Dixmier-Moeglin equivalence for Leavitt path algebras

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Overview

General results about prime and primitive ideals

A always denotes an algebra over the field K.

Goal: Distinguish the primitive ideals of A from the rest of $\mathrm{Spec}(A)$.

One possible approach: topological

$P \in \mathrm{Spec}(A)$ is called *locally closed* in case

$$\bigcap \{Q \in \mathrm{Spec}(A) \mid Q \supsetneqq P\} \supsetneqq P.$$

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Theorem. (Dixmier and Moeglin, separately, late 1970's.) Let $U(\mathcal{L})$ be the enveloping algebra of a finite dimensional complex Lie algebra $\mathcal{L}$, and let $P \in \mathrm{Spec}(U(\mathcal{L}))$. Then P is primitive if and only if P is locally closed.

General results about prime and primitive ideals

Another possible approach: Analyze the ring of quotients $Q(A/P)$ of A/P.

Denote by $Z(Q(A/P))$ the center of $Q(A/P)$. (The "extended centroid" of P.) This is a field extension of K.

$P \in \mathrm{Spec}(A)$ is called *rational* in case $Z(Q(A/P))$ is an algebraic extension of K.

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General results about prime and primitive ideals

Definition

Let K be a field and let A be a K-algebra. We say that A *satisfies the Dixmier-Moeglin equivalence on prime ideals* if for $P \in \mathrm{Spec}(A)$ the following conditions are equivalent:

1. P is primitive;
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So $U(\mathcal{L})$ satisfies the Dixmier-Moeglin (DM) equivalence.

Indeed, many algebras satisfy the DM equivalence.

DM Equivalence for Leavitt path algebras

The main goal of this talk is to establish:

Theorem

(A-, Jason Bell, K.M. Rangaswamy) Let K be any field, and E any finite graph. Then $L_K(E)$ satisfies the Dixmier-Moeglin equivalence on prime ideals.

Leavitt path algebras

Recall: Start with a (finite) graph $E = (E^0, E^1, r, s)$, and field K.

Construct the "double graph" (or "extended graph") $\widehat{E}$, and then the path algebra $K\widehat{E}$.

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Consider these relations in $K\widehat{E}$:

(CK1) $e^*e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$.

(CK2) $v = \sum_{\{e \in E^1 | s(e)=v\}} ee^*$ for all $v \in E^0$

(just at those vertices v which are not *sinks*)

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Definition

The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

Leavitt path algebras: Examples

Example 1.

$$E = \ \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

Then $L_K(E) \cong M_n(K)$.

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Then $L_K(E) \cong M_n(K)$.

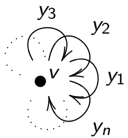
Example 2.

$$E = R_1 = \bullet^{v} \circlearrowright x$$

Then $L_K(E) \cong K[x, x^{-1}]$.

Leavitt path algebras: Examples

Example 3.

$E = R_n =$ 

Then $L_K(E) \cong L_K(1, n)$, the *Leavitt algebra of type* $(1, n)$.

Leavitt path algebras: basic properties

There is a natural $\mathbb{Z}$-grading on $L_K(E)$, generated by defining

$$\deg(v) = 0, \quad \deg(e) = 1, \quad \deg(e^*) = -1$$

Graded ideals

If $v, w \in E^0$ write $v \geq w$ in case there is a path which starts at v and ends at w. (This includes the case $v = w$.)

A subset H of E^0 is **hereditary** if whenever $v \in H$ and $v \geq w$, then $w \in H$.

A subset S of E^0 is **saturated** if whenever $v \in E^0$ is a non-sink, and the range vertices of *all* of the edges emitted by v are in S, then v is in S as well.

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Lemma: If I is an ideal of $L_K(E)$, then $I \cap E^0$ is hereditary and saturated.

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Proof. For hereditary: If v connects to w by an edge e, and $v \in I$, then $w = e^*e = e^*ve \in I$. (Induction for $v \geq w$.)

Graded ideals

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Proof. For hereditary: If v connects to w by an edge e, and $v \in I$, then $w = e^*e = e^*ve \in I$. (Induction for $v \geq w$.)

For saturated: If v is not a sink, then

$$v = \sum_{\{e \in E^1 | s(e) = v\}} ee^* = \sum_{\{e \in E^1 | s(e) = v\}} er(e)e^* \in I. \quad \square$$

$\mathrm{Spec}(L_K(E))$

Proposition. An ideal I of $L_K(E)$ is graded if and only if $I = < I \cap E^0 >$.

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Corollary:

$$J(L_K(E)) = \{0\} \quad \text{for any graph } E.$$

$\mathrm{Spec}(L_K(E))$

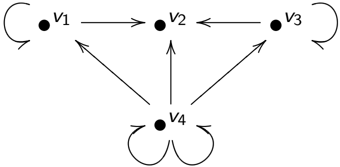
For $H \subseteq E^0$ the graph $E^0 \setminus H$ is formed by throwing out all vertices in H, as well as all edges starting and/or ending in H.

If H is a hereditary saturated subset of E^0, then $< H >$ is a graded ideal of $L_K(E)$, and we get

$$L_K(E)/ < H > \cong \; L_K(E^0 \setminus H).$$

Graded ideals

Example. $E : =$



There are six hereditary saturated subsets in E^0:

$$H_1 = \emptyset \qquad H_2 = \{v_1, v_2\} \qquad H_3 = \{v_2, v_3\}$$

$$H_4 = \{v_1, v_2, v_3\} \qquad H_5 = \{v_2\} \qquad H_6 = E^0$$

$\mathrm{Spec}(L_K(E))$

Much of the heavy lifting needed to describe $\mathrm{Spec}(L_K(E))$ is done in APPSM:

G. Aranda Pino, E. Pardo, and M. Siles Molina,
Prime spectrum and primitive Leavitt path algebras,
Indiana Univ. Math. J. **58**(2), 869–890 (2009).

$\mathrm{Spec}(L_K(E))$

Definition

A nonempty subset $M \subseteq E^0$ is a maximal tail if:

(MT1) If $v \in E^0$, $w \in M$ and $v \geq w$ then $v \in M$;

(MT2) if $v \in M$ and $s^{-1}(v) \neq \emptyset$ then there is $e \in E^1$ such that $s(e) = v$ and $r(e) \in M$; and

(MT3) for every $v, w \in M$ there exists $y \in M$ such that $v \geq y$ and $w \geq y$.

$\mathcal{M}(E)$ denotes the set of maximal tails of E.

$\mathrm{Spec}(L_K(E))$

Proposition: If $H \subseteq E^0$ is hereditary and saturated, then $< H >$ is a (graded) prime ideal if and only if $M = E^0 \setminus H$ is nonempty and satisfies (MT3).

So we can identify the graded prime ideals in $L_K(E)$.

$\mathrm{Spec}(L_K(E))$

More specifically, there is a bijective correspondence between the set of graded prime ideals of $L_K(E)$ and maximal tails of E, given by

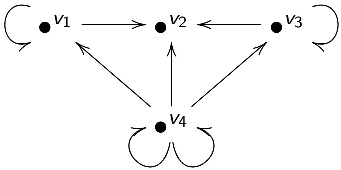
$$P \in \mathrm{gradedSpec}(L_K(E)) \mapsto E^0 \setminus \left(P \cap E^0\right).$$

The inverse of this map is given by

$$M \in \mathcal{M}(E) \mapsto \sum_{v \in E^0 \setminus M} < v > .$$

$\mathrm{Spec}(L_K(E))$

Example, revisited. $E: =$



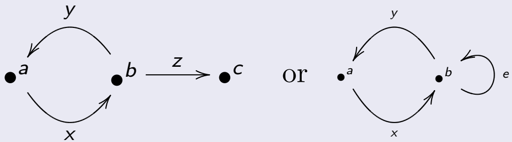
Of the six hereditary saturated subsets H_i in E^0, four have nonempty complements $M_i = E^0 \setminus H_i$ which satisfy (MT3):

$$H_1 = \emptyset \quad H_2 = \{v_1, v_2\} \quad H_3 = \{v_2, v_3\} \quad H_4 = \{v_1, v_2, v_3\}$$

$\mathrm{Spec}(L_K(E))$

Definitions

An *exit* for a cycle:



A graph F has Condition (L) in case every cycle in F has an exit in F.

$\mathrm{Spec}(L_K(E))$

Definitions

$\mathcal{M}_\gamma(E) \subseteq \mathcal{M}(E)$ denotes those maximal tails M for which every cycle in M has an exit in M (i.e., M has Condition (L).)

$\mathcal{M}_\tau(E) = \mathcal{M}(E) \setminus \mathcal{M}_\gamma(E)$.

In the previous example: $M_1, M_4 \in \mathcal{M}_\gamma$, while $M_2, M_3 \in \mathcal{M}_\tau$.

$\mathrm{Spec}(L_K(E))$

Observations:

1. There are many non-graded prime ideals in $K[x, x^{-1}] = L_K(R_1) = L_K(\bullet \circlearrowleft)$, these are indexed by the irreducible polynomials in $K[x, x^{-1}]$.

2. For any graph E, if c is a cycle without exits based at the vertex v, then

$$vL_K(E)v \cong K[x, x^{-1}].$$

$\mathrm{Spec}(L_K(E))$

A description of the *non*-graded prime ideals is also given in APPSM:

There is a bijective correspondence

$$\mathrm{nongradedSpec}(L_K(E)) \to \mathcal{M}_\tau(E) \times \mathrm{maxSpec}(K[x, x^{-1}]).$$

$\mathrm{Spec}(L_K(E))$

Specifically:

(1) If P is a non-graded prime ideal of $L_K(E)$, then $M := E^0 \setminus (P \cap E^0)$ is in $\mathcal{M}_\tau(E)$.

(2) If $M \in \mathcal{M}_\tau(E)$ then there is a *unique* cycle (up to cyclic permutation) $c = c(M)$ in M that does not have an exit in M.

(3) In this situation, there exists an irreducible $f_P(x) \in K[x, x^{-1}]$ such that $f_P(c) \in P$.

$\mathrm{Spec}(L_K(E))$

The correspondence is given by:

$$P \in \mathrm{nongrSpec}(L_K(E)) \mapsto \left(E^0 \setminus \left(P \cap E^0\right), < f_P(x) >\right)$$

and

$$(M, < f(x) >) \in \mathcal{M}_\tau(E) \times \mathrm{maxSpec}(K[x, x^{-1}])$$

$$\mapsto \sum_{v \in E^0 \setminus M} < v > + < f(c(M)) > .$$

$\mathrm{Spec}(L_K(E))$

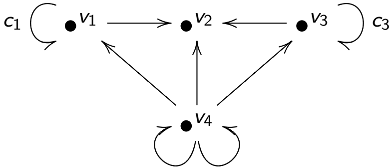
Theorem. [APPSM, Theorem 3.8] There is a bijection

$$\mathrm{Spec}(L_K(E)) \;\leftrightarrow\; M_\gamma(E) \sqcup M_\tau(E) \sqcup (M_\tau(E) \times \mathrm{maxSpec}(K[x, x^{-1}]))$$

$\mathrm{Spec}(L_K(E))$

Example, third visit:

$E :=$



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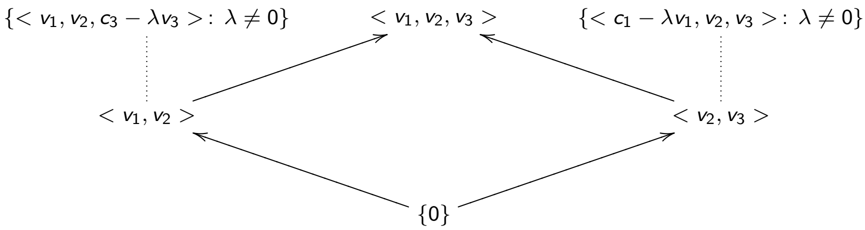

$\mathrm{Spec}(L_K(E))$

$$\mathrm{Spec}(L_K(E)) \leftrightarrow M_\gamma(E) \sqcup M_\tau(E) \sqcup (M_\tau(E) \times \mathrm{maxSpec}(K[x, x^{-1}]))$$

Can we use this correspondence to identify the primitive ideals?

If $P =< H >$ corresponds to an element of $M_\gamma(E)$, then $L_K(E)/P \cong L_K(E^0 \setminus H)$, and M has Condition (L). So P is primitive [APPSM Theorem 4.6].

If $P =< H >$ corresponds to an element of $M_\tau(E)$, then P is not primitive [APPSM Lemma 4.1].

$\mathrm{Spec}(L_K(E))$

Proposition. If $P \in \mathrm{Spec}(L_K(E))$ corresponds to an element of $M_\tau(E) \times \mathrm{maxSpec}(K[x, x^{-1}])$, then P is primitive.

Proof.

$$P = \sum_{v \in E^0 \setminus M} < v > \; + \; < f(c(M)) >$$

for some irreducible $f(x) \in K[x, x^{-1}]$, where $c(M)$ is a cycle without exits based at the vertex w. If $u := w + P$, then

$$u(L_K(E)/P)u \cong K[x, x^{-1}]/ < f(x) >,$$

a field. So $L_K(E)/P$ has a primitive corner. $\square$

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Graded:

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Primitive:

$$M_\gamma(E) \qquad\qquad \sqcup \ (M_\tau(E) \times \mathrm{maxSpec}(K[x, x^{-1}]))$$

$\mathrm{Spec}(L_K(E))$

We get some immediate consequences.

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Proposition. For any graph E, $L_K(E)$ is a Jacobson ring. (That is, the Jacobson radical of each prime homomorphic image of $L_K(E)$ is zero.)

Proof. If P is not primitive then P is graded, so $P = < H >$, and

$$L_K(E)/P \cong L(E^0 \setminus H)$$

is a Leavitt path algebra, so has Jacobson radical zero.

$\mathrm{Spec}(L_K(E))$

"More Vertices" Proposition. Let $P \in \mathrm{Spec}(L_K(E))$. Then P is primitive if and only if for every $Q \in \mathrm{Spec}(L_K(E))$ having $P \subsetneqq Q$, there exists a vertex v with $v \in Q \setminus P$.

Locally closed prime ideals

Main Theorem: Part I of DM Equivalence. Let E be a finite graph, and K any field. Let $P \in \mathrm{Spec}(L_K(E))$. Then these are equivalent:

1. P is primitive.
2. P is locally closed; i.e., $\bigcap\{Q \in \mathrm{Spec}(A) \mid Q \supsetneqq P\} \supsetneqq P$.

"Locally closed" portion of DM Theorem

Proof. Suppose P is primitive. By More Vertices Proposition,

$$\bigcap\{Q \mid Q \in \mathrm{Spec}(L(E)), Q \supsetneq P\} \supseteq \bigcap\{P+ < v > \mid v \notin P\}.$$

The second intersection contains $\prod_{v \notin P}(P+ < v >)$. Since P is prime and none of the $P+ < v >$ is contained in P, this product must contain P strictly.

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Suppose on the other hand P is not primitive. Since $L_K(E)$ is Jacobson, then P is not locally closed.

(This is well-known, see e.g. Brown / Goodearl.) $\square$

Rational prime ideals

The Martindale ring of quotients of a prime ring.

Definition

Let K be a field and let A be a prime K-algebra. The (right) Martindale ring of quotients of A, denoted $Q_M(A)$, consists of equivalence classes of pairs (I, f) where $\{0\} \neq I \trianglelefteq A$ and $f \in \mathrm{Hom}_A(I_A, A_A)$.

Here

$$(I, f) \sim (I', f') \quad \text{in case} \quad f = f' \text{ on } I \cap I'.$$

Addition and multiplication in $Q_M(A)$ are given by

$$(I, f) + (J, g) \; = \; (I \cap J, f + g), \qquad (I, f) \cdot (J, g) \; = \; (JI, f \circ g).$$

Rational prime ideals

Definition

Let K be a field and let A be a prime K-algebra. The *extended centroid of* A is defined to be $Z(Q_M(A))$.

One can show:

$$Z(Q_M(A)) = \{[(I,f)] \mid \{0\} \neq I \trianglelefteq A, f \in \mathrm{Hom}_A({}_AI_A, {}_AA_A)\}.$$

"Rational" portion of DM Theorem

Main Theorem: Part II of DM Equivalence. Let E be a finite graph, and K any field. Let $P \in \mathrm{Spec}(L_K(E))$. Then these are equivalent:

1. P is primitive.
2. P is rational; i.e., $Z(Q(L_K(E)/P))$ is algebraic over K.

"Rational" portion of DM Theorem

Key ingredients of the Proof.

Suppose P is not primitive. Then $L_K(E)/P \cong L_K(M)$, where M is a graph which contains a cycle c without exits, based at v.

We build an element of $Z(Q(L_K(M)))$ which is transcendental over K. Consider

$$f : < v > \to L_K(M) \qquad f : avb \mapsto acb$$

for all $a, b \in L_K(M)$, and extend K-linearly.

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for all $a, b \in L_K(M)$, and extend K-linearly.

If f is well-defined, then f is clearly a $L_K(M)$-$L_K(M)$ bimodule homomorphism, and so in $Z(Q(L_K(M)))$.

"Rational" portion of DM Theorem

It suffices to show that whenever $\sum_{i=1}^{m} a_i v b_i = 0$, then $\sum_{i=1}^{m} a_i c b_i = 0$.

Assume K is infinite. By contradiction, suppose there are elements $a_1, \ldots, a_m, b_1, \ldots, b_m$ in $L_K(M)$ such that $\sum_{i=1}^{m} a_i v b_i = 0$, while $y := \sum_{i=1}^{m} a_i c b_i$ is nonzero. Then we get

$$y = \sum_{i=1}^{m} a_i (c - \lambda v) b_i$$

for all $\lambda \in K$. In particular, y is in the ideal $< c - \lambda v >$ for all $\lambda \in K$, and so

$$\bigcap_{\lambda \in K} < c - \lambda v >$$

is a nonzero ideal of $L_K(M)$.

"Rational" portion of DM Theorem

Since $L_K(M)$ is a prime ring,

$$v \cdot [\bigcap_{\lambda \in K} < c - \lambda v >] \cdot v \neq \{0\}.$$

Using $vL_K(M)v \cong K[x, x^{-1}]$ this gives

$$\bigcap_{\lambda \in K} K[x, x^{-1}](x - \lambda)K[x, x^{-1}] \neq \{0\}.$$

But K infinite gives this last intersection is 0 in $K[x, x^{-1}]$, a contradiction.

(For the finite K case, embed K in an infinite K' ...)

"Rational" portion of DM Theorem

So f is a bimodule homomorphism, which yields

$$[(<v>, f)] \in Z(Q(L_K(M))).$$

Now it's not hard (again using $vL_K(M)v \cong K[x, x^{-1}]$) to show that $[(<v>), f)]$ is transcendental over K.

Thus P is not rational.

"Rational" portion of DM Theorem

Conversely, show that if P is primitive then P is rational.

"Rational" portion of DM Theorem

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Let K' be an uncountable purely transcendental field extension of K. We have that

$$L_{K'}(E) \cong L_K(E) \otimes_K K', \quad \text{and}$$

$$P' := P \otimes_K K' \text{ is a primitive ideal of } L_{K'}(E).$$

"Rational" portion of DM Theorem

Define $R = L_K(E)/P$ and $R' = L_{K'}(E)/P'$.

Because E is finite we have $\dim_{K'}(L_{K'}(E))$ is at most countable. Since R' is primitive and $\dim_{K'} R' < \mathrm{card}(K')$, $Z(Q(R'))$ is algebraic over K'. (Again see e.g. Brown / Goodearl.)

But there's a natural embedding

$$\Phi : Z(Q(R)) \otimes_K K' \to Z(Q(R'))$$

defined as: $\Phi([(I, f)]) = [(I', f')]$, where

$$f' = f \otimes \mathrm{id} : I \otimes_K K' \to R \otimes_K K' = R'.$$

Using this, one gets that $Z(Q(R))$ is algebraic over K as well, so that P is rational. $\square$

What posets arise as $\mathrm{gradedSpec}(L_K(E))$?

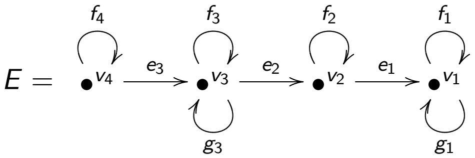
1. An analysis of the graded prime ideals.

Question 1: Given an arbitrary finite poset $\mathcal{P}$, can we find a graph E for which $\mathrm{gradedSpec}(L_K(E)) \cong \mathcal{P}$?

Question 2: Given a poset $\mathcal{P}$ and a subposet $\mathcal{P}' \subseteq \mathcal{P}$, can we find a graph E for which $\Psi : \mathrm{gradedSpec}(L_K(E)) \to \mathcal{P}$ is an isomorphism having the property that $Q \in \mathrm{gradedSpec}(L_K(E))$ is primitive precisely when $\Psi(Q) \in \mathcal{P}'$?

What posets arise as $\mathrm{gradedSpec}(L_K(E))$?

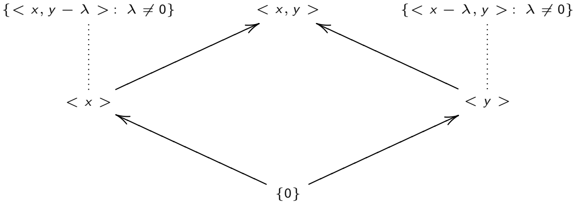
UCCS algebra group has started to look at this. We know how to do the case when the partially ordered set is totally ordered. The idea (couched as an Example):



has $\mathrm{gradedSpec}(L_K(E)) \ = \ \{1 \leq 2 \leq 3 \leq 4\}$, with the primitives corresponding to the vertices at which there are two loops.

Observations about $\mathrm{Spec}(L_K(E))$

2. Here is the prime spectrum of the quantum plane:



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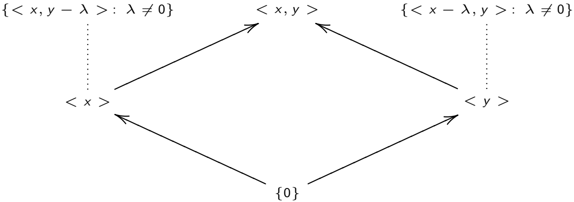
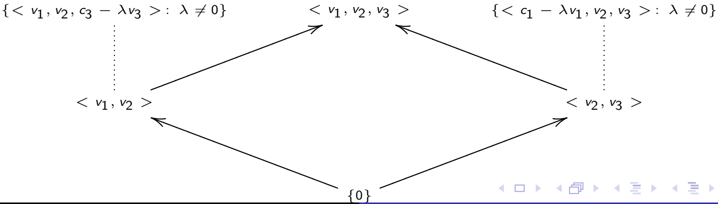


Here is the prime spectrum of $L_{\mathbb{C}}(E)$ of our previous example:



Observations about $\mathrm{Spec}(L_K(E))$

Can we somehow build graphs E for which $\mathrm{Spec}(L_{\mathbb{C}}(E))$ matches $\mathrm{Spec}(R)$ for well-studied finitely generated $\mathbb{C}$-algebras R which satisfy the DM equivalence on prime ideals?

The strata of $\mathrm{Spec}(L_K(E))$

3. The bijective correspondence

$$\mathrm{Spec}(L_K(E)) \;\leftrightarrow\; M_\gamma(E) \;\sqcup\; M_\tau(E) \;\sqcup\; (M_\tau(E) \times \mathrm{maxSpec}(K[x, x^{-1}]))$$

can recast in the language of *stratification*.

For each $P \in \mathrm{Spec}(L_K(E))$ let P_0 denote the largest graded ideal that is contained in P. Then P_0 is in fact a graded prime ideal. Given a maximal tail M, define

$$\mathrm{Spec}_M(L_K(E)) = \{P \in \mathrm{Spec}(L_K(E)) \;:\; P \cap E^0 = E^0 \setminus M\}.$$

Call $\mathrm{Spec}_M(L_K(E))$ the *stratum corresponding to M*.

The strata of $\mathrm{Spec}(L_K(E))$

Theorem

(Stratification for Leavitt path algebras) Let K be a field, E any finite graph. Then

$$\mathrm{Spec}(L_K(E)) = \bigsqcup_{M \in \mathcal{M}(E)} \mathrm{Spec}_M(L_K(E)).$$

Moreover,

$$\mathrm{Spec}_M(L_K(E)) \cong \begin{cases} \mathrm{Spec}(K[x, x^{-1}]) & \text{if } M \in \mathcal{M}_\tau(E), \\ \mathrm{Spec}(K) & \text{if } M \in \mathcal{M}_\gamma(E). \end{cases}$$

In particular, the prime spectrum is the union of a finite number of strata, with each stratum homeomorphic to a 0- or 1-dimensional torus, and the primitive ideals are precisely those that are maximal in their stratum.

Coffee?