A class of special solutions for the ultradiscrete Painlevé II equation

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Abstract. A class of special solutions are constructed in an intuitive way for the ultradiscrete analog of $q$-Painlevé II equation. The solutions are classified into four groups depending on the function-type and the system parameter.

1 Introduction

Uladiscretization [1] is a limiting procedure transforming a given difference equation into a cellular automaton, in which dependent variables also take discrete values. To apply this procedure, we first replace a dependent variable $x_n$ in the equation by

$$x_n = e^{X_n/\varepsilon},$$

where $\varepsilon$ is a positive parameter. Next, we apply $\varepsilon \log$ to both sides of the equation and take the limit $\varepsilon \to +0$. Then, using identity

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{X/\varepsilon} + e^{Y/\varepsilon}) = \max(X, Y),$$

the original difference equation is approximated by a piecewise linear equation which can be regarded as a time evolution rule for a cellular automaton. In many examples, cellular automata obtained by this systematic method preserve the essential properties of the original equations, such as the qualitative behavior of exact solutions. However, the ansatz (1) is only possible if the variable $x_n$ is positive definite. This restriction is called ‘negative problem’.

From theoretical and application points of view, it is an interesting problem to study ultradiscrete analogs of special functions and their defining equations, including the Painlevé equations. Ultradiscrete analogs for some of the Painlevé equations are proposed, for example, in [2, 3, 4]. However, the class of ultradiscretizable Painlevé equations has been restricted because of the negative problem. Some attempts resolving this problem are reported, for example, in [5, 6, 7]. The authors and coworkers study in [5] an ultradiscrete Painlevé II equation with sinh ansatz and discuss its special solution of Bi function type.

In order to overcome the negative problem, a new method ‘ultradiscretization with parity variables’ (p-ultradiscretization) is proposed in [8]. The procedure keeps track of the sign of original variables. By using this method, the authors and coworkers present [9] a p-ultradiscrete analog of the $q$-Painlevé II equation ($q$-PII),

$$a_2^2 z(\tau) = \frac{a_2^2 z(\tau)}{\tau - z(\tau)},$$

In [9], we also discuss a series of special solutions corresponding to that of $q$-PII written in the determinants of size $N$. However, the resulting solutions are reduced to only one solution for the p-ultradiscrete Painlevé II (udPII) equation. In this paper, we construct other series of special solutions for udPII and discuss their structure. In section 2, we introduce the results in [9] for the p-ultradiscrete Airy equation. Then, we construct special solutions for udPII in section 3. Finally, concluding remarks are given in section 4.
2 Ultradiscrete Airy equation with parity variables

We start with a $q$-difference analog of the Airy equation

$$w(q\tau) - \tau w(\tau) + w(q^{-1}\tau) = 0,$$

which reduces to the Airy equation

$$\frac{d^2v}{ds^2} + sv = 0$$

in a continuous limit.

In order to ultradiscretize (3), we put $\tau = q^m$ and $q = e^{Q/\varepsilon}$ ($Q < 0$). Furthermore, we introduce an ansatz for $p$-ultradiscretization,

$$w(q^m) = \{s(\omega_m) - s(-\omega_m)\}e^{W_m/\varepsilon},$$

where $\omega_m \in \{+1, -1\}$ denotes the sign of $w(q^m)$ and $s(\omega)$ is defined by

$$s(\xi) = \begin{cases} 
1 & (\xi = +1) \\
0 & (\xi = -1).
\end{cases}$$

Taking the ultradiscrete limit, we obtain a $p$-ultradiscrete analog of the Airy equation

$$\max(W_{m+1} + S(\omega_{m+1}), mQ + W_m + S(-\omega_m), W_{m-1} + S(-\omega_{m-1}))$$

$$= \max(W_{m+1} + S(-\omega_{m+1}), mQ + W_m + S(\omega_m), W_{m-1} + S(-\omega_{m-1})), \quad (4)$$

where $S(\omega)$ is defined by

$$S(\omega) = \begin{cases} 
0 & (\omega = +1) \\
-\infty & (\omega = -1).
\end{cases}$$

An ultradiscretized variable is represented by a pair of $\omega_m$ and $W_m$, which is denoted as $W_n = (\omega_m, W_m)$ in what follows. It is possible to rewrite the implicit form (4) into explicit forward schemes

$$\omega_{m+1} = \begin{cases} 
\frac{\omega_m - \omega_{m-1}}{2} + \frac{\omega_m + \omega_{m-1}}{2}\text{sgn}(F_m) & (\omega_m = -\omega_{m-1} \text{ or } F_m \neq 0) \\
\text{indefinite} & (\omega_m = \omega_{m-1} \text{ and } F_m = 0)
\end{cases}$$

$$W_{m+1} = \begin{cases} 
\max(mQ + W_m, W_{m-1}) & (\omega_m = -\omega_{m-1} \text{ or } F_m \neq 0) \\
\leq W_{m-1} & (\omega_m = \omega_{m-1} \text{ and } F_m = 0)
\end{cases},$$

where $F_m := mQ + W_m - W_{m-1}$. Note that we generally have both of unique and indeterminate schemes depending on given values of $(\omega_m, W_m)$ and $(\omega_{m-1}, W_{m-1})$. The explicit backward schemes are obtained by replacing $m \pm 1$ with $m \mp 1$, respectively.

We find two typical solutions of (4). One is an Ai-function-type solution for $W_0 = (+1, 0)$ and $W_1 = (+1, 0)$,

$$u_{\text{Ai}}(m) = (\omega_m, W_m) = \begin{cases} 
(-1)^{\frac{m(m-1)}{2}}, 0 & (m \geq 0) \\
+1, \frac{m(m-1)}{2}Q & (m \leq -1),
\end{cases}$$

and the other is a Bi-function-type for $W_0 = (+1, 0)$ and $W_1 = (-1, 0)$,

$$u_{\text{Bi}}(m) = (\omega_m, W_m) = \begin{cases} 
(-1)^{\frac{m(m+1)}{2}}, 0 & (m \geq 0) \\
+1, -\frac{m(m+1)}{2}Q & (m \leq -1).
\end{cases}$$

They show similar behavior as those of the Ai and Bi functions, respectively.
3 Ultradiscrete Painlevé II equation with parity variables

For the following discussion, we first introduce the results for (2). It has been shown in [10] that

\[
    z^{(N)}(\tau) = \begin{cases} 
    \frac{g^{(N)}(\tau)g^{(N+1)}(q\tau)}{q^Ng^{(N)}(q\tau)g^{(N+1)}(q\tau)} & (N \geq 0) \\
    \frac{g^{(N)}(\tau)g^{(N+1)}(q\tau)}{q^{N+1}g^{(N)}(q\tau)g^{(N+1)}(q\tau)} & (N < 0) 
\end{cases}
\] (5)

solves (2) with \( \alpha = q^{2N+1} \), where the functions \( g^{(N)}(t) \) \( (N \in \mathbb{Z}) \) satisfy the bilinear equations

\[
    q^{2N}g^{(N+1)}(q^{-1}\tau)g^{(N)}(q^2\tau) - q^{N}rg^{(N+1)}(\tau)g^{(N)}(q\tau) + g^{(N+1)}(q\tau)g^{(N)}(\tau) = 0 \quad (6)
\]

\[
    q^{2N}g^{(N+1)}(q^{-1}\tau)g^{(N)}(q\tau) - q^{2N}rg^{(N+1)}(\tau)g^{(N)}(\tau) + g^{(N+1)}(q\tau)g^{(N)}(q^{-1}\tau) = 0 \quad (7)
\]

for \( N \geq 0 \) and

\[
    q^{2N+2}g^{(N+1)}(q^{-1}\tau)g^{(N)}(q^2\tau) - q^{N+1}rg^{(N+1)}(\tau)g^{(N)}(q\tau) + g^{(N+1)}(q\tau)g^{(N)}(\tau) = 0 \quad (8)
\]

\[
    q^{2N+2}g^{(N+1)}(q^{-1}\tau)g^{(N)}(q\tau) - q^{2N+1}rg^{(N+1)}(\tau)g^{(N)}(\tau) + g^{(N+1)}(q\tau)g^{(N)}(q^{-1}\tau) = 0 \quad (9)
\]

for \( N < 0 \). It is also known that \( g^{(N)}(\tau) \) are written in terms of the Casorati determinant of size \( |N| \) whose elements are represented by the solutions of (3).

In order to construct ultradiscrete analogs of these equations, we put \( \tau = q^m \), \( q = e^{Q/\varepsilon}(Q < 0) \) and \( a = e^{A/\varepsilon} \). Furthermore, we introduce

\[
    z(q^m) = (s(\zeta_m) - s(-\zeta_m))e^{Z_m/\varepsilon}
\]

\[
    g^{(N)}(q^m) = (s(\gamma_m^{(N)}) - s(-\gamma_m^{(N)}))e^{G_m^{(N)}/\varepsilon}.
\]

Then (2) is reduced to udPPII,

\[
    \max \left[ Z_{m+1} + 3Z_m + Z_{m-1} + \max \left\{ \left( S(-\zeta_{m+1}) + S(\zeta_m) + S(\zeta_{m-1}), \right) \right. \right.
\]

\[
    S(-\zeta_{m+1}) + S(\zeta_m) + S(-\zeta_{m-1}), S(-\zeta_{m+1}) + S(-\zeta_m) + S(\zeta_{m-1}),
\]

\[
    S(\zeta_{m+1}) + S(-\zeta_m) + S(-\zeta_{m-1}) \right], Z_{m+1} + 2Z_m + S(\zeta_m),
\]

\[
    2Z_m + Z_{m-1} + S(\zeta_m), Z_m + S(\zeta_m), Z_m + A + 2mQ + S(\zeta_m),
\]

\[
    Z_{m+1} + 2Z_m + Z_{m-1} + mQ + \max \left\{ \left( S(-\zeta_{m+1}) + S(\zeta_m), S(\zeta_{m+1}) + S(-\zeta_{m-1}), S(-\zeta_{m+1}) + S(-\zeta_m), S(-\zeta_{m+1}) + S(-\zeta_m) \right) \right. \}
\]

\[
    Z_{m+1} + Z_m + mQ + \max \left\{ \left( S(-\zeta_m), S(\zeta_m), S(-\zeta_m) \right) \right. \}
\]

\[
    \left. Z_{m+1} + Z_m + mQ + \max \left\{ \left( S(-\zeta_{m-1}), S(\zeta_m), S(-\zeta_{m-1}), S(-\zeta_m) \right) \right. \}
\]

\[
    \left. Z_m + Z_{m-1} + mQ + \max \left\{ \left( S(-\zeta_m), S(\zeta_{m-1}), S(-\zeta_m), S(-\zeta_{m-1}) \right) \right. \right], mQ \right].
\] (10)
For (6) and (7), we have their ultradiscrete analogs

\[
\begin{align*}
\max \left[ 2NQ + G_{m-1}^{(N+1)} + G_m^{(N)} \right.
+ \max \left\{ S(\gamma_{m-1}^{(N+1)}), S(\gamma_{m+2}^{(N)}), S(-\gamma_{m-1}^{(N+1)}), S(-\gamma_{m+2}^{(N)}) \right\}, \\
(N + m)Q + G_{m+1}^{(N+1)} + G_m^{(N)} \\
+ \max \left\{ S(\gamma_{m}^{(N+1)}), S(-\gamma_{m-1}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(\gamma_{m+1}^{(N)}) \right\}, \\
G_{m+1}^{(N+1)} + G_m^{(N)} + \max \left\{ S(\gamma_{m+1}^{(N+1)}), S(\gamma_{m}^{(N)}), S(-\gamma_{m+1}^{(N+1)}), S(-\gamma_{m}^{(N)}) \right\} \right]
\end{align*}
\]

\[= \max \left[ 2NQ + G_{m-1}^{(N+1)} + G_m^{(N)} \right.
+ \max \left\{ S(\gamma_{m-1}^{(N+1)}), S(\gamma_{m+1}^{(N)}), S(-\gamma_{m-1}^{(N+1)}), S(-\gamma_{m+1}^{(N)}) \right\}, \\
(2N + m)Q + G_{m+1}^{(N+1)} + G_m^{(N)} \\
+ \max \left\{ S(\gamma_{m}^{(N+1)}), S(-\gamma_{m}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(\gamma_{m}^{(N)}) \right\}, \\
G_{m+1}^{(N+1)} + G_m^{(N)} + \max \left\{ S(\gamma_{m+1}^{(N+1)}), S(\gamma_{m}^{(N)}), S(-\gamma_{m+1}^{(N+1)}), S(-\gamma_{m-1}^{(N)}) \right\} \right]
\] (11)

and

\[
\begin{align*}
\max \left[ 2NQ + G_{m-1}^{(N+1)} + G_m^{(N)} \right.
+ \max \left\{ S(\gamma_{m-1}^{(N+1)}), S(\gamma_{m+1}^{(N)}), S(-\gamma_{m-1}^{(N+1)}), S(-\gamma_{m+1}^{(N)}) \right\}, \\
(2N + m)Q + G_{m+1}^{(N+1)} + G_m^{(N)} \\
+ \max \left\{ S(\gamma_{m}^{(N+1)}), S(-\gamma_{m}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(\gamma_{m}^{(N)}) \right\}, \\
G_{m+1}^{(N+1)} + G_m^{(N-1)} + \max \left\{ S(\gamma_{m+1}^{(N-1)}), S(\gamma_{m-1}^{(N)}), S(-\gamma_{m+1}^{(N+1)}), S(-\gamma_{m-1}^{(N)}) \right\} \right]
\end{align*}
\]

\[= \max \left[ 2NQ + G_{m-1}^{(N+1)} + G_m^{(N)} \right.
+ \max \left\{ S(\gamma_{m-1}^{(N+1)}), S(-\gamma_{m-1}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(\gamma_{m+1}^{(N)}) \right\}, \\
(2N + m)Q + G_{m+1}^{(N+1)} + G_m^{(N)} \\
+ \max \left\{ S(\gamma_{m}^{(N+1)}), S(\gamma_{m}^{(N)}), S(-\gamma_{m+1}^{(N+1)}), S(-\gamma_{m}^{(N)}) \right\}, \\
G_{m+1}^{(N+1)} + G_m^{(N)} + \max \left\{ S(\gamma_{m+1}^{(N+1)}), S(-\gamma_{m-1}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(\gamma_{m-1}^{(N)}) \right\} \right]
\] (12)
respectively. For (8) and (9), we have

\[
\max \left[ 2(N + 1)Q + G_{m+1}^{(N+1)} + G_{m+2}^{(N)} + \max \{S(\gamma_{m+1}^{(N+1)}), S(\gamma_{m+2}^{(N+1)}), S(-\gamma_{m-1}^{(N+1)}), S(-\gamma_{m+2}^{(N+1)}) \}, \right.
\]

\[
(N + m + 1)Q + G_{m+1}^{(N+1)} + G_{m+1}^{(N)} + \max \{S(\gamma_{m+1}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(\gamma_{m+1}^{(N)}) \},
\]

\[
G_{m+1}^{(N+1)} + G_{m+1}^{(N)} + \max \{S(\gamma_{m+1}^{(N+1)}), S(\gamma_{m}^{(N)}), S(-\gamma_{m+1}^{(N+1)}), S(-\gamma_{m}^{(N)}) \} \right]
\]

\[
= \max \left[ 2(N + 1)Q + G_{m+1}^{(N+1)} + G_{m+2}^{(N)} + \max \{S(\gamma_{m-1}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(\gamma_{m-1}^{(N)}) \}, \right.
\]

\[
(N + m + 1)Q + G_{m+1}^{(N+1)} + G_{m}^{(N)} + \max \{S(\gamma_{m}^{(N+1)}), S(-\gamma_{m}^{(N+1)}), S(\gamma_{m}^{(N)}) \},
\]

\[
G_{m+1}^{(N+1)} + G_{m}^{(N)} + \max \{S(\gamma_{m+1}^{(N+1)}), S(-\gamma_{m}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(\gamma_{m}^{(N)}) \} \right] 
\]

(13)

and

\[
\max \left[ 2(N + 1)Q + G_{m-1}^{(N+1)} + G_{m+1}^{(N)} + \max \{S(\gamma_{m-1}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(\gamma_{m}^{(N+1)}) \}, \right.
\]

\[
(2N + m + 1)Q + G_{m+1}^{(N+1)} + G_{m}^{(N)} + \max \{S(\gamma_{m}^{(N+1)}), S(-\gamma_{m}^{(N+1)}), S(\gamma_{m}^{(N)}) \},
\]

\[
G_{m+1}^{(N+1)} + G_{m-1}^{(N)} + \max \{S(\gamma_{m+1}^{(N+1)}), S(-\gamma_{m+1}^{(N+1)}), S(-\gamma_{m}^{(N)}) \} \right] 
\]

(14)

respectively. Finally, the transformations (5) are reduced to

\[
\psi_{m}^{(N)} = \gamma_{m}^{(N)} \gamma_{m+1}^{(N+1)} \gamma_{m+1}^{(N+1)} \gamma_{m+1}^{(N)} \gamma_{m+1}^{(N)} \]  
(15)

\[
Z_{m}^{(N)} = G_{m}^{(N)} + G_{m+1}^{(N+1)} - G_{m+1}^{(N+1)} - G_{m+1}^{(N)} - NQ \]  
(16)

for \( N \geq 0 \) and

\[
\psi_{m}^{(N)} = \gamma_{m}^{(N)} \gamma_{m+1}^{(N+1)} \gamma_{m+1}^{(N+1)} \gamma_{m+1}^{(N)} \gamma_{m+1}^{(N)} \]  
(17)

\[
Z_{m}^{(N)} = G_{m}^{(N)} + G_{m+1}^{(N+1)} - G_{m+1}^{(N+1)} - G_{m+1}^{(N)} - (N + 1)Q \]  
(18)

for \( N < 0 \). If we find solutions for the ultradiscrete bilinear equations, special solutions for udPII are obtained through (15) – (18).

Hereafter we consider only the case of \( A = (2N+1)Q \) in (10), which corresponds to \( a = q^{2N+1} \) in the discrete system. Firstly, we present the results reported in [9], that is, the Ai-function-type solutions for \( N \geq 0 \). Solutions of (13) and (14) are given by \( g_{m}^{(N)} = (\gamma_{m}^{(N)}, G_{m}^{(N)}) \) for
N = 0, 1, 2, . . . , where

\[
\gamma_m^{(N)} = \begin{cases} 
\gamma_0^{(N)} (-1)^{-\frac{m(m+1)N}{2}} & (m \geq 0) \\
\gamma_0^{(N)} & (m \leq -1)
\end{cases}
\]

\[
G_m^{(N)} = \begin{cases} 
\frac{mN(N-1)}{2} Q + G_0^{(N)} & (m \geq 0) \\
\frac{mN(m+N-2)}{2} Q + G_0^{(N)} & (m \leq -1)
\end{cases}
\]

Since \(G_m^{(N)} \to -\infty\) as \(m \to -\infty\) in the same way as the \(uA\) function, we call these solutions the \(Ai\)-function-type solutions. From these solutions, we have only one special solution of \(udP\) with \(A = (2N+1)Q\) for \(N = 0, 1, 2, . . . \)

\[
Z_m^{(N)} = (\zeta_m^{(N)}, \zeta_m^{(N)}) = \begin{cases} 
((-1)^m, 0) & (m \geq 0) \\
(+1, mQ) & (m \leq -1)
\end{cases}
\]

which does not depend on \(N\). We note that \(Z_m^{(N)} \to \infty\) as \(m \to -\infty\).

Secondly, we investigate \(Bi\)-function-type solutions for \(N \geq 0\). We find that \(G_m^{(0)} = (+1, 0)\) and \(G_m^{(1)} = uBi(m)\) solve (11) and (12) with \(N = 0\). By using this result, we inductively construct solutions \(G_m^{(N+1)}\) of the equations with \(N \geq 1\) for a given function \(G_m^{(N)}\) and assigned values of \(G_0^{(N+1)}\) and \(G_1^{(N+1)}\). We further assume that \(G_0^{(N+1)}\) and \(G_1^{(N+1)}\) are chosen so that \(G_m^{(N+1)}\) for any \(m\) are uniquely determined in (11) and (12). Then we have the following solutions

\[
G_m^{(N)} = (\gamma_m^{(N)}, G_m^{(N)}),
\]

where

\[
\gamma_m^{(N)} = \begin{cases} 
(-1)^{-\frac{m(m+1)N}{2}} \gamma_0^{(N)} & (m \geq 0) \\
(-1)^{-\frac{m(m-1)(m+1)}{24}} \gamma_0^{(N)} & (-2 \geq m \geq -2N - 1, N: even) \\
(-1)^{-\frac{m(m+1)(m+2)(m+3)}{24}} \gamma_0^{(N)} & (-2 \geq m \geq -2N - 1, N: odd) \\
(-1)^{-\frac{N(N-1)}{2}} \gamma_0^{(N)} & (m \leq -2N - 2),
\end{cases}
\]

\[
G_m^{(N)} = \begin{cases} 
\frac{mN(N-1)}{2} Q + G_0^{(N)} & (m \geq 0) \\
\frac{mN(N-1)}{2} Q + \frac{(m-2)m(2m+1)}{24} Q + G_0^{(N)} & (m = -2, -4, . . . , -2N) \\
\frac{mN(N-1)}{2} Q + \frac{(m-1)(m+1)(2m-3)}{24} Q + G_0^{(N)} & (m = -3, -5, . . . , -2N - 1) \\
\frac{mN(m+N)}{2} Q - \frac{N(N-1)(N+1)}{24} Q + G_0^{(N)} & (m \leq -2N - 2).
\end{cases}
\]

Since \(G_m^{(N)} \to \infty\) as \(m \to -\infty\) in the same way as the \(uBi\) function, we call these solutions the \(Bi\)-function-type solutions.

By substituting these solutions into (15) and (16), we obtain special solutions of \(udP\),

\[
Z_m^{(N)} = (\zeta_m^{(N)}, \zeta_m^{(N)}) = \begin{cases} 
((-1)^m, 0) & (m \geq -2N - 1) \\
(+1, -(m + 2N + 1)Q) & (m \leq -2N - 2).
\end{cases}
\]

We notice that (19) and (20) have different asymptotic behavior in \(Z_m^{(N)}\) for \(m \to -\infty\) and in the phases for \(m > 0\). Furthermore, we remark that (20) have \(N\)-dependence.

Thirdly, we study \(Ai\)-function-type solutions for \(N < 0\). We find that \(G_m^{(0)} = (+1, 0)\) and \(G_m^{(-1)} = uAi(m - 1)\) solve (13) and (14) with \(N = -1\). Starting from these simple solutions, we
inductively find the solutions \( G_m^{(N)} = (\gamma_m^{(N)}, G_m^{(N)}) \) for \( N = 0, -1, -2, \ldots \), where

\[
\gamma_m^{(N)} = \begin{cases} 
(-1)^{(m+N)(m+N-1)/2} \gamma_1^{(N)} & (m \geq -N) \\
\gamma_1^{(N)} & (m \leq -N)
\end{cases}
\]

\[
G_m^{(N)} = \begin{cases} 
\frac{N(N+1)(m+N-1)Q + G_1^{(N)}}{2} & (m \geq -N) \\
-\frac{N(m+N-1)(m-1)Q + G_1^{(N)}}{2} & (m \leq -N).
\end{cases}
\]

Substituting these \( G_m^{(N)} \) into (17) and (18), we have special solutions of uBiPII with \( A = (2N + 1)Q \) for \( N = -1, -2, \ldots \),

\[
Z_m^{(N)} = \left( \zeta_m^{(N)}, Z_m^{(N)} \right) = \begin{cases} 
((-1)^{m-1}, 0) & (m \geq -N) \\
(+1, -(m + 2N + 1)Q) & (m \leq -N - 1).
\end{cases}
\]

Typical behavior of these solutions is shown in Figure 1. They converge to 0 as \( m \to -\infty \) and oscillate for \( m \geq -N \). It is interesting to note that (21) constructed from the uAi function has essentially the same structure as (20) constructed from the uBi function.

![Graph](image1.png)

**Figure 1.** Behavior of special solutions (21) with \( Q = -1 \). (a) and (b) are for \( N = -1 \) and \( N = -4 \), respectively.

Finally, we study Bi-function-type solutions for \( N < 0 \). We find that \( G_m^{(0)} = (+1, 0) \) and \( G_m^{(-1)} = \text{uBi}(m - 1) \) solve (13) and (14) with \( N = -1 \). We construct solutions \( G_m^{(N)} \) of the equations with \( N \leq -2 \) for a given function \( G_m^{(N+1)} \) and assigned values of \( G_1^{(N)} \) and \( G_2^{(N)} \). We further assume that \( G_1^{(N)} \) and \( G_2^{(N)} \) are chosen so that \( G_m^{(N)} \) for \( m \leq 0 \) are uniquely determined in (13) and (14). We again inductively obtain the solutions \( G_m^{(N)} = (\gamma_m^{(N)}, G_m^{(N)}) \), where

\[
\gamma_m^{(N)} = \begin{cases} 
(-1)^{N(N+2)/8} \gamma_1^{(N)} & (m \geq 3 - N, N : even) \\
(-1)^{m+N-3(m+N-4)/8} \gamma_{N-1}^{(N-1)} + \gamma_1^{(N)} & (m \geq 3 - N, N : odd) \\
(-1)^{N+2(N+3)(N+4)(N+5)/24} \gamma_{N-1}^{(N)} & (N - 1 \leq m \leq 2 - N) \\
(-1)^{N+2(N+3)(N+4)(N+5)/24} \gamma_{N-1}^{(N)} & (m \leq N - 2),
\end{cases}
\]

\[
p_{N,m} = \frac{(m-N-3)(m-N-4)(m-N-5)(m-N-6)}{24} + \frac{(N+2)(N+3)(N+4)(N+5)}{24},
\]
that, although (22) is similar to (19), (22) has more complicated internal structure. From these solutions, we have special solutions of udPII,

\[
G_m^{(N)} = \begin{cases} 
\frac{(m-1)N(N+1)}{2}Q - \frac{N(N+2)(2N-1)}{24}Q + G_1^{(N)} \\
\frac{(m-1)N(N+1)}{2}Q - \frac{(N-1)(N+1)(2N-3)}{24}Q + G_1^{(N)} \\
\frac{(m-1)N(m+3N+2)}{4}Q + \frac{(m-2)m(2m+1)}{24}Q + G_1^{(N)} \\
\frac{(m-1)N(m+3N+2)}{4}Q + \frac{(m-1)(m+1)(2m-3)}{24}Q + G_1^{(N)} \\
mN(m+N)Q + \frac{N(2N^2-15N-14)}{24}Q + G_1^{(N)} \\
mN(m+N)Q + \frac{N(2N^2-17N+3)}{24}Q + G_1^{(N)}
\end{cases}
\]

\(m \geq 3 - N, N: even\)
\(m \geq 3 - N, N: odd\)
\(N - 1 \leq m \leq 2 - N, m: even, N: even\)
\(N - 1 \leq m \leq 2 - N, m: even, N: odd\)
\(N - 1 \leq m \leq 2 - N, m: odd\)
\(m \leq N - 2, N: even\)
\(m \leq N - 2, N: odd\).

From these solutions, we have special solutions of udPII,

\[
Z_m^{(N)} = \left(\zeta_m^{(N)}, Z_m^{(N)}\right) = \begin{cases} 
((-1)^{m}, 0) & (m \geq -N - 1) \\
(-1)^{\frac{m-N}{2}}, \frac{m+N}{2} & (|m| \leq -N - 2, m - N: even) \\
(-1)^{\frac{m-N-1}{2}}, \frac{m+N+1}{2} & (|m| \leq -N - 2, m - N: odd) \\
(+1, mQ) & (m \leq N + 1).
\end{cases}
\]

Typical behavior of these solutions is shown in Figure 2. Note that (21) and (22) have different asymptotic behavior in \(Z_m^{(N)}\) as \(m \to -\infty\) and in the phases for \(m \geq -N\). We also comment that, although (22) is similar to (19), (22) has more complicated internal structure.

**Figure 2.** Behavior of special solutions (22) with \(Q = -1\). (a) and (b) are for \(N = -1\) and \(N = -6\), respectively.
4 Concluding Remarks

In this paper we have presented a class of special solutions for the p-ultradiscrete analog of $q$-PII. The solutions are classified into four groups: Ai-function-type and Bi-function-type solutions for the system parameter $N \geq 0$, and those for $N < 0$. In the preceding paper [9] are given only the Ai-function-type solutions for $N \geq 0$, which do not depend on $N$. Three other groups which are newly given in this paper do depend on $N$. Moreover, the solutions of each group have different structures. For example, we observe differences between the Ai- and Bi-function-type solutions in their asymptotic amplitude and phases, which may reflect the structure of solutions of difference and continuous equations. The Bi-function-type solutions for $N \geq 0$ have fairly complicated internal structure, although we do not know the origin of these structures yet. At any rate, these results may indicate the richness of solution space of the ultradiscrete equation.

For the continuous and discrete Airy equations, linear combination of Ai and Bi functions give their general solutions. In the ultradiscrete case, $\max(f, g)$ corresponds to the linear combination of functions $f$ and $g$. Hence, we believe that the cases we treated in this paper cover quite wide class of special solutions of the ultradiscrete equations.

Our method of constructing solutions is intuitive and purely based on the ultradiscrete equations. We believe that the solutions we obtain correspond to those of $q$-PII represented by the Casorati determinant of size $|N|$ whose elements are given by the $q$-difference Ai or Bi function. It is a future problem to clarify the relationship between discrete and ultradiscrete solutions through a limiting procedure. It is also a future problem to construct p-ultradiscrete analogs of other Painlevé equations and their special solutions.
Do ultradiscrete systems with parity variables satisfy the singularity confinement criterion?


Ultradiscrete singularity confinement test, which is an integrability detector for ultradiscrete equations with parity variables, is applied to various ultradiscrete equations. The ultradiscrete equations exhibit singularity structures analogous to those of the discrete counterparts. Exact solutions to linearisable ultradiscrete equations are also constructed to explain the singularity structures.
I. ULTRADISCRETISATION WITH PARITY VARIABLES

In order to show how ultradiscretisation with parity variables works in a simple case, we choose the discrete Riccati equation

\[ x_{n+1} = a + b - \frac{ab}{x_n}. \]  

Using the discrete Cole-Hopf transformation

\[ x_n = \frac{u_n}{u_{n-1}}, \]

the mapping (1) can be transformed into the second order linear difference equation

\[ u_{n+1} - (a + b)u_n + abu_{n-1} = 0. \]

The general solution to (3), neglecting the case \( a = b \), is given by

\[ u_n = c_1 a^n + c_2 b^n \quad \text{for } a \neq b. \]

Substituting (4) back into (2) gives the following general solution

\[ x_n = \frac{a^n + cb^n}{a^{n-1} + cb^{n-1}}, \]

where \( c = c_2/c_1 \). Note that \( x_n \to \max(a,b) \) as \( n \to \infty \).

We now turn our attention to ultradiscretising (1). For brevity, we shall assume \( a, b > 0 \). The first step in ultradiscretisation of (1) with parity variables is to introduce a new dependent variable \( \sigma_n \in \{-1, 1\} \) (the parity variable), and write \( x_n = \sigma_n \tilde{x}_n \) where \( \tilde{x}_n > 0 \) is another dependent variable (the amplitude variable which is always positive). We further introduce a function \( s \) defined by

\[ s(\rho) = \begin{cases} 
1 & \text{if } \rho = 1 \\
0 & \text{if } \rho = -1 
\end{cases} \]

and write \( \sigma_n = s(\sigma_n) - s(-\sigma_n) \). Once we collect non-negative terms to each side of equality and use the exponential ansatz \( a = e^{A/\delta} \), \( b = e^{B/\delta} \) and \( \tilde{x} = e^{X/\delta} \), we obtain the following implicit equation for \( \sigma_n \) and \( X_n \) in the limit \( \delta \to +0 \):

\[
\max\{X_{n+1} + X_n + \max\{S(\sigma_{n+1}) + S(\sigma_n), S(-\sigma_{n+1}) + S(-\sigma_n)\},
\max(A, B) + X_n + S(-\sigma_n), A + B\}
= \max\{X_{n+1} + X_n + \max\{S(\sigma_{n+1}) + S(-\sigma_n), S(-\sigma_{n+1}) + S(\sigma_n)\},
\max(A, B) + X_n + S(\sigma_n)\},
\]

(7)
where the function $S$ is defined by

$$S(\rho) = \begin{cases} 0 & \text{if } \rho = 1 \\ -\infty & \text{if } \rho = -1. \end{cases}$$

(8)

Equation (7) can be rewritten as the following pair of equations:

$$\sigma_{n+1} = \begin{cases} \frac{1}{2}[1 - \sigma_n + (1 + \sigma_n)\text{sgn}\{\max(A, B) - A - B + X_n\}] & \text{if } \sigma_n = -1 \text{ or } X_n \neq A + B - \max(A, B) \\ \text{indeterminate} & \text{if } \sigma_n = 1 \text{ and } X_n = A + B - \max(A, B) \end{cases}$$

(9)

and

$$X_{n+1} = \begin{cases} \max(A, B, A + B - X_n) & \text{if } \sigma_n = -1 \text{ or } X_n \neq A + B - \max(A, B) \\ \leq \max(A, B) & \text{if } \sigma_n = 1 \text{ and } X_n = A + B - \max(A, B), \end{cases}$$

(10)

where the signum function “sgn” is defined by

$$\text{sgn}(K) = \begin{cases} 1 & \text{if } K > 0 \\ 0 & \text{if } K = 0 \\ -1 & \text{if } K < 0. \end{cases}$$

(11)

We will refer to (9) and (10) as the ultradiscrete Riccati equations. In what follows, we use the notation $\mathcal{X}_n := (\sigma_n, X_n)$ to denote the pair of dependent variables $\sigma_n$ and $X_n$. Writing $u_n = \tau_n \bar{u}_n$ where $\tau_n \in \{-1, 1\}$ and $\bar{u}_n > 0$, and applying the same procedure to (2), we obtain the ultradiscrete form of the Cole-Hopf transformation:

$$\sigma_n = \tau_n \tau_{n-1}$$

(12)

and

$$X_n = U_n - U_{n-1}.$$  

(13)
Finally, substituting (12), (13) into the ultradiscrete Riccati equations (9) and (10), we obtain the following pair of equations:

\[
\tau_{n+1} = \begin{cases} 
\frac{1}{2} \left[ \tau_n - \tau_{n-1} + (\tau_n + \tau_{n-1}) \text{sgn} \{ \max(A, B) - A - B + U_n - U_{n-1} \} \right] \\
\text{if } \tau_n = -\tau_{n-1} \text{ or } U_n \neq A + B - \max(A, B) + U_{n-1} \\
\text{indeterminate} \\
\text{if } \tau_n = \tau_{n-1} \text{ and } U_n = A + B - \max(A, B) + U_{n-1} 
\end{cases}
\] (14)

and

\[
U_{n+1} = \begin{cases} 
\max \{ \max(A, B) + U_n, A + B + U_{n-1} \} \\
\text{if } \tau_n = -\tau_{n-1} \text{ or } U_n \neq A + B - \max(A, B) + U_{n-1} \\
\leq \max(A, B) + U_n \\
\text{if } \tau_n = \tau_{n-1} \text{ and } U_n = A + B - \max(A, B) + U_{n-1}. 
\end{cases}
\] (15)

Equations (14), (15) will be referred to as the linearised ultradiscrete Riccati equations.

Given the initial values \( U_0 = (\tau_0, U_0) \) and \( U_1 = (\tau_1, U_1) \), the solutions to the linearised ultradiscrete Riccati equations (14) and (15) are as follows, where for brevity, we consider only the case \( A \neq B \).

1. If \( \tau_1 = -\tau_0 \) or \( U_1 \neq A + B - \max(A, B) + U_0 \), then

\[
\tau_n = \frac{1}{2} \left[ \tau_1 - \tau_0 + (\tau_1 + \tau_0) \text{sgn} \{ \max(A, B) - A - B + U_1 - U_0 \} \right]
\] for \( n \geq 2 \)

and

\[
U_n = (n - 2) \max(A, B) + \max \{ \max(A, B) + U_1, A + B + U_0 \}
\] for \( n \geq 2. \) (16)

2. Given some integer \( k(\geq 1) \), if \( \tau_n = \tau_0 \) and \( U_n = n \{ A + B - \max(A, B) \} + U_0 \) for \( 1 \leq n \leq k \) but \( \tau_{k+1} = -\tau_0 \) or \( U_{k+1} \neq (k + 1) \{ A + B - \max(A, B) \} + U_0 \), then

\[
\tau_n = \begin{cases} 
\tau_0 & \text{for } 1 \leq n \leq k \\
\text{indeterminate} & \text{for } n = k + 1 \\
\frac{1}{2} ((\tau_{k+1} + \tau_0) \text{sgn} \{ (k + 1)(\max(A, B) - A - B) + U_{k+1} - U_0 \} + \tau_{k+1} - \tau_0) & \text{for } n \geq k + 2 
\end{cases}
\] (18)
and

\[
U_n \begin{cases} 
= n \{ A + B - \max(A, B) \} + U_0 & \text{for } 1 \leq n \leq k \\
\leq \max(A, B) + k \{ A + B - \max(A, B) \} + U_0 & \text{for } n = k + 1 \\
= \max[\max(A, B) + U_{k+1}, A + B + k \{ A + B - \max(A, B) \} + U_0] \\
+ (n - k - 2) \max(A, B) & \text{for } n \geq k + 2.
\end{cases}
\]  

(19)

3. Otherwise,

\[\tau_n = \tau_0 \quad \text{for } n \geq 1\]  

(20)

and

\[U_n = n \{ A + B - \max(A, B) \} + U_0 \quad \text{for } n \geq 1.\]  

(21)

In all cases, the parity variable \(\tau\) tends to a constant while the amplitude variable \(U\) either grows linearly or tends to a constant, depending on the signs of \(A\) and \(B\).

Substituting (16)-(21) back into the ultradiscrete Cole-Hopf transformation (12) and (13), we obtain the following solution to the ultradiscrete Riccati equations (9) and (10):

1. If \(\sigma_1 = -1\) or \(X_1 \neq A + B - \max(A, B)\), we have

\[
\sigma_n = \begin{cases} 
\frac{1}{2} [1 - \sigma_1 + (1 + \sigma_1) \text{sgn} \{ \max(A, B) - A - B + X_1 \}] & \text{for } n = 2 \\
1 & \text{for } n \geq 3
\end{cases}
\]  

(22)

and

\[X_n = \begin{cases} 
\max(A, B, A + B - X_1) & \text{for } n = 2 \\
\max(A, B) & \text{for } n \geq 3.
\end{cases}\]  

(23)

2. Given some integer \(k \geq 1\), if \(\sigma_n = 1\) and \(X_n = A + B - \max(A, B)\) for \(1 \leq n \leq k\), but \(\sigma_{k+1} = -1\) or \(X_{k+1} \neq A + B - \max(A, B)\), we have

\[
\sigma_n = \begin{cases} 
1 & \text{for } 1 \leq n \leq k \\
\text{indeterminate} & \text{for } n = k + 1 \\
\frac{1}{2} [1 - \sigma_{k+1} + (1 + \sigma_{k+1}) \text{sgn} \{ \max(A, B) - A - B + X_{k+1} \}] & \text{for } n = k + 2 \\
1 & \text{for } n \geq k + 3
\end{cases}
\]  

(24)
3. Otherwise, we have

\[
\sigma_n = 1 \quad \text{for } n \geq 1
\]  \hspace{1cm} (26)

and

\[
X_n = A + B - \max(A, B) \quad \text{for } n \geq 1.
\]  \hspace{1cm} (27)

In all cases, we see that the solution tends to a constant value. This is analogous to the long time behaviour of the solutions of continuous and discrete Riccati equations.
II. SINGULARITY CONFINEMENT CRITERION FOR ULTRADISCRETE SYSTEMS

The singularity confinement criterion for ultradiscrete systems (with parity variables) was introduced by some of the present authors as an integrability detector i.e. as a criterion which would allow one to distinguish between integrable and nonintegrable systems. The pertinence of this criterion was displayed prominently\(^5\) where we have detailed its workings.

As an illustrative example, let us consider the autonomous limit of multiplicative dP\(^{12}\)

\[ x_{n+1}x_n x_{n-1} = ax_n + 1. \]  \hspace{1cm} (28)

This equation has an invariant and is integrable. A singular point of a second order difference equation is defined as the particular point \(x_n\) such that \(\partial x_{n+1}/\partial x_{n-1} = 0\) and \(\partial x_{n-k}/\partial x_{n-1} \neq 0\) (\(\forall k \geq 1\)) hold for generic \(x_{n-1}\). The singularity pattern is defined as the set of values \(\{x_n, x_{n+1}, \ldots\}\) such that \(\partial x_{n+k}/\partial x_{n-1} = 0\) (\(k \geq 1\)) hold. By this definition, the (unique) singular point of (28) is \(x_n = -1/a\) and its confined singularity pattern is given by \(\{-1/a, 0, \infty, \infty, 0, -1/a\}\) where regular values extend outside of this sequence.

Equation (28) can be ultradiscretised with parity variables for \(a > 0\) to the following pair of equations:

\[
\sigma_{n+1} = \begin{cases} 
\frac{\sigma_n - 1}{2} \{1 + \sigma_n + (1 - \sigma_n)\text{sgn}(A + X_n)\} & \text{if } \sigma_n = 1 \text{ or } X_n \neq -A \\
\text{indeterminate} & \text{if } \sigma_n = -1 \text{ and } X_n = -A
\end{cases} \]

and

\[
X_{n+1} = \begin{cases} 
\max(A, -X_n) - X_{n-1} & \text{if } \sigma_n = 1 \text{ or } X_n \neq -A \\
\leq A - X_{n-1} & \text{if } \sigma_n = -1 \text{ and } X_n = -A.
\end{cases} \]

In what follows, we assume \(A > 0\). The singular point of a second order ultradiscrete equation with parity variables is defined as the particular point \(X_n\) such that \(X_{n+1}\) is not uniquely determined for generic initial value \(X_{n-1}\). The singularity pattern is defined as the set of values \(\{X_{n+1}, X_{n+2}, \ldots\}\) that depend on the indeterminacy \(X_{n+1}\). By this definition, the singular point of (29) and (30) is \(X_n = (-1, -A)\), and taking this singular point gives rise to the singularity \(X_{n+1} = (\sigma_{n+1}, X_{n+1})\) where \(\sigma_{n+1}\) = indeterminate and \(X_{n+1} \leq A - X_{n-1}\).

Just as in the discrete case, to test (29) and (30) for ultradiscrete singularity confinement we iterate with an initial value \(X_0 = (\sigma_0, X_0)\) and the singular point \(X_1 = (-1, -A)\), and see how the singularity propagates. We see that the range \(X_2 \leq A - X_0\) depends on the initial
value $X_0$, and consequently the subsequent evolution depend on $X_0$ and how we choose the quantity $X_2$. For the case $X_0 < -A$, there exist the following three distinct singularity patterns.

1. If we choose $X_2 < -A$, then

\begin{align*}
\mathcal{X}_0 &= (\sigma_0, X_0) \\
\mathcal{X}_1 &= (-1, -A) \\
\mathcal{X}_2 &= (\sigma_2, X_2), \quad \text{where } \sigma_2 = \text{indeterminate and } X_2 < -A \\
\mathcal{X}_3 &= (-\sigma_2, A - X_2) \\
\mathcal{X}_4 &= (\sigma_2, A - X_2) \\
\mathcal{X}_5 &= (-\sigma_2, X_2) \\
\mathcal{X}_6 &= (-1, -A) \\
\mathcal{X}_7 &= (\sigma_7, X_7), \quad \text{where } \sigma_7 = \text{indeterminate and } X_7 \leq A - X_2.
\end{align*}

2. If we choose $-A < X_2 < 2A$, then

\begin{align*}
\mathcal{X}_0 &= (\sigma_0, X_0) \\
\mathcal{X}_1 &= (-1, -A) \\
\mathcal{X}_2 &= (\sigma_2, X_2), \quad \text{where } \sigma_2 = \text{indeterminate and } -A < X_2 < 2A \\
\mathcal{X}_3 &= (-1, 2A) \\
\mathcal{X}_4 &= (\sigma_2, A - X_2) \\
\mathcal{X}_5 &= (-1, -A) \\
\mathcal{X}_6 &= (\sigma_6, X_6), \quad \text{where } \sigma_6 = \text{indeterminate and } X_6 \leq X_2.
\end{align*}
3. If we choose $2A < X_2 \leq A - X_0$, then

\begin{align*}
X_0 &= (\sigma_0, X_0) \\
X_1 &= (-1, -A) \\
X_2 &= (\sigma_2, X_2), \text{ where } \sigma_2 = \text{indeterminate and } 2A < X_2 \leq A - X_0 \\
X_3 &= (-1, 2A) \\
X_4 &= (\sigma_2, A - X_2) \\
X_5 &= (-\sigma_2, -3A + X_2) \\
X_6 &= (\sigma_2, X_2) \\
X_7 &= (-\sigma_2, 4A - X_2) \\
& \vdots
\end{align*}

(33)

In (31), we see that the reappearance of the singular point at $X_6$ gives rise to a new indeterminate value $X_7$. The subsequent values are independent of the singularity $X_2$, and we call such a singularity pattern confined. Similarly, the singularity in (32) is also confined. In (33), on the other hand, the dependence on $X_2$ propagates indefinitely, and we call such a singularity pattern unconfined.

For other initial values $X_0$ we obtain only confined patterns regardless of the choice of $X_2$, namely (31) and (32) for the case $-A < X_0 < 2A$, and (31) for the case $X_0 > 2A$.

The integrability criterion is applied to our example in the following way. For $X_0 > -A$, the singularity is confined. For $X_0 < -A$, there exist both confined and unconfined singularity patterns, but the singularity can be made confined provided we restrict the range of the singularity $X_2 \leq A - X_0$ to $X_2 < 2A$. In short, for any given initial value $X_0$, the singularity is confined with a restriction on the singularity $X_2$ whenever necessary. In general, whenever, perhaps with the appropriate constraint which force a singularity to confine, we have only confined singularities for any initial value of $X_0$, we consider that our integrability criterion is satisfied. Therefore we conclude that (29) and (30) satisfy the integrability criterion.
III. NONINTEGRABLE ULTRADISCRETE SYSTEMS WITH CONFINED SINGULARITIES: HIETARINTA-VIALLET EQUATION

We start with the well-known Hietarinta-Viallet mapping\(^9\):

\[ x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2}. \]  

(34)

The singularity pattern of (34) is \(\{0, \infty, \infty, 0\}\). Despite this confined singularity pattern the mapping was shown to be nonintegrable.

Note that the singular point \(x_n = 0\) of (34) vanishes in taking an ultradiscrete limit. We therefore use a change of variable \(\hat{x}_n = x_n + 1\) to shift this singular point to \(\hat{x}_n = 1\), and ultradiscretize the resulting equation for \(\hat{x}\) with parity variables. This leads to the following pair of equations:

\[
\sigma_{n+1} = \begin{cases} 
\text{indeterminate} & \text{if } \sigma_n = 1 \text{ and } X_n = 0 \\
\text{indeterminate} & \text{if } \sigma_n = -1 \text{ and } X_n = 0 \geq X_{n-1} \\
\text{indeterminate} & \text{if } \sigma_n = \sigma_{n-1} \text{ and } X_n = X_{n-1} > 0 \\
\text{indeterminate} & \text{if } \sigma_{n-1} = 1 \text{ and } X_{n-1} = 0 > X_n \\
\frac{1}{4}[1 + \text{sgn}(\max(X_n, X_{n-1}))][\sigma_n - \sigma_{n-1} + (\sigma_n + \sigma_{n-1})\text{sgn}(X_n - X_{n-1})] + \frac{1}{2}[1 - \text{sgn}(\max(X_n, X_{n-1}))] & \text{otherwise}
\end{cases}
\]

(35)

and

\[
X_{n+1} = \begin{cases} 
= \text{arbitrary} & \text{if } \sigma_n = 1 \text{ and } X_n = 0 \\
\leq 0 & \text{if } \sigma_n = -1 \text{ and } X_n = 0 \geq X_{n-1} \\
\leq X_n & \text{if } \sigma_n = \sigma_{n-1} \text{ and } X_n = X_{n-1} > 0 \\
\leq 0 & \text{if } \sigma_{n-1} = 1 \text{ and } X_{n-1} = 0 > X_n \\
= \max(X_n, X_{n-1}, 0) & \text{otherwise}.
\end{cases}
\]

(36)

We will call (35), (36) ultradiscrete Hietarinta-Viallet equations. The (unique) singular point is \(X_n = (1, 0)\). As usual we iterate with an initial value \(X_0 = (\sigma_0, X_0)\) and the singular point \(X_1 = (1, 0)\), which gives \(X_2 = (\sigma_2, X_2)\) where \(\sigma_2 = \text{indeterminate}\) and \(X_2 = \text{arbitrary}\). For any \(X_0\), the singularity patterns vary depending on the choice of \(X_2\). If we choose \(X_2 < 0\), then we have \(X_3 = (\sigma_3, X_3)\) where \(\sigma_3 = \text{indeterminate}\) and \(X_3 \leq 0\), and the subsequent values depend on the choice of this indeterminacy. If we choose \(X_2 > 0\), then we have
\( \mathcal{X}_4 = (\sigma_4, X_4) \) where \( \sigma_4 \) = indeterminate and \( X_4 \leq X_2 \), and again the subsequent values depend on the choice of this indeterminacy.

1. If we choose \( X_2 < 0 \) and \( X_3 < 0 \), then

\[
\begin{align*}
\mathcal{X}_0 &= (\sigma_0, X_0) \\
\mathcal{X}_1 &= (1, 0) \\
\mathcal{X}_2 &= (\sigma_2, X_2), \quad \text{where } \sigma_2 = \text{indeterminate and } X_2 < 0 \\
\mathcal{X}_3 &= (\sigma_3, X_3), \quad \text{where } \sigma_3 = \text{indeterminate and } X_3 < 0 \\
\mathcal{X}_4 &= (1, 0) \\
\mathcal{X}_5 &= (\sigma_5, X_5), \quad \text{where } \sigma_5 = \text{indeterminate and } X_5 = \text{arbitrary}.
\end{align*}
\]

(37)

2. If we choose \( X_2 < 0 \), \( X_3 = (1, 0) \), then

\[
\begin{align*}
\mathcal{X}_0 &= (\sigma_0, X_0) \\
\mathcal{X}_1 &= (1, 0) \\
\mathcal{X}_2 &= (\sigma_2, X_2), \quad \text{where } \sigma_2 = \text{indeterminate and } X_2 < 0 \\
\mathcal{X}_3 &= (1, 0) \\
\mathcal{X}_4 &= (\sigma_4, X_4), \quad \text{where } \sigma_4 = \text{indeterminate and } X_4 = \text{arbitrary}.
\end{align*}
\]

(38)

3. If we choose \( X_2 < 0 \), \( X_3 = (-1, 0) \), then

\[
\begin{align*}
\mathcal{X}_0 &= (\sigma_0, X_0) \\
\mathcal{X}_1 &= (1, 0) \\
\mathcal{X}_2 &= (\sigma_2, X_2), \quad \text{where } \sigma_2 = \text{indeterminate and } X_2 < 0 \\
\mathcal{X}_3 &= (-1, 0) \\
\mathcal{X}_4 &= (\sigma_4, X_4), \quad \text{where } \sigma_4 = \text{indeterminate and } X_4 \leq 0.
\end{align*}
\]

(39)
4. If we choose $X_2 > 0$, $X_4 \neq 0$ or $X_1 \neq (\sigma_2, X_2)$, then

$$X_0 = (\sigma_0, X_0)$$
$$X_1 = (1, 0)$$
$$X_2 = (\sigma_2, X_2), \text{ where } \sigma_2 = \text{indeterminate and } X_2 > 0$$
$$X_3 = (\sigma_2, X_2)$$
$$X_4 = (\sigma_4, X_4), \text{ where } \sigma_4 = \text{indeterminate and } X_4 \leq X_2$$
$$X_5 = (-\sigma_2, X_2)$$
$$X_6 = (-\sigma_2, X_2)$$

5. If we choose $X_2 > 0$, $X_4 = (1, 0)$, then

$$X_0 = (\sigma_0, X_0)$$
$$X_1 = (1, 0)$$
$$X_2 = (\sigma_2, X_2), \text{ where } \sigma_2 = \text{indeterminate and } X_2 > 0$$
$$X_3 = (\sigma_2, X_2)$$
$$X_4 = (1, 0)$$
$$X_5 = (\sigma_5, X_5), \text{ where } \sigma_5 = \text{indeterminate and } X_5 = \text{arbitrary.}$$

6. If we choose $X_2 > 0$, $X_4 = (-1, 0)$, then

$$X_0 = (\sigma_0, X_0)$$
$$X_1 = (1, 0)$$
$$X_2 = (\sigma_2, X_2), \text{ where } \sigma_2 = \text{indeterminate and } X_2 > 0$$
$$X_3 = (\sigma_2, X_2)$$
$$X_4 = (-1, 0)$$
$$X_5 = (\sigma_5, X_5), \text{ where } \sigma_5 = \text{indeterminate and } X_5 \leq 0.$$
7. If we choose $X_2 > 0$ and $\mathcal{X}_4 = (\sigma_2, X_2)$, then

\[
\begin{align*}
\mathcal{X}_0 &= (\sigma_0, X_0) \\
\mathcal{X}_1 &= (1, 0) \\
\mathcal{X}_2 &= (\sigma_2, X_2), \quad \text{where } \sigma_2 = \text{indeterminate and } X_2 > 0 \\
\mathcal{X}_3 &= (\sigma_2, X_2) \\
\mathcal{X}_4 &= (\sigma_2, X_2) \\
\mathcal{X}_5 &= (\sigma_5, X_5), \quad \text{where } \sigma_5 = \text{indeterminate and } X_5 \leq X_2
\end{align*}
\] (43)

In (37)-(39) the singularity $\mathcal{X}_2$ is confined, but in (40) and (43) the singularity propagates indefinitely. For any $X_0$, the singularity is confined provided we impose the constraint $X_2 < 0$, and therefore the ultradiscrete Hietarinta-Viallet equations satisfy the integrability criterion.

However, one remark needs to be addressed here. That is, the confined singularity patterns (37)-(39) do not correspond to the one obtained from the difference equation (34). The ultradiscrete equivalent of the singularity pattern of (34) is (41). It is interesting to note that the ultradiscrete Hietarinta-Viallet equations satisfy the integrability criterion but its confined singularity structure is not the one found from the difference equation.
IV. INTEGRABLE ULTRADISCRETE SYSTEMS WITH UNCONFINED SINGULARITIES: TSUDA ET AL. LINEARISABLE EQUATION

Tsuda et al. introduced the following linearisable mapping

\[ x_{n+1} = ax_{n-1} \frac{x_n + a}{x_n + 1}. \]  \hspace{1cm} (44)

Equation (44) has one singular point \( x_n = -1 \) with a confined singularity pattern \( \{ -1, \infty, -a \} \), and another singular point \( x_n = -a \) with an unconfined pattern \( \{ -a, 0, -a^3, 0, -a^5, \ldots \} \).

Since there exists one singular point whose singularity pattern is unconfined, the integrability criterion is violated.

Ultradiscretising (44) with parity variables for \( a > 0 \) leads to the pair

\[ \sigma_{n+1} = \begin{cases} \frac{\sigma_n - 1}{2} \{ \sigma_n + 1 + (\sigma_n - 1)\text{sgn}(X_n)\text{sgn}(A - X_n) \} & \text{if } \sigma_n = 1, \text{ or } X_n \neq 0 \text{ and } X_n \neq A \\ \text{indeterminate} & \text{if } \sigma_n = -1 \text{ and } X_n = 0 \\ \text{indeterminate} & \text{if } \sigma_n = -1 \text{ and } X_n = A \end{cases} \]  \hspace{1cm} (45)

and

\[ X_{n+1} = \begin{cases} A + X_{n-1} + \max(X_n, A) - \max(X_n, 0) & \text{if } \sigma_n = 1, \text{ or } X_n \neq 0 \text{ and } X_n \neq A \\ \geq A + X_{n-1} + \max(A, 0) & \text{if } \sigma_n = -1 \text{ and } X_n = 0 \\ \leq 2A + X_{n-1} - \max(A, 0) & \text{if } \sigma_n = -1 \text{ and } X_n = A. \end{cases} \]  \hspace{1cm} (46)

In what follows, we assume \( A > 0 \). The singularity pattern through the first singular point \( X_1 = (-1, 0) \) under the condition \( X_0 > -A \) is given below:

\[ X_0 = (\sigma_0, X_0) \]
\[ X_1 = (-1, 0) \]
\[ X_2 = (\sigma_2, X_2), \text{ where } \sigma_2 = \text{indeterminate and } X_2 \geq 2A + X_0 \]  \hspace{1cm} (47)
\[ X_3 = (-1, A) \]
\[ X_4 = (\sigma_4, X_4), \text{ where } \sigma_4 = \text{indeterminate and } X_4 \leq A + X_2. \]

The singularity in (47) is confined. For \( X_0 < -A \), there also exist unconfined patterns but the confined pattern (47) is obtained provided we choose \( X_2 > A \). Thus, as far as this first
singular point is concerned, the singularity is confined.

On the other hand, the singularity pattern through the second singular point \( X_1 = (-1, A) \) with \( X_0 < -2A \) is as follows.

\[
X_0 = (\sigma_0, X_0)\\
X_1 = (-1, A)\\
X_2 = (\sigma_2, X_2), \text{ where } \sigma_2 = \text{indeterminate and } X_2 \leq A + X_0 \\
X_3 = (-1, 3A)\\
X_4 = (\sigma_2, A + X_2)\\
X_5 = (-1, 5A)\\
X_6 = (\sigma_2, 2A + X_2) \\
\vdots
\]

As expected from the discrete case, the singularity in (48) is not confined. Since there exists one case for the initial value \( X_0 \) such that no confined singularity pattern exists, (45) and (46) do not satisfy the integrability criterion.

The general expression for the pattern (47) can be found from the exact solutions of the ultradiscrete equations (45), (46).

V. CONCLUSION

In this paper, we have investigated the integrability detector for ultradiscrete systems with parity variables. This detector is a transposition of the singularity cofinement discrete integrability detector to an ultradiscrete setting. The singularity confinement criterion is based on the requirement that every spontaneously appearing singularity disappears after a certain number of iteration steps. In the case of ultradiscrete systems with parity variables under consideration, the situation is somewhat more complex. If for all initial conditions only confined singularities exist, then the system does satisfy the confinement criterion, while if for some initial condition there exist only unconfined singularities, then the confinement criterion is not satisfied. The situation is more subtle when for some initial condition there exist both confined and unconfined singularity patterns. The way to apply our criterion in this case is to examine the unconfined singularities and the conditions for their existence:
if with the application of suitable constraints that are independent of the initial condition we can make the unconfined singularities disappear, then we claim that the singularity confinement criterion is satisfied.

One advantage of this integrability detector is that it is applicable to a wider class of ultradiscrete equations, including the ones obtained from difference equations of the form $x_{n+1} + x_{n-1} = f(x_n)$. The Hietarinta-Viallet equation (34) is such an example. The criterion works in close parallel to that of the discrete case, and consequently there exist nonintegrable ultradiscrete equations the singularities of which are confined. At that level, the results based on our method are in perfect parallel to the ones obtained with the Joshi-Larfortune approach (but we must stress once more that our method has a much wider applicability).

The application of our criterion to linearisable systems led mostly to results which were expected but also to, on the surface, a puzzling one. Just as in the discrete case, the projective systems satisfy the integrability requirement, while systems linearisable through some more complicated methods need not. A most interesting result was the one obtained for the limit $n \to \infty$ of the Gambier mapping. As explained, the interpretation, in the discrete case, was that the mapping does confine, albeit after an infinite number of steps (which in principle can be interpreted as “non confinement”) and in the light of this, the confinement property in the ultradiscrete case should not be astonishing. It is worth pointing out that we were able to construct the exact solution of the linearisable equations and thus make explicit the structure of their singularities (the analogous approach in the discrete case is much harder and in some cases next to impossible).

Just as with the singularity confinement test for discrete equations, the ultradiscrete singularity confinement furnishes only a local information about the solution which may be insufficient for the detection of integrability of the equation. For the latter it is important to combine the local singularity structure with the global behaviour of the solution. Such an approach has been already proposed by Halburd and Southall in a “standard” ultradiscrete setting. It would be interesting to investigate the possibility of extending their approach to the case of ultradiscrete systems with parity variables.
Appendix A: Solutions to (45) and (46)

It was shown\textsuperscript{13} that (44) could be linearised by the change of variable

\[ y_n = (x_n + a)(x_{n-1} + 1) \]  

(A1)

to the following first order equation for \( y \):

\[ y_{n+1} - ay_n + a(a - 1) = 0, \]  

(A2)

whose general solution is given by

\[ y_n = a^{n-1}y_1 + a - a^n. \]  

(A3)

To solve (A1) for \( x_n \), we use the Cole-Hopf transformation

\[ x_n = \frac{u_{n+1}}{u_n} - 1, \]  

(A4)

which transforms (A1) into the following second order linear equation for \( u \):

\[ u_{n+1} + (a - 1)u_n - y_n u_{n-1} = 0, \]  

(A5)

where \( y_n \) is given by (A3). Substituting the solution of (A5) into (A4) gives the solution to (44).

To find the ultradiscrete counterpart of above, we start with the ultradiscretisation of (A3) with parity variables. Writing \( y_n = \tau_n \tilde{y}_n \) where \( \tau_n \in \{-1,1\} \) and \( \tilde{y}_n > 0 \), and using the exponential ansatz \( a = e^{A/\delta} \) and \( \tilde{y} = e^{Y/\delta} \), we obtain the following solution in the limit \( \delta \to +0 \) for \( Y_1 > A > 0 \):

\[ \tau_n = \tau_1 \quad \text{for } n \geq 2 \]  

(A6)

and

\[ Y_n = (n - 1)A + Y_1 \quad \text{for } n \geq 2. \]  

(A7)

Next, we write \( u_n = \rho_n \tilde{u}_n \) where \( \rho_n \in \{-1,1\} \) and \( \tilde{u}_n > 0 \), and use the ansatz \( \tilde{u} = e^{U/\delta} \) to the linearised equation (A5). In taking the limit \( \delta \to +0 \), we obtain the following equations:

\[ \rho_{n+1} = \begin{cases} 
\frac{1}{2}\{\tau_n\rho_{n-1} - \rho_n + (\tau_n\rho_{n-1} + \rho_n)\text{sgn}(-A + Y_n + U_{n-1} - U_n)\} \\
\text{if } \rho_n = -\tau_n\rho_{n-1} \text{ or } A + U_n \neq Y_n + U_{n-1} \\
\text{indeterminate} \\
\text{if } \rho_n = \tau_n\rho_{n-1} \text{ and } A + U_n = Y_n + U_{n-1} 
\end{cases} \]  

(A8)
where \(\tau_n\) and \(Y_n\) are given by (A6) and (A7), respectively.

We present below the solutions to (A8) and (A9) for particular initial values \(U_0\) and \(U_1\). The solutions for other initial values can be obtained similarly.

1. If \(0 < U_1 - U_0 < Y_1 - A\), then

\[
\rho_n = \begin{cases} 
\frac{n^2}{4}\rho_0 & \text{for } n = 2, 4, 6, \ldots \\
\frac{n^2}{4}\rho_1 & \text{for } n = 3, 5, 7, \ldots
\end{cases}
\]

and

\[
U_n = \begin{cases} 
\frac{n(n-2)}{4}A + \frac{n}{2}Y_1 + U_0 & \text{for } n = 2, 4, 6, \ldots \\
\frac{(n-1)^2}{4}A + \frac{n-1}{2}Y_1 + U_1 & \text{for } n = 3, 5, 7, \ldots
\end{cases}
\]

2. If \(\rho_1 = \tau_1\rho_0\) and \(U_1 - U_0 = Y_1 - A\), then we have \(\rho_2\) indeterminate and \(U_2 \leq A + U_1\).

The subsequent values for \(\rho_2 = -\rho_1\) or \(U_2 \neq U_1\) are given by

\[
\rho_n = \begin{cases} 
\text{indeterminate} & \text{for } n = 2 \\
\frac{n^2}{4}\rho_1 & \text{for } n = 3, 5, 7, \ldots \\
\frac{n^2}{4}\left\{\rho_2 - \rho_1 + (\rho_2 + \rho_1)\text{sgn}(U_2 - U_1)\right\} & \text{for } n = 4, 6, 8, \ldots
\end{cases}
\]

and

\[
U_n = \begin{cases} 
\leq A + U_1 & \text{for } n = 2 \\
\frac{(n-1)^2}{4}A + \frac{n-1}{2}Y_1 + U_1 & \text{for } n = 3, 5, 7, \ldots \\
\frac{n(n-2)}{4}A + \frac{n-2}{2}Y_1 + \max(U_2, U_1) & \text{for } n = 4, 6, 8, \ldots
\end{cases}
\]

In both cases, the solution \(U_n\) exhibits a quadratic growth in the coefficient of \(A\).

The Cole-Hopf transformation (A4) is ultradiscretised to the following pair of equations:

\[
\sigma_n = \begin{cases} 
\frac{1}{2}\left\{\rho_{n+1}\rho_n - 1 + (\rho_{n+1}\rho_n + 1)\text{sgn}(U_{n+1} - U_n)\right\} & \text{if } \rho_{n+1} = -\rho_n \text{ or } U_{n+1} \neq U_n \\
\text{indeterminate} & \text{if } \rho_{n+1} = \rho_n \text{ and } U_{n+1} = U_n
\end{cases}
\]
and
\[ X_n \begin{cases} = \max(U_{n+1} - U_n, 0) & \text{if } \rho_{n+1} = -\rho_n \text{ or } U_{n+1} \neq U_n, \\ \leq 0 & \text{if } \rho_{n+1} = \rho_n \text{ and } U_{n+1} = U_n, \end{cases} \]  
(A15)

where \( \rho_n \) and \( U_n \) are given by (A10)-(A13).

Substituting (A10)-(A13) into the ultradiscrete Cole-Hopf transformation (A14) and (A15), we can obtain the following solutions to (45) and (46) for given initial values \( X_0 \) and \( X_1 \).

1. If \( X_0 = (\sigma_0, X_0) \) with \( \sigma_0 = \) arbitrary and \( X_0 > 0 \), \( X_1 = (\sigma_1, X_1) \) with \( \sigma_1 = \) arbitrary and \( X_1 > A \), then

\[ \sigma_n = \begin{cases} \sigma_0 & \text{for } n = 2, 4, 6, \ldots \\ \sigma_1 & \text{for } n = 3, 5, 7, \ldots \end{cases} \]  
(A16)

and

\[ X_n = \begin{cases} \frac{n}{2}A + X_0 & \text{for } n = 2, 4, 6, \ldots \\ \frac{n-1}{2}A + X_1 & \text{for } n = 3, 5, 7, \ldots \end{cases} \]  
(A17)

2. If \( X_0 = (\sigma_0, X_0) \) with \( \sigma_0 = \) arbitrary and \( X_0 > 0 \), and \( X_1 = (1, A) \), then

\[ \sigma_n = \begin{cases} \sigma_0 & \text{for } n = 2, 4, 6, \ldots \\ 1 & \text{for } n = 3, 5, 7, \ldots \end{cases} \]  
(A18)

and

\[ X_n = \begin{cases} \frac{n}{2}A + X_0 & \text{for } n = 2, 4, 6, \ldots \\ \frac{n+1}{2}A & \text{for } n = 3, 5, 7, \ldots \end{cases} \]  
(A19)

3. If \( X_0 = (\sigma_0, X_0) \) with \( \sigma_0 = \) arbitrary and \( X_0 > 0 \), and \( X_1 = (\sigma_1, X_1) \) with \( \sigma_1 = \) arbitrary and \( 0 < X_1 < A \), then

\[ \sigma_n = \begin{cases} \sigma_0 \sigma_1 & \text{for } n = 2, 4, 6, \ldots \\ \sigma_1 & \text{for } n = 3, 5, 7, \ldots \end{cases} \]  
(A20)

and

\[ X_n = \begin{cases} \frac{n+2}{2}A + X_0 - X_1 & \text{for } n = 2, 4, 6, \ldots \\ \frac{n-1}{2}A + X_1 & \text{for } n = 3, 5, 7, \ldots \end{cases} \]  
(A21)

4. If \( X_0 = (\sigma_0, X_0) \) with \( \sigma_0 = \) arbitrary and \( X_0 > 0 \), and \( X_1 = (-1, 0) \), then

\[ \sigma_n = \begin{cases} \text{indeterminate} & \text{for } n = 2 \\ -1 & \text{for } n = 3, 5, 7, \ldots \\ -\sigma_0 & \text{for } n = 4, 6, 8, \ldots \end{cases} \]  
(A22)

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and

\[
X_n \left\{ \begin{array}{ll}
\geq 2A + X_0 & \text{for } n = 2 \\
= \frac{n-1}{2}A & \text{for } n = 3, 5, 7, \ldots \\
= \frac{n+2}{2}A + X_0 & \text{for } n = 4, 6, 8, \ldots 
\end{array} \right. 
\]  \tag{A23}

The variables \(\sigma_n\) and \(X_n\) of the confined pattern (47) obey (A22) and (A23), respectively. We see that the singularity \(X_2\) disappears and the initial value \(X_0\) is recovered.