Non-stable K-theory and graph algebras

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Outline

1. Structure of Projectives
2. Quiver Algebras
3. Are graph monoids enough?
4. Separated graphs

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Introduction

A ring R is *(von Neumann) regular* if for all $x \in R$ there is $y \in R$ such that $x = xyx$. In R every right ideal is generated by an idempotent: $xR = (xy)R$. The set $L(R_R)$ of principal right ideals of R is a **lattice**, which is **modular** and **complemented**.

Von Neumann showed a **realization result**:

Theorem (Von Neumann coordinatization theorem)

For every complemented modular lattice L with a "homogeneous basis" of order ≥ 4 there is a regular ring R, unique up to isomorphism, such that $L \cong L(R_R)$.

Under the influence of Goodearl and Handelman, the subject became close to the study of the structure of finitely generated projective modules over regular rings, and its interconnections with the ring structure.

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The Monoid of Projective Modules

Definition

Let R be a ring. The monoid of projective modules, $\mathcal{V}(R)$, is the set of isomorphism classes of finitely generated projective left R-modules. We endow $\mathcal{V}(R)$ with the structure of an abelian monoid by imposing the operation $[P] + [Q] = [P \oplus Q]$.

The Realization Theorem for Hereditary Rings

Theorem (Bergman'74, Bergman-Dicks'78)

Let M be an abelian monoid with a distinguished element $\mathbf{1} \neq 0$, such that:

1. $(\forall x, y \in M)(x + y = 0) \Rightarrow x = y = 0$ *(conical).*
2. $(\forall x \in M)(\exists y \in M, n \geqslant 0)$ *such that* $x + y = n\mathbf{1}$ *(order-unit).*

Then there exists a hereditary K-algebra R, such that $\mathcal{V}(R) \cong M$ as monoids with order-unit.

The Realization Problem for Regular Rings

For a regular ring R, $\mathcal{V}(R)$ is a refinement monoid.

Let M be an abelian monoid. Then M is a refinement monoid in case any equality $a+b=c+d$ admits a refinement:

	c	d
a	x	y
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Fundamental Open Problem (Goodearl 1995)

What abelian monoids appear as $\mathcal{V}(R)$ for a regular ring R?

R1. Realization Problem for von Neumann Regular Rings

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Fundamental Open Problem (Goodearl 1995)

What abelian monoids appear as $\mathcal{V}(R)$ for a regular ring R?

R2. Realization Problem for von Neumann Regular Rings

Is every countable refinement conical abelian monoid with order-unit realizable by a von Neumann regular ring?

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Separative Rings

Definition (A, Goodearl, O'Meara, Pardo 1998)

A class $\mathcal{C}$ of modules is called *separative* if for all $A, B \in \mathcal{C}$ we have

$$A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B.$$

A ring R is *separative* if $FP(R)$ is a separative class.

SP. Separativity Problem for von Neumann Regular Rings

Is every von Neumann regular ring separative?

We have

$(R2$ has positive answer $) \implies (SP$ has a negative answer $)$.

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The Definitions

Definition

A quiver (oriented graph) E is a 4-tuple (E^0, E^1, r, s) where E^0 is the set of vertices, E^1 is the set of arrows and $r, s\colon E^1 \to E^0$ are the incidence maps.

E is *row-finite* in case $|s^{-1}(v)| < \infty$ for every $v \in E^0$.

Definition

A path in a quiver E is either an ordered sequence of arrows $\alpha = e_1 \cdots e_n$ with $r(e_t) = s(e_{t+1})$ for $1 \leqslant t < n$, or a path of length 0 (trivial path) corresponding to a vertex $v \in E^0$.

We denote the set of all paths in E by $\mathrm{Path}(E)$.

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Examples

The n-rose quiver, R_n, is given by one vertex and n arrows:

Paths in R_n are words in the set $\{e_1, \ldots, e_n\}$.
The n-line quiver, A_n, is the following quiver:

$$\bullet^{v_1} \xrightarrow{f_1} \bullet^{v_2} \xrightarrow{f_2} \bullet^{v_3} \longrightarrow \cdots \longrightarrow \bullet^{v_{n-1}} \xrightarrow{f_{n-1}} \bullet^{v_n}$$

Paths in A_n are sequences $f_i f_{i+1} \cdots f_{i+k}$ or vertices v_i.

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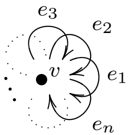

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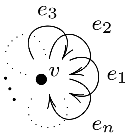

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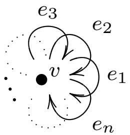

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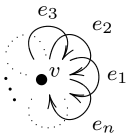

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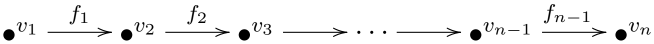

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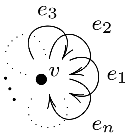

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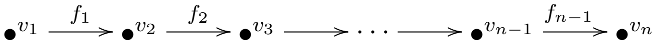

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Given a quiver E and a field K, one may associate to it the **path algebra** $P_K(E)$, but this is not interesting for our purposes:

1. $P_K(E)$ is far from being regular.
2. $\mathcal{V}(P_K(E)) = (\mathbb{Z}^+)^{(E^0)}$ is far from being an interesting monoid.

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Instead we may associate to a row-finite quiver E the **Leavitt path algebra** $L_K(E)$ (Abrams-Aranda 2005, A-Moreno-Pardo 2007), which is the K-algebra with generators $E^0 \sqcup E^1 \sqcup (E^1)^*$ and relations:

- (V) $uv = \delta_{uv} u$ for all $u, v \in E^0$.
- (E1) $s(e)e = er(e) = e$ for all $e \in E^1$.
- (E2) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$.
- (CK1) $f^* e = \delta_{ef} r(e)$ for all $e, f \in E^1$.
- (CK2) $v = \sum_{e \in s^{-1}(v)} ee^*$ for all $v \in E^0$ such that $s^{-1}(v) \neq \emptyset$.

Examples

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- For the n-rose quiver we have that $P_K(R_n)$ is the free algebra $K\langle e_1, \dots, e_n\rangle$.
- For the n-line quiver we have that $P_K(A_n)$ is a triangular matrix ring of size $n \times n$:

$$\begin{pmatrix} K & \dots & K \\ & \ddots & \vdots \\ 0 & & K \end{pmatrix}$$

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- For the n-rose quiver we have that $L_K(R_n)$ is the *classical Leavitt algebra of type (1,n)*

$$L(1,n) = K\langle x_1, \dots, x_n, y_1, \dots, y_n \mid y_i x_j = \delta_{i,j}, \sum_{i=1}^{n} x_i y_i = 1 \rangle$$

- For the n-line quiver we have that $L_K(A_n)$ is a **full** matrix ring of size $n \times n$:

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The Monoid of a row-finite Quiver

For a row-finite quiver E, let $M(E)$ be the abelian monoid generated by $\{a_v \mid v \in E^0\}$, with the relations

$$a_v = \sum_{\{e \in E^1 \mid s(e)=v\}} a_{r(e)} \quad \text{for every } v \in E^0 \text{ with } s^{-1}(v) \neq \emptyset.$$

Theorem (Ara, Moreno, Pardo'07)

For every row-finite graph E there is a natural isomorphism

$$M(E) \cong \mathcal{V}(L_K(E)).$$

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For the quiver A_n, we have $M(A_n) = \langle a \mid \rangle = \mathbb{Z}^+$.

More generally, if E if finite and has no oriented cycles, then

$$M(E) = (\mathbb{Z}^+)^r,$$

where r is the number of sinks of E.

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The monoid $M(E)$ is a conical refinement monoid.

However $L_K(E)$ is in general not a regular ring. In fact by results of Abrams and Rangaswami,

$$L_K(E) \text{ regular} \iff E \text{ is acyclic} .$$

Idea: To embed $L_K(E)$ into a “quotient ring" which is regular.

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$$\begin{array}{ccc} K & \longrightarrow & K[x] \\ \big\downarrow & & \\ K[x^{-1}] & & \end{array}$$

$$\begin{array}{ccc} K^{(E^0)} & \longrightarrow & P(E) \\ \big\downarrow & & \\ P(E^*) & & \end{array}$$

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K & \longrightarrow & K[x] & \longrightarrow & K_{\mathrm{rat}}[x] & \longrightarrow & K[[x]] \\
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The Algebra of Formal Power Series of a Quiver

The augmentation homomorphism $\varepsilon\colon P(E) \to K^{(E^0)} \subseteq P(E)$ is defined by $\varepsilon\left(\sum_{\gamma\in E^*} \lambda_\gamma \gamma\right) = \sum_{\gamma\in E^0} \lambda_\gamma \gamma$.

Definition

Let $I = \ker(\varepsilon)$ be the augmentation ideal of $P(E)$. Then the formal power series algebra of the quiver E, $P((E))$, is the I-adic completion of $P(E)$, that is $P((E)) \cong \varprojlim P(E)/I^n$.

Remark

An element in $P((E))$ can be written in a unique way as a possibly infinite sum $\sum_{\gamma\in \text{Path}(E)} \lambda_\gamma \gamma$ with $\lambda_\gamma \in K$.

We will identify $P(E)$ with its image in $P((E))$.

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- For the n-rose quiver we have that $P((R_n))$ is the (usual) formal power series algebra $K\langle\langle e_1, \ldots, e_n \rangle\rangle$.
- For the n-line quiver the augmentation ideal is:

$$\begin{pmatrix} 0 & K & \dots & K \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & K \\ 0 & \dots & \dots & 0 \end{pmatrix},$$

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The Algebra of Rational Series of a Quiver

Given a ring S and a subring $R \subseteq S$, we denote by $\Sigma(R \subseteq S)$ the set of all square matrices over R which are invertible over S.

Definition

- A subring $R \subseteq S$ is rationally closed in S if $\Sigma(R \subseteq S) = GL(R)$, that is, if $GL(R) = M(R) \cap GL(S)$.
- Given a subring $R \subseteq S$ the rational closure of R in S is the smallest subring of S containing R and rationally closed.

Definition

The algebra of rational series of the quiver E, $P_{\text{rat}}(E)$, is the rational closure of $P(E)$ in $P((E))$.

The Algebra of Rational Series of a Quiver

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The Universal Localization

Let R be a ring and Σ a set of homomorphisms in the category of finitely generated projective left R-modules. We say that a ring homomorphism $f\colon R \to S$ is Σ-inverting if for every $\alpha \in \Sigma$, $\alpha \otimes_f S$ is invertible.

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Let R be a ring and Σ a set of homomorphisms in the category of f.g. projective left R-modules. The universal localization of R is a ring $\Sigma^{-1}R$ with a ring homomorphism $\lambda_\Sigma\colon R \to \Sigma^{-1}R$ such that is universal Σ-inverting.

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Let $\Sigma = \Sigma(P(E) \subseteq P((E)))$. Then $P_{\mathrm{rat}}(E)$ coincides with the universal localization of $P(E)$ with respect to Σ.

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The Diagram II

We have a commutative diagram:

$$\begin{array}{ccccccc} K^{(E^0)} & \longrightarrow & P(E) & \xrightarrow{\iota_\Sigma} & P_{\mathrm{rat}}(E) & \longrightarrow & P((E)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P(E^*) & \longrightarrow & L(E) & \xrightarrow{\iota_\Sigma} & Q(E) & \longrightarrow & U(E), \end{array}$$

$Q(E) = \Sigma^{-1}L(E)$ is the regular algebra of the quiver E

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Theorem (A-Brustenga, 2007)

Let E be a row-finite quiver. Then $Q(E)$ and $U(E)$ are von Neumann regular rings.

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The Diagram of Monoids

In summary, we have the following:

$$\begin{array}{ccccc}
(\mathbb{Z}^+)^d \cong \mathcal{V}(P(E)) & \xrightarrow{\cong} & \mathcal{V}(P_{\mathrm{rat}}(E)) & \xrightarrow{\cong} & \mathcal{V}(P((E))) \\
\downarrow & & \downarrow & & \downarrow \\
M(E) \cong \mathcal{V}(L(E)) & \xrightarrow{\cong} & \mathcal{V}(Q(E)) & \xrightarrow{\cong} & \mathcal{V}(U(E))
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Outline

1. Structure of Projectives
2. Quiver Algebras
3. Are graph monoids enough?
4. Separated graphs

Graph monoids: a large class, but not big enough

Theorem (Brookfield 2001)

If M is a finitely generated refinement monoid then M is separative.

Since every graph monoid (of a row-finite graph) is a direct limit of finitely generated graph monoids, we get

Corollary

Every graph monoid is separative.

But there are even finitely generated conical refinement monoids which are not graph monoids.

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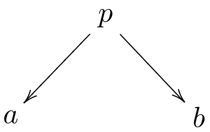
An example

The simplest refinement monoid which is not a graph monoid is the following [A, Perera, Wehrung, 2008]:

$$M = \langle p, a, b \mid p = p + a = p + b \rangle .$$



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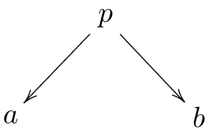

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Outline

1. Structure of Projectives
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Separated graphs (joint work with Ken Goodearl)

Definition

A *separated graph* is a pair (E, C) where E is a graph, $C = \bigsqcup_{v \in E^0} C_v$, and C_v is a partition of $s^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v:

$$s^{-1}(v) = \bigsqcup_{X \in C_v} X.$$

(In case v is a sink, we take C_v to be the empty family of subsets of $s^{-1}(v)$.)
The constructions we introduce revert to existing ones in case $C_v = \{s^{-1}(v)\}$ for each $v \in E^0$. We refer to a *non-separated graph* in that situation.

The Leavitt path algebra of a separated graph

Definition

The *Leavitt path algebra of the separated graph* (E, C) with coefficients in the field K, is the K-algebra $L_K(E, C)$ with generators $\{v, e, e^* \mid v \in E^0, e \in E^1\}$, subject to the following relations:

(V) $vv' = \delta_{v,v'}v$ for all $v, v' \in E^0$,

(E1) $s(e)e = er(e) = e$ for all $e \in E^1$,

(E2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$,

(SCK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in X$, $X \in C$, and

(SCK2) $v = \sum_{e\in X} ee^*$ for every finite set $X \in C_v$, $v \in E^0$.

For a row-finite quiver E, the Leavitt path algebra $L_K(E)$ is just $L_K(E, C)$ where $C_v = \{s^{-1}(v)\}$ if $s^{-1}(v) \neq \emptyset$ and $C_v = \emptyset$ if $s^{-1}(v) = \emptyset$.

Despite the great similarity in the definitions, the Leavitt path algebras of separated graphs encompase a much larger class of algebras than Leavitt path algebras do, for instance free products of Leavitt path algebras and algebras closely related to the Leavitt algebras $L_K(m, n)$ for $1 < m \leq n$.

Example

Assume that (E, C) is a separated graph and that $|E^0| = 1$. Then we have

$$L_K(E, C) \cong *_{X \in C} \, L_K(|X|),$$

that is, $L_K(E, C)$ is a free product over K of classical Leavitt algebras of type $(1, |X|)$, for $X \in C$.

Leavitt (1962) defined algebras $L_K(m,n)$ for $1 \leq m \leq n$ in the following way:
$L_K(m,n)$ is the K-algebra with generators

$$\{X_{ji}, X_{ji}^* : 1 \leq j \leq m, 1 \leq i \leq n\}$$

and defining relations:

$$XX^* = I_m, \quad X^*X = I_n,$$

where $X = (X_{ji})$.

Example

Let $1 \leq m \leq n$. Let us consider the separated graph $(E(m,n), C(m,n))$, where $E(m,n)$ is the graph consisting of two vertices v, w and with

$$E(m,n)^1 = \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\},$$

with $s(\alpha_i) = s(\beta_j) = v$ and $r(\alpha_i) = r(\beta_j) = w$ for all i, j, and $C(m,n)$ consists of two elements $X = \{\alpha_1, \dots, \alpha_n\}$ and $Y = \{\beta_1, \dots, \beta_m\}$.

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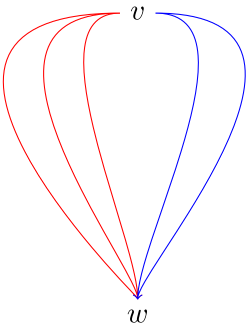

Figure: The separated graph $(E(2,3), C(2,3))$

As remarked by E. Pardo, we have

Lemma

There is a natural isomorphism

$$\gamma \colon L(m,n) \to wA_{m,n}w$$

given by

$$\gamma(X_{ji}) = \beta_j^* \alpha_i, \quad \gamma(X_{ji}^*) = \alpha_i^* \beta_j$$

Note that

$$\gamma(\sum_{i=1}^{n} X_{ji} X_{ki}^*) = \sum_{i=1}^{n} \beta_j^* \alpha_i \alpha_i^* \beta_k = \beta_j^* \beta_k = \delta_{jk} w$$

and similarly $\gamma(\sum_{j=1}^{m} X_{ji}^* X_{jk}) = \delta_{ik} w$ so γ is a well-defined homomorphism, which is shown to be an isomorphism.

Since $v \sim n \cdot w \sim m \cdot w$, we get from the above

$$A_{m,n} \cong M_{n+1}(wA_{m,n}w) \cong M_{n+1}(L(m,n)) \cong M_{m+1}(L(m,n)).$$

Definition

(E, C) is *finitely separated* in case $|X| < \infty$ for all $X \in C$.

Definition

Let (E, C) be a finitely separated graph. The *monoid* of (E, C) is the abelian monoid $M(E, C)$ with generators $\{a_v \mid v \in E^0\}$ and relations

$$a_v = \sum_{e \in X} a_{r(e)}, \qquad \forall X \in C_v, \forall v \in E^0.$$

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Proposition

If M is any conical abelian monoid, then there exists a finitely separated graph (E, C) such that

$$M \cong M(E, C) \cong \mathcal{V}(L_K(E, C)).$$

Example

In the example $M = \langle a, b \mid 2a = a + 2b \rangle$, we have two generators a, b and one relation $R : 2a = a + 2b$.

Figure: $M(E, C) = \langle R, a, b \mid R = 2a, R = a + 2b \rangle \cong M$.

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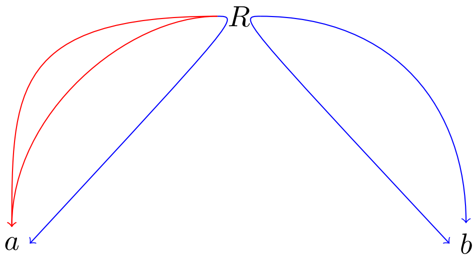

Figure: $M(E, C) = \langle R, a, b \mid R = 2a, R = a + 2b \rangle \cong M$.

Definition (Wehrung)

A monoid homomorphism $\psi : M \to F$ is *unitary* provided

1. ψ is injective;
2. $\psi(M)$ is *cofinal* in F, that is, for each $u \in F$ there is some $v \in M$ with $u \leq \psi(v)$;
3. whenever $u, u' \in M$ and $v \in F$ with $\psi(u) + v = \psi(u')$, we have $v \in \psi(M)$.

Embedding separated graph monoids into refinement separated graph monoids

Theorem

Given a finitely separated graph (E, C) there exists another separated graph (E_+, C^+) and a suitable embedding of separated graphs $\iota\colon (E, C) \to (E_+, C^+)$ such that:

1. $M(E_+, C^+)$ *is a refinement monoid.*
2. $M(\iota)\colon M(E, C) \to M(E_+, C^+)$ *is a unitary embedding.*
3. $L(\iota)\colon L_K(E, C) \to L_K(E_+, C^+)$ *is an algebra embedding.*

Unfortunately, $L_K(E,C)$ is not a regular ring even when $M(E,C) \cong \mathcal{V}(L_K(E,C))$ is a refinement monoid.

What about embedding it into a suitable universal localization?

We don't know, but a first difficulty is that although $L_K(E,C)$ is *generated* by partial isometries (elements w such that $w = ww^*w$), since $e = ee^*e$ and $e^* = e^*ee^* \;\; \forall e \in E^1$, the **products of these generators are not partial isometries in general.** (Although are so in the non-separated case.)

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Strategy:

1. To create a new algebra $\mathcal{R}$ from $L_K(E, C)$ such that all monomials of $\mathcal{R}$ are partial isometries.
2. To find a suitable set Σ so that the universal localization $\Sigma^{-1}\mathcal{R}$ is regular.
3. Of course, with controlled monoids!

Thank you very much for your attention!!!