le l-m-1, \\ 2, & i = l-m, j = k+1 \quad \text{or } i = k+1, \ j = l-m, \\ 4m, & j = 2l - m - i + 1, \quad l-m+2 \le i \le l-1, \\ 2, & i = l, j = l-m+1 \quad \text{or } i = l-m+1, \ j = l, \end{cases}$$

where $C = 4(l - m - k)$. *The entries of the matrix* $(g^{ij}(t))$ *and the Christoffel symbols* $\Gamma^{ij}_m(t)$ *are weighted homogeneous polynomials in* $t^1, \ldots, t^l, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$.

Main theorem 3. *For any fixed integer $0 \leq m \leq l-k$, there exists a unique Frobenius structure of charge $d=1$ on the orbit space $\mathcal{M} \setminus \{t^{l-m}=0\} \cup \{t^l=0\}$ of $\widetilde{W}^{(k)}(C_l)$ weighted homogeneous polynomial in $t^1, t^2, \cdots, t^l, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$ such that*

1. *The unity vector field e coincides with $\sum_{j=k}^{l} a_j \frac{\partial}{\partial y^j} = \frac{\partial}{\partial t^k}$;*
2. *The Euler vector field has the form*

$$E = \sum_{\alpha=1}^{l} \tilde{d}_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{\partial}{\partial t^{l+1}}$$

3. *The intersection form of the Frobenius structure coincides with the metric $(g^{ij}(t))$.*

Main theorem 4. *The Frobenius manifold structures that we obtain in this way from* B_l *and* D_l*, by fixing the* k*-th vertex of the corresponding Dynkin diagram, are isomorphic to the one that we obtain from* C_l *by choosing the* k*-th vertex of the Dynkin diagram of* C_l*.*

Example. $[C_5, k=1, m=2]$Let R be the root system of type C_5, take $k=1, m=2$, then

$$\begin{aligned}
F = \ & \frac{1}{2}\, t_6 {t_1}^2 + \frac{1}{2}\, t_1 t_2 t_3 + \frac{1}{2}\, t_1 t_4 t_5 - \frac{1}{72}\, {t_3}^4 {t_5}^4 - \frac{1}{8}\, t_2 t_3 t_4 t_5 \\
& - \frac{1}{2268}\, {t_5}^8 - \frac{1}{36288}\, {t_3}^8 - \frac{1}{48}\, {t_3}^2 {t_2}^2 - \frac{1}{48}\, {t_4}^2 {t_5}^2 + \frac{1}{24}\, {t_5}^4 t_2 t_3 \\
& + \frac{1}{96}\, {t_3}^4 t_4 t_5 + \frac{1}{1440}\, {t_3}^5 t_2 + \frac{1}{360}\, t_4 {t_5}^5 + t_2 t_3 e^{t_6} - t_4 t_5 e^{t_6} \\
& - \frac{2}{3}\, {t_5}^4 e^{t_6} + \frac{1}{6}\, {t_3}^4 e^{t_6} + \frac{1}{2}\, e^{2t_6} + \frac{1}{48}\, \frac{{t_2}^3}{t_3} + \frac{1}{192}\, \frac{{t_4}^3}{t_5}.
\end{aligned}$$

The Euler vector field is given by

$$E = t_1\partial_1 + \frac{3}{4}t_2\partial_2 + \frac{1}{4}t_3\partial_3 + \frac{3}{4}t_4\partial_4 + \frac{1}{4}t_5\partial_5 + \partial_6.$$

thanks

Appendix. Main techniques to obtain flat coordinates

The first step: $y \to \tau$

$$\sum_{j=0}^{l} \theta^j u^{l-j} = \sum_{j=0}^{l-m} \varpi^j (u+2)^m (u-2)^{l-m-j} - \sum_{j=l-m+1}^{l} \varpi^j (u+2)^{l-j} (u-2)^{j-k-1} .$$

where

$$\theta^j = \begin{cases} e^{k\, y^{l+1}}, \\ y^j e^{(k-j) y^{l+1}}, \\ y^j, \end{cases} \qquad \varpi^j = \begin{cases} e^{k\, \tau^{l+1}}, & j = 0, \\ \tau^j e^{(k-j)\tau^{l+1}}, & j = 1, \cdots, k-1, \\ \tau^j, & j = k, \cdots, l. \end{cases}$$

The second step: $\tau \to z$

$$z^{l+1} = \tau^{l+1},\ z^j = \tau^j + p_j(\tau^1, \dots, \tau^{j-1}, e^{\tau^{l+1}}),\ 1 \le j \le k,$$

$$z^j = \tau^j + \sum_{s=j+1}^{l-m} c_s^j \tau^s, \quad k+1 \le j \le l-k-m,$$

$$z^j = \tau^j + \sum_{s=j+1}^{l} h_s^j \tau^s, \quad l-k-m+1 \le j \le l,$$

where p_j are some weighted homegeoneous polynomials and c_s^j and h_s^j are determined by the following function respectively

$$\cosh\left(\frac{\sqrt{t}}{2}\right)\left(\frac{2\sinh\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t}}\right)^{2i-1}, \quad \left(\frac{\tanh(\sqrt{t})}{\sqrt{t}}\right)^{2i-1}.$$

The third step: $z \to w$

$$
\begin{aligned}
&w^i = z^i, \quad i = 1, \ldots, k, \ l+1, \\
&w^{k+1} = z^{k+1}(z^{l-m})^{-\frac{1}{2(l-m-k)}}, \\
&w^s = z^s(z^{l-m})^{-\frac{s-k}{l-m-k}}, \ s = k+2, \cdots, l-m-1, \\
&w^{l-m} = (z^{l-m})^{\frac{1}{2(l-m-k)}}, \\
&w^{l-m+1} = z^{l-m+1}(z^l)^{-\frac{1}{2m}}, \\
&w^r = z^r(z^l)^{-\frac{r+m-l}{m}}, \ r = l-m+2, \cdots, l-1, \\
&w^l = (z^l)^{\frac{1}{2m}}.
\end{aligned}
$$

The last step: $w \to t$

$$\begin{aligned}
&t^1 = w^1, \dots, t^k = w^k,\ \ t^{l+1} = w^{l+1},\\
&t^{k+1} = w^{k+1} + w^{l-m}\, h_{k+1}(w^{k+2}, \dots, w^{l-m-1}),\\
&t^j = w^{l-m}(w^j + h_j(w^{j+1}, \dots, w^{l-m-1})),\ \ k+2 \le j \le l-m-1,\\
&t^{l-m+1} = w^{l-m+1} + w^l\, h_{l-m+1}(w^{l-m+2}, \dots, w^{l-1}),\\
&t^s = w^l(w^s + h_s(w^{s+1}, \dots, w^{l-1})),\ \ l-m+2 \le s \le l-1\\
&t^{l-m} = w^{l-m}, \quad t^l = w^l.
\end{aligned}$$

Here $h_{l-m-1} = h_{l-1} = 0$, h_j are weighted homogeneous polynomials of degree $\frac{k\,(l-m-j)}{l-m-k}$ for $j = k+1, \dots, l-m-2$ and h_s are weighted homogeneous polynomials of degree $\frac{k\,(l-s)}{m}$ for $s = l-m+2, \dots, l-1$.

Due to the above construction, we can associate the following natural degrees to the flat coordinates

$$\tilde{d}_j = \deg t^j := \frac{j}{k}, \quad 1 \leq j \leq k,$$
$$\tilde{d}_s = \deg t^s := \frac{2l - 2m - 2s + 1}{2(l - m - k)}, \quad k + 1 \leq s \leq l - m,$$
$$\tilde{d}_\alpha = \deg t^\alpha := \frac{2l - 2\alpha + 1}{2m}, \quad l - m + 1 \leq \alpha \leq l,$$
$$\tilde{d}_{l+1} = \deg t^{l+1} := 0.$$

Main mathematical applications of Frobenius manifolds

★ The theory of Gromov - Witten invariants,

★ Singularity theory,

★ Hamiltonian theory of integrable hierarchies,

★ Differential geometry of the orbit spaces of reflection groups and of their extensions $\rightsquigarrow$ semisimple Frobenius manifolds.

[B.Dubrovin's conjecture] The monodromy group is a discrete group for a solution of WDVV equations with good properties.

Example 1. [$W(M)$=Coxeter group A_1] $n = 1$, $M = \mathbb{R}$, $t = t^1$,

$$F(t) = \frac{1}{6}t^3, \quad E = t\partial_t, \quad e = \partial_t, \quad \eta^{11} =< \partial_t, \partial_t >= 1.$$

$\rightsquigarrow$ dispersionless KdV hierarchy $\rightsquigarrow$ Witten Conjecture.

Example 2. [$W(M)$=extended affine Weyl group $\widetilde{W}(A_1)$]
Quantum cohomology of $\mathbb{P}^1$:

$$F = \frac{1}{2}t_1^2 t_2 + e^{t_2}, E = t_1\partial_1 + 2\partial_2, e = \partial_1.$$

$\rightsquigarrow$ dispersionless extended Toda hierarchy $\rightsquigarrow$ Toda Conjecture.

§2. Frobenius manifolds and Coxeter groups

Let W be a finite irreducible *Coxeter group*.

$$W \curvearrowright V \qquad \rightsquigarrow \qquad W \curvearrowright S(V)$$

[Chevalley Theorem]. *The ring $S(V)^W$ of W-invariant polynomial functions on V*

$$\mathbb{C}[x_1, \cdots, x_n]^W \simeq \mathbb{C}[y^1, \cdots, y^n],$$

where $y^i = y^i(x_1, \cdots, x_n)$ are certain homogeneous W-invariant polynomials of degree $\deg y^i = d_i, \; i = 1, \cdots, n.$

The maximal degree h is called the Coxeter number. We use the ordering of the invariant polynomials

$$\deg y^n = d_n = h > d_{n-1} > \cdots > d_1 = 2.$$

The degrees satisfy the *duality condition*

$$d_i + d_{n-i+1} = h + 2, \quad i = 1, \cdots, n.$$

$$W \curvearrowright V \qquad \rightsquigarrow \qquad W \curvearrowright V \otimes \mathbb{C}$$

$$\mathcal{M} = V \otimes \mathbb{C}/W \qquad \text{affine algebraic variety}$$

$$S(V)^W \qquad \text{the coordinate ring of} \quad \mathcal{M}$$

$V \rightsquigarrow$ flat manifold $\quad (V, \{x_1, \cdots, x_n\}, (dx_a, dx_b)^* = \delta_{ab})$

$\rightsquigarrow (\mathcal{M} \setminus \Sigma, g^{ij}(y))$

$$g^{ij}(y) := (dy^i, dy^j)^* = \sum_{a,b=1}^{n} \frac{\partial y^i}{\partial x_a} \frac{\partial y^j}{\partial x_b} \delta_{ab}$$

Lemma.[K.Saito etc 1980]

1. The metric $(g^{ij}(y))$ is flat on $\mathcal{M} \setminus \Sigma$.

2. These $g^{ij}(y)$ are at most linear w.r.t y^n.

Write

$$e := \frac{\partial}{\partial y^n}.$$

Introduce a new metric,

$$\eta^{ij}(y) := \langle dy^i, dy^j \rangle = \mathcal{L}_e g^{ij}(y) = \frac{\partial g^{ij}(y)}{\partial y^n}.$$

Theorem. [K.Saito etc. 1980, B.Dubrovin 1992]

The metrics $\langle\ ,\ \rangle$ and $(\ ,\ )^*$ form a flat pencil of metrics. Moreover, there exist homogeneous polynomials

$$t^1(x), \cdots, t^n(x)$$

of degrees $d_1, \cdots, d_n$ respectively such that the matrix

$$\langle dt^i,\ dt^j \rangle := \eta^{ij} = \frac{\partial g^{ij}(t)}{\partial t^n}$$

is a constant nondegenerate matrix.

Theorem.[B.Dubrovin, 1992] *There exists a unique Frobenius structure of charge* $d = 1 - \frac{2}{h}$ *on the orbit space* $\mathcal{M}$ *polynomial in* $t^1, t^2, \cdots, t^n$ *such that*

1. *The unity vector field e coincides with* $\frac{\partial}{\partial y^n} = \frac{\partial}{\partial t^n}$;
2. *The Euler vector field has the form*

$$E = \sum_{\alpha=1}^{n} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}.$$

Theorem. [B.Dubrovin's conjecture, 1996. C.Hertling, 1999]

Any irreducible semisimple polynomial Frobenius manifold with positive invariant degrees is isomorphic to the orbit space of a finite Coxeter group.