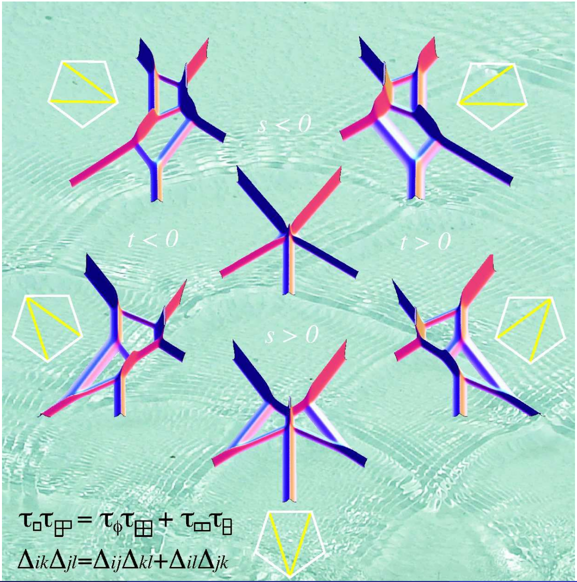
KP solitons, total positivity, and cluster algebras

Lauren K. Williams,
UC Berkeley



(joint work with Yuji Kodama)

Plan of the talk

Understanding regular soliton solutions to the KP equation via total positivity for the Grassmannian and cluster algebras.

- Background on total positivity on the Grassmannian
- Background on the KP equation
- Tight connection between KP solitons and total positivity
- Application 1: classification of soliton graphs
- Application 2: connection to cluster algebras
- Application 3: the inverse problem for soliton graphs

Total positivity on the Grassmannian

The real Grassmannian and its non-negative part

The Grassmannian $Gr_{k,n}(\mathbb{R}) = \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

Given $I \in \binom{[n]}{k}$, the Plücker coordinate $\Delta_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The totally non-negative part of the Grassmannian $(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ where all Plucker coordinates $\Delta_I(A)$ are non-negative.

Context

1930's: Beginning of classical theory of totally positive matrices, matrices with all minors positive.

1990's: Lusztig developed total positivity in Lie theory. Provided part of the motivation for Fomin-Zelevinsky's cluster algebras.

2001-2006: Postnikov defined and studied the totally non-negative part of the real Grassmannian.

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Total positivity on the Grassmannian

Postnikov studied $(Gr_{k,n})_{\geq 0}$, found a nice cell decomposition, and showed that cells were in bijection with several interesting combinatorial objects.

Theorem (Postnikov)

Given a subset $\mathcal{M}$ of $\binom{[n]}{k}$, define the positroid cell

$$\mathcal{S}_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid \Delta_I(A) > 0 \textit{ iff } I \in \mathcal{M}\}.$$

If $\mathcal{S}_{\mathcal{M}}^{tnn} \neq \emptyset$, then it is a cell (homeomorphic to an open ball).

We will be interested in only the *irreducible* positroid cells.
(in row-echelon form, its elements don't contain all-zero column or a row which contains all zeros besides the pivot)

Total positivity on the Grassmannian

Theorem (Postnikov)

The (irreducible) positroid cells of $(Gr_{k,n})_{\geq 0}$ are in bijection with (and labeled by):

- *Derangements $\pi \in S_n$ with k excedances*
- *(Irreducible) ⅃-diagrams L contained in $k \times (n-k)$ rectangle*
- *(Irred.) Equivalence classes of reduced plabic graphs G of type (k, n)*

(3,4,5,1,2) =

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 5 & 1 & 2 \end{array}$$

If $\mathcal{S}_{\mathcal{M}}^{tnn}$ is labeled by the derangement π, we also refer to the cell as $\mathcal{S}_{\pi}^{tnn}$. (Similarly for L, G.)

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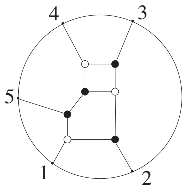

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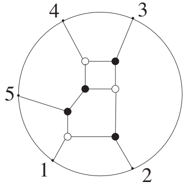

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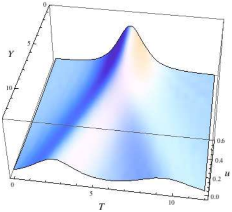
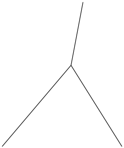
The Kadomtsev-Petviashvili equation

The KP equation

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+3\frac{\partial^2 u}{\partial y^2}=0$$

- Proposed by Kadomtsev and Petviashvili in 1970, in order to study the stability of the one-soliton solution of the Korteweg-de Vries (KdV) equation under the influence of weak transverse perturbations.
- Gives an excellent model to describe shallow water waves





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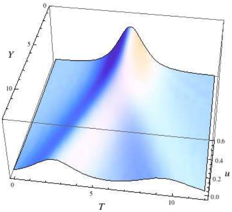

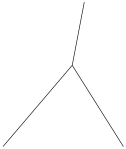

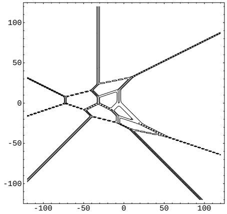

Soliton solutions to the KP equation

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From $A \in Gr_{k,n}(\mathbb{R})$, can construct τ_A, and then a solution u_A of the KP equation.
(cf Sato, Hirota, Satsuma, Freeman-Nimmo, ...)

The τ function τ_A is given by a Wronskian

Fix real parameters κ_j such that $\kappa_1 < \kappa_2 < \cdots < \kappa_n$.
Define $E_j(t_1, \ldots, t_n) := \exp(\kappa_j t_1 + \kappa_j^2 t_2 + \cdots + \kappa_j^n t_n)$.
For $J = \{j_1, \ldots, j_k\} \subset [n]$, define $E_J := E_{j_1} \ldots E_{j_k} \prod_{\ell < m}(\kappa_{j_m} - \kappa_{j_\ell})$.
The τ-function is

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A solution $u_A(x, y, t)$ of the KP equation

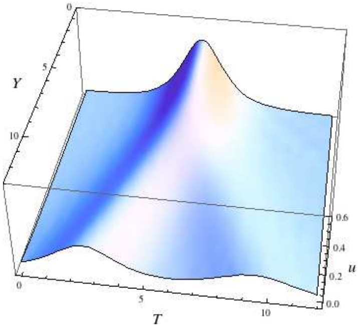
Set $x = t_1, y = t_2, t = t_3$ (treat other t_i's as constants). Then

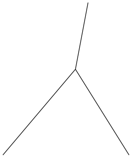
$$u_A(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \tau_A(x, y, t) \tag{1}$$

is a solution to the KP equation.
If all $\Delta_I(A) \geq 0$, this solution is everywhere regular.

The contour plot of $u_A(x, y, t)$

We analyze $u_A(x, y, t)$ by fixing t, and drawing its *contour plot* $\mathcal{C}_t(u_A)$ for fixed times t – this will approximate the subset of $\mathbb{R}^2$ where $u_A(x, y, t)$ takes on its maximum values.





Definition of the contour plot as tropical curve

Choose $A \in S_{\mathcal{M}}^{tnn}$.

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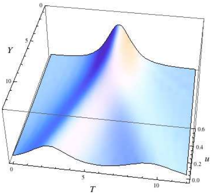
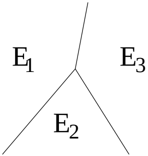
The *contour plot* $\mathcal{C}_t(u_A)$ is the subset of $\mathbb{R}^2$ where $f_A(x,y,t)$ is not linear.





One term E_I dominates u_A in each region of the complement of $\mathcal{C}_t(u_A)$.

Label each region by the dominant exponential.

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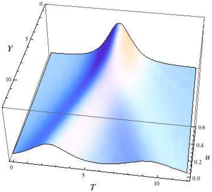

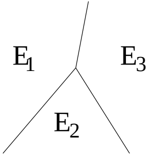

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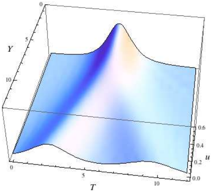

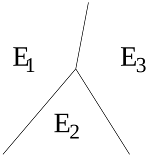

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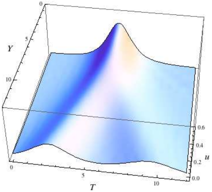

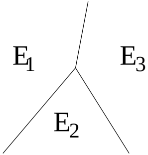

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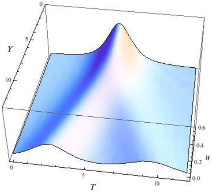

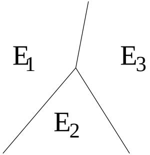

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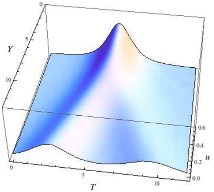

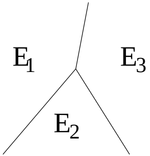

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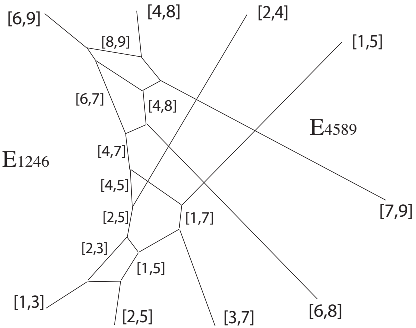
Visualizing soliton solutions to the KP equation

Generically, interactions of *line-solitons* are trivalent or are *X-crossings*.

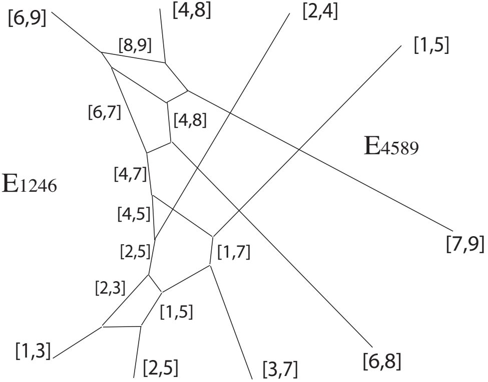


If two adjacent regions are labeled E_I and E_J, then $J = (I \setminus \{i\}) \cup \{j\}$.[a]
The line-soliton between the regions has slope $\kappa_i + \kappa_j$; label it $[i, j]$.

[a]Phase shifts negligable if differences $\kappa_{i+1} - \kappa_i$ are similar and graph on large scale.

Soliton graphs

We associate a *soliton graph* $G_t(u_A)$ to a contour plot $\mathcal{C}_t(u_A)$ by: forgetting lengths and slopes of edges, and marking a trivalent vertex black or white based on whether it has a unique edge down or up.



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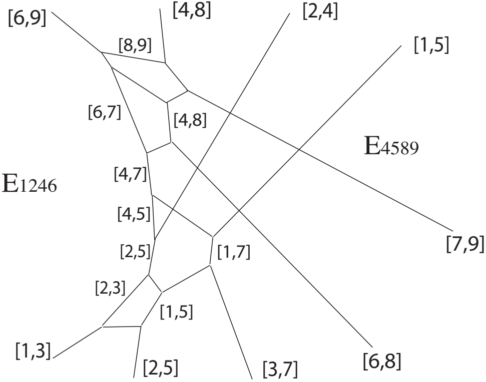

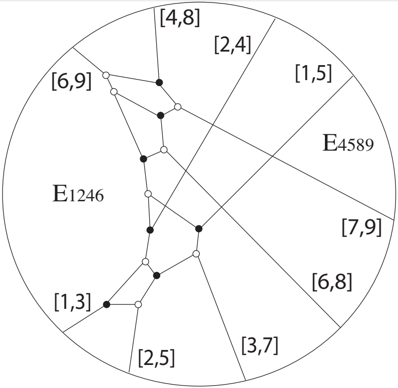

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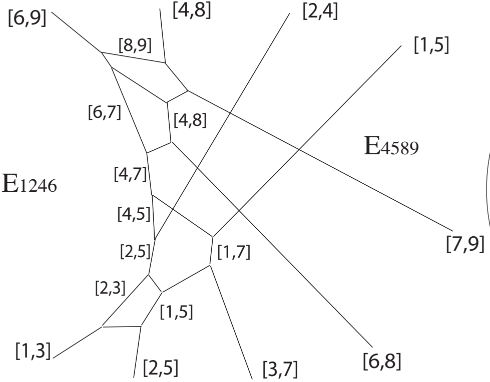

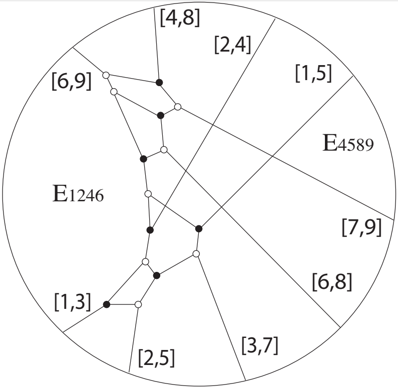

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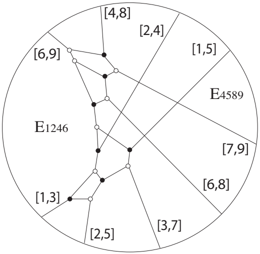
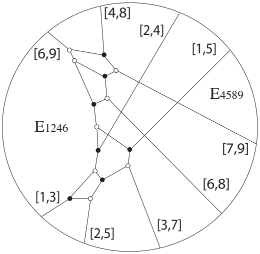


Let A be an element of an (irreducible) positroid cell in $(Gr_{kn})_{\geq 0}$. What can we say about the soliton graph $G_t(u_A)$?

Metatheorem (Kodama-W.)

Which cell A lies in determines the asymptotics of $G_t(u_A)$ as $y \to \pm\infty$ and $t \to \pm\infty$. Use the derangement and ⅃*-diagram labeling the cell.*



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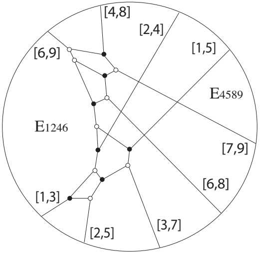
How the positroid cell determines asymptotics at $y \to \pm\infty$

Recall: positroid cells in $(Gr_{kn})_{\geq 0} \leftrightarrow$ derangements $\pi \in S_n$ with k exc.

Theorem (Biondini-Chakravarty, Chakravarty-Kodama + Kodama-W.)

Let A lie in the irreducible positroid cell $\mathcal{S}_\pi^{tnn}$ of $(Gr_{kn})_{\geq 0}$.

For any t, the soliton graph $G_t(u_A)$ has precisely:
k line-solitons at $y >> 0$, labeled by the excedances $[i, \pi(i)]$ of π, and
$n-k$ line-solitons at $y << 0$ labeled by the nonexcedances $[i, \pi(i)]$.



$G_t(u_A)$ where $A \in \mathcal{S}_\pi^{tnn}$ for $\pi = (5, 4, 1, 8, 2, 9, 3, 6, 7)$.

How the positroid cell determines asymptotics at $t \longrightarrow -\infty$

Recall: positroid cells in $(Gr_{k,n})_{\geq 0} \leftrightarrow$ Ⅎ-diagrams contained in $k \times (n-k)$ rectangle

Definition

A Ⅎ-diagram is a filling of the boxes of a Young diagram by $+$'s and 0's such that: there is no 0 with a $+$ above it in the same column, and a $+$ to its left in the same row.

0	0	+	+	+
0	0	0	+	+
+	+	+	+	
+	+	+		

How the positroid cell determines asymptotics at $t \longrightarrow -\infty$

Theorem (Kodama-W.)

Let L be a ⅃-diagram. The following procedure realizes the soliton graph $G_t(u_A)$ for $A \in \mathcal{S}_L^{tnn}$ and $t << 0$.

0	0	+	+	+
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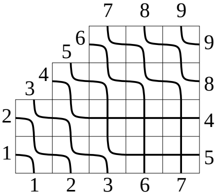

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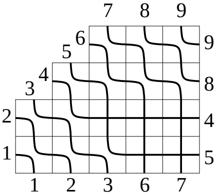

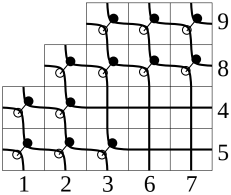

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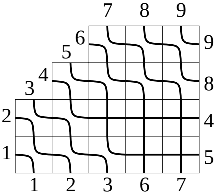

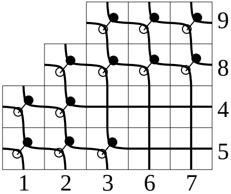

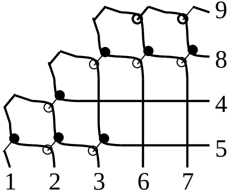

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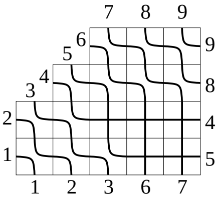
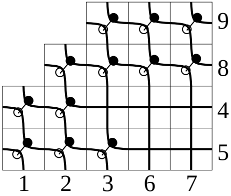
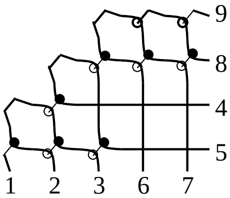
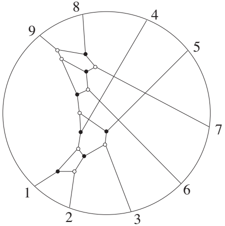








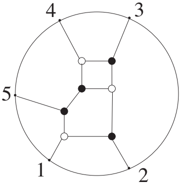
Plabic graphs and soliton graphs

Recall: Positroid cells $\leftrightarrow$ equivalence classes of reduced plabic graphs.

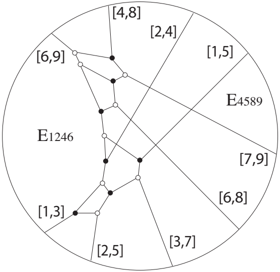
We've seen that derangements and Ꮈ-diagrams are very useful for understanding soliton graphs. What about plabic graphs?

Definition

A *plabic graph* is a planar undirected graph G inside a disk with n *boundary vertices* $1, \ldots, n$ in counterclockwise order around the disk's boundary, such that each boundary vertex i is incident to a single edge. Interior vertices are colored black or white.



Soliton graph → *generalized* plabic graph



A *generalized plabic graph* is a graph in a disk with *n boundary vertices* $1, \ldots, n$ *labeled in any order* around the bdry, s.t. each bdry vertex has degree 1. *Two edges may cross.* Interior vertices colored black or white.

Associate a *generalized plabic graph* to each soliton graph by:

- Labeling each bdry vertex incident to the line-soliton $[i, \pi(i)]$ by $\pi(i)$.
- Forgetting the labels of line-solitons and regions.

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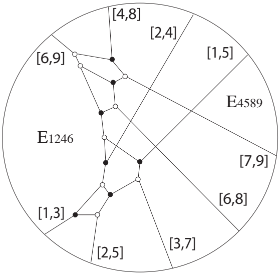

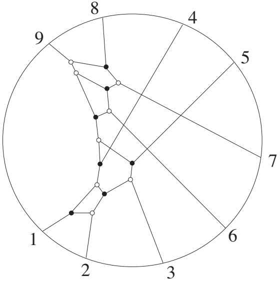

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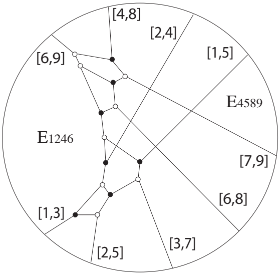

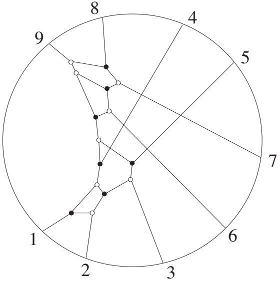

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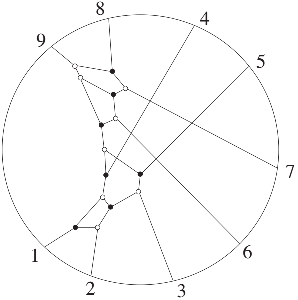
Passing from soliton graph $\rightarrow$ generalized plabic graph does not lose any information!

Theorem (Kodama-W.)

We can reconstruct the labels of line-solitons by following the "rules of the road." From the boundary vertex i, turn right at black and left at white. Label each edge along trip with i, and each region to the left of trip by i.



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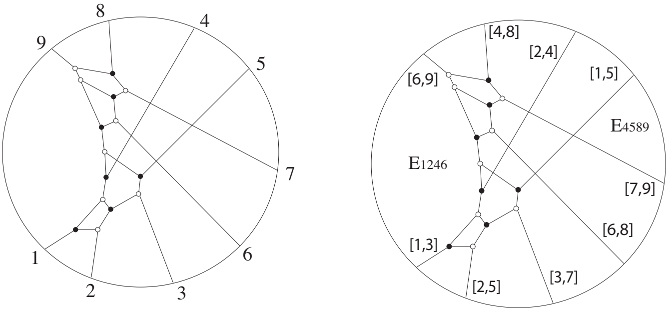

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Cluster algebras and the Grassmannian

Definition (Cluster algebra, cf Fomin and Zelevinsky)

A *cluster algebra* is "a kind of commutative ring with a great deal of structure." It has a distinguished family of generators called *cluster variables*, which are naturally grouped into generating sets called *clusters*.

Can be viewed as kind of discrete dynamical system; close relation with T-systems, Q-systems, etc. Dynamics is governed by *quiver mutation*.

Theorem (J. Scott)

The coordinate ring $\mathbb{C}[Gr_{k,n}]$ of the Grassmannian has a natural cluster algebra structure. The Plücker coordinates comprise some of the cluster variables.

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Soliton graphs and clusters

Definition (The totally positive Grassmannian $(Gr_{k,n})_{>0}$)

$(Gr_{k,n})_{>0}$ is the set of $A \in Gr_{k,n}$ s.t. $\Delta_I(A) > 0 \forall I$.

Theorem (Kodama-W.)

Let $A \in (Gr_{k,n})_{>0}$, and consider the soliton graph $G_t(u_A)$. If it is generic (all vertices trivalent), then the set of dominant exponentials labeling $G_t(u_A)$ forms a cluster for the cluster algebra associated to the Grassmannian.

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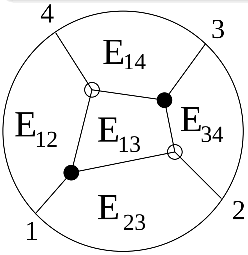

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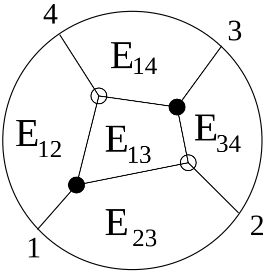

Corollary

Let $A \in (Gr_{k,n})_{>0}$, and consider the soliton graph $G_t(u_A)$. If it is generic then the set of dominant exponentials labeling $G_t(u_A)$ represent a set $\mathcal{C}$ of algebraically independent Plücker coordinates. Moreover, every Plücker coordinate can be written as a Laurent polynomial in the elements of $\mathcal{C}$.

Inverse problem

Given a time t together with the contour plot of a soliton solution of KP, can one reconstruct the point of $(Gr_{k,n})_{\geq 0}$ which gave rise to the solution?

Theorem (Kodama-W.)

1. If we know that $t << 0$ sufficiently small, we can solve the inverse problem, no matter what cell of $(Gr_{k,n})_{\geq 0}$ the element A came from.
2. If the contour plot is generic and came from a point of the TP Grassmannian, we can solve the inverse problem, regardless of time t.

Proof of 1: uses our description of soliton graphs at $t << 0$, and work of Kelli Talaska.

Proof of 2: uses result that the set of dominant exponentials labeling such a contour plot forms a cluster in the cluster algebra associated to $Gr_{k,n}$.

Application: solving the inverse problem for soliton graphs

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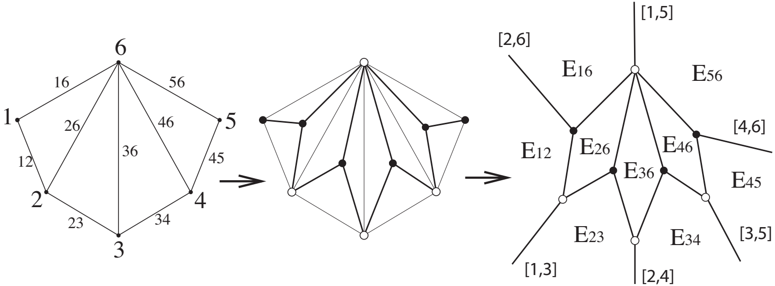
Application: classification of soliton graphs for $(Gr_{2,n})_{>0}$

Up to graph-isomorphism, the soliton graphs for $(Gr_{2,n})_{>0}$ are in bijection with triangulations of an n-gon.



Theorem (Kodama-W.)

Every plabic graph obtained via the above algorithm is a soliton graph $G_t(u_A)$ for some $A \in (Gr_{2,n})_{>0}$. Conversely, all (generic) soliton graphs for $A \in (Gr_{2,n})_{>0}$ can be produced from a triangulation of an n-gon as above.

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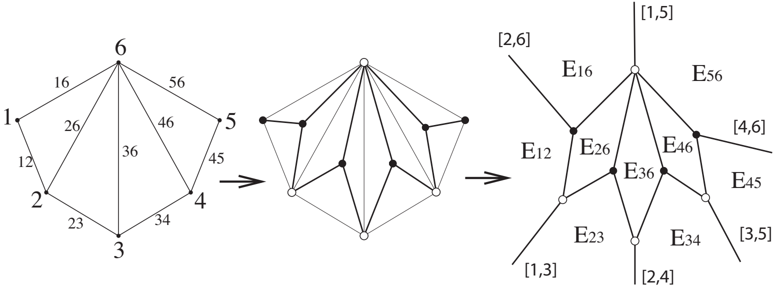

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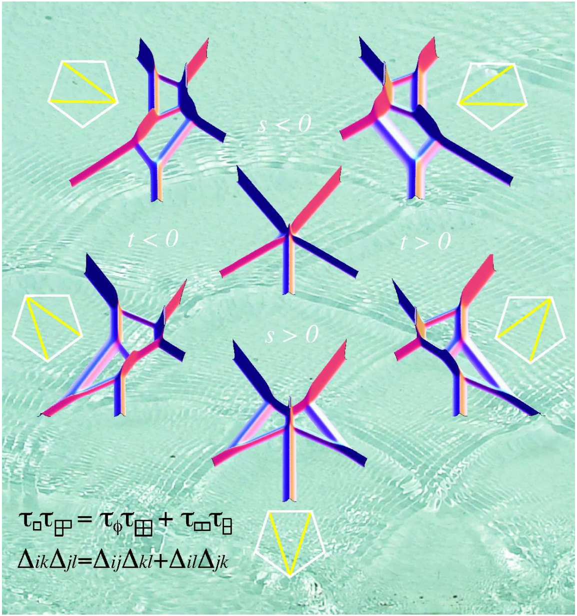
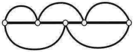
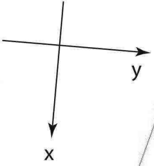
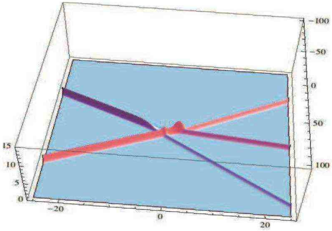


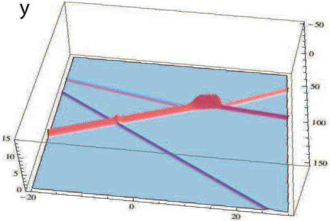
Thanks for listening! (movies?)

- KP solitons, total positivity, and cluster algebras (K. + W.), PNAS, May 11, 2011.
- KP solitons and total positivity for the Grassmannian (K. + W.), `http://front.math.ucdavis.edu/1106.0023`.









Why look at asymptotics as $y \to \pm\infty$ and not $x \to \pm\infty$?

The equation for a line-soliton separating dominant exponentials E_I and E_J is where $I = \{i, m_2, \ldots, m_k\}$ and $J = \{j, m_2, \ldots, m_k\}$ is

$$x + (\kappa_i + \kappa_j)y + (\kappa_i^2 + \kappa_i\kappa_j + \kappa_j^2)t = constant.$$

So we may have line-solitons parallel to the y-axis, but never to the x-axis. (κ_i's are fixed)