Twisted Differential Operators and Hochschild Homology

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1. Introduction

Notation

X , smooth irreducible algebraic curve over $\mathbb{C}$

$\mathcal{O}_X$, the ring of regular functions on X

$\mathcal{D}_X$, the ring of differential operators on X

For most of the talk, we will assume that X is affine, with $\bar{X} \setminus X = \{\infty\}$.

Example 1. If $X = \mathbb{A}^1$ then $\mathcal{D}_X \cong A_1(\mathbb{C}) := \mathbb{C}\langle x,\, y\rangle/(xy - yx - 1)$.

Morita classification of rings of differential operators on curves:

Theorem 1 (Smith and Stafford, 1988; Cannings and Holland, 1994). *Let $\mathcal{D}$ be a domain ($\mathbb{C}$-algebra) Morita equivalent to $\mathcal{D}_X$. Then there is a (singular) curve Y and rank one torsion-free coherent sheaf $\mathcal{L}$ on Y such that*

$$\mathcal{D} \cong \mathcal{D}_Y(\mathcal{L}) \ ,$$

where $\mathcal{D}_Y(\mathcal{L})$ is the ring of twisted differential operators on Y with coefficients in $\mathcal{L}$. Conversely, if Y is a curve such that

(a) $\tilde{Y} \cong X$,

(b) $\pi : \tilde{Y} \to Y$ *is (set-theoretically) bijective,*

then $\mathcal{D}_Y(\mathcal{L})$ is Morita equivalent to $\mathcal{D}_X$ for any rank one torsion-free coherent sheaf $\mathcal{L}$ on Y.

Remarks:

1. If $\mathcal{D}$ is not a domain, then $\mathcal{D} \sim \mathcal{D}_X$ iff $\mathcal{D} \cong M_r(\mathcal{D}_X)$ for some $r \geq 2$.

2. If X and Y are two $smooth$ curves then $\mathcal{D}_X \sim \mathcal{D}_Y$ iff $\mathcal{D}_X \cong \mathcal{D}_Y$.

Thus, the class of domains Morita equivalent to $\mathcal{D}_X$ is precisely $\{\mathcal{D}_Y(\mathcal{L})\}$, where Y is a curve satisfying (a) and (b) above.

Problem: Classify the algebras $\{\mathcal{D}_Y(\mathcal{L})\}$ up to isomorphism.

For the affine line, a complete answer is known, and it is surprisingly simple (perhaps, deceptively so...)

Theorem 2 (Kouakou, 1994; Wilson and B., 1999). *Let $X = \mathbb{A}^1$. To each algebra $\mathcal{D}_Y(\mathcal{L})$ in the Morita class of $\mathcal{D}_X$ one can assign a (unique) non-negative integer $n = n(Y, \mathcal{L})$, such that*

$$\mathcal{D}_Y(\mathcal{L}) \cong \mathcal{D}_{Y'}(\mathcal{L}') \iff n(Y, \mathcal{L}) = n(Y', \mathcal{L}') .$$

In other words, the domains Morita equivalent to $A_1(\mathbb{C})$ are classified, up to isomorphism, by $\mathbb{Z}_{\geq 0}$, with $n = 0$ corresponding to $A_1(\mathbb{C})$.

Remark: For $\mathcal{L} = \mathcal{O}_Y$, we call $n(Y, \mathcal{L})$ the *differential genus* of Y (but it now appears that there is a better name...)

For an arbitrary curve X, the answer is more complicated (to the best of our knowledge, it has not appeared in the literature so far). The invariant $n(Y, \mathcal{L})$ in Theorem 2 is most naturally defined in terms of 'moduli spaces' of projective $\mathcal{D}$-modules on X, and is not easily "seen" in terms of $(Y, \mathcal{L})$. For a general X, the situation is similar.

Motivation:

1. If $Y \subset \mathbb{C}^2$ is a plane curve (as in Theorem 2) with a single cusp at the origin, the number $n(Y, \mathcal{L})$ (and hence the corresponding isomorphism class $[\mathcal{D}_Y(\mathcal{L})]$) is a *topological* invariant of the complement $\mathbb{C}^2 \setminus Y$: in fact, it coincides with the Casson invariant of a knot associated to $\mathbb{C}^2 \setminus Y$. It is an interesting question whether this is true in the global case. (Classically, due to Zariski, it is known that the topology of $\mathbb{C}^2 \setminus Y$ depends on the position of singularities of Y.)

2. Analogy with the gluing construction for holonomic $\mathcal{D}$-modules and perverse sheaves (Beilinson, Verdier). V. Ginzburg's question/suggestion: is there a relevant version of nearby and vanishing cycles functors?

2. Projective $\mathcal{D}$-modules on curves

Aim: Classify f.g. projective $\mathcal{D}$-modules on X. It suffices to describe the rank one projectives: we let $\mathcal{I}(\mathcal{D})$ denote the space of isomoprhism classes of such modules. Equivalently, $\mathcal{I}(\mathcal{D})$ is the space of isomorphism classes of (nonzero) left ideals of $\mathcal{D}_X$. Alternative approaches: Ben-Zvi and Nevins (2007); Chalykh and B. (2008).

Stable classification. Let $K_0(X)$ and $\mathrm{Pic}(X)$ be the Grothendieck and Picard group of X. We have

$$\mathrm{rk} \oplus \det : \; K_0(X) \xrightarrow{\sim} \mathbb{Z} \oplus \mathrm{Pic}(X) \ .$$

Consider

$$\gamma : \mathcal{I}(\mathcal{D}) \xrightarrow{\mathrm{can}} K_0(\mathcal{D}) \xrightarrow{\sim} K_0(X) \xrightarrow{\det} \mathrm{Pic}(X) \ ,$$

where $\simeq$ is the Quillen isomoprhism induced by the inclusion $\mathcal{O} \hookrightarrow \mathcal{D}$.

Proposition 1. *Let M be a projective $\mathcal{D}$-module of rank 1 equipped with a good filtration. Let $\overline{\mathcal{D}} := \mathsf{gr}(D)$, and let $\overline{M}$ be the graded $\overline{\mathcal{D}}$-module associated to M. Then*

(*a*) *there is a unique ideal $\mathcal{I}_M \subseteq \mathcal{O}$, such that $\overline{M}$ is isomorphic to a sub-$\overline{\mathcal{D}}$-module of $\overline{\mathcal{D}}\mathcal{I}_M$ of finite codimension;*

(*b*) *the class $[\mathcal{I}_M] \in \mathrm{Pic}(X)$ and the codimension $n := \dim_{\mathbb{C}} \overline{\mathcal{D}}\mathcal{I}_M/\overline{M}$ are independent of the choice of filtration, and we have*

$$\gamma[M] = [\mathcal{I}_M] \ .$$

(*c*) *if M and N are two projective $\mathcal{D}$-modules of rank 1, then*

$$[\, M \,] = [\, N \,] \ \textit{in}\ K_0(\mathcal{D}) \quad \Longleftrightarrow \quad [\mathcal{I}_M] = [\mathcal{I}_N] \ \textit{in}\ \mathrm{Pic}(X) \ .$$

Thus, $\gamma : \mathcal{I}(\mathcal{D}_X) \to \mathrm{Pic}(X)$ is a fibration, with fibres being precisely the *stable* isomorphism classes of projectives of $\mathcal{D}$. The stably free projectives M are characterized by the property that $\overline{M}$ is isomorphic to an ideal in $\overline{\mathcal{D}}$ of finite codimension (over $\mathbb{C}$).

We refine the K-theoretic classification by describing the fibres of γ.

3. Global Calogero-Moser spaces

Double derivations (Crawley-Boevey, 1999). Let A be any unital associative algebra. Then $A^{\otimes 2}$ has two commuting bimodule structures:

$$a.(x \otimes y).b = ax \otimes yb \qquad \text{and} \qquad a.(x \otimes y).b = xb \otimes ay \ .$$

Let $\mathbb{D}\mathrm{er}(A) := \mathrm{Der}(A, A^{\otimes 2})$ for the first ('outer') structure and define

$$\Pi^{\lambda}(A) := T_A \mathbb{D}\mathrm{er}(A) / \langle \Delta_A - \lambda \rangle \ , \quad \lambda \in A \ .$$

where $\Delta_A \in \mathbb{D}\mathrm{er}(A)$ is given by $x \to x \otimes 1 - 1 \otimes x$, $x \in A$.

Remark: $\Pi^{\lambda}(A)$ depends only on the class of λ in $\mathrm{HH}_0(A)$. It is sometimes convenient to let $\lambda \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{C}$ relating K_0 to HH_0 via the trace map.

Geometric meaning (Crawley-Boevey, Etingof and Ginzburg, 2007): Assume that A is formally smooth (quasi-free). Then

$$\mathrm{Rep}_n[T_A \mathbb{D}\mathrm{er}(A)] \cong T^*\mathrm{Rep}_n(A)\ , \quad \forall n \in \mathbb{N}\ .$$

If λ is constant, then $\Pi^\lambda(A) = \mu^{-1}(\lambda)$ is a fibre algebra for the moment map w.r.t. the natural GL_n-action on $T^*\mathrm{Rep}_n(A)$:

$$\mu : T^*\mathrm{Rep}_n(A) \to \mathfrak{gl}_n\ , \quad \varrho \mapsto \Delta_A(\varrho)\ .$$

Relative version: If B is an algebra over a f.d. semisimple algebra $S = \bigoplus_{i=1}^N \mathbb{C}\, e_i$, then we define

$$\Pi_S^{\boldsymbol{\lambda}}(B) := T_B \mathbb{D}\mathrm{er}_S(B)/\langle \Delta_{B,S} - \boldsymbol{\lambda} \rangle\ , \quad \boldsymbol{\lambda} \in S\ ,$$

where $\Delta_{B,S} := \mathrm{ad}(\mathbf{e}) \in \mathbb{D}\mathrm{er}_S(B)$ with $\mathbf{e} = \sum_{i=1}^N e_i \otimes e_i\ .$

Example 2. If Q be a finite quiver (oriented graph) with vertex set Q_0 and arrow set Q_1, let $B := \mathbb{C}Q$, $S = \bigoplus_{i \in Q_0} e_i$. Then

$$\Pi_S^{\boldsymbol{\lambda}}(B) \cong \frac{\mathbb{C}\bar{Q}}{\sum_{a \in Q_1}[a,\, a^*] = \sum_{i \in Q_0} \lambda_i\, e_i} \;, \quad \boldsymbol{\lambda} = \sum_{i \in Q_0} \lambda_i e_i \;.$$

Example 3. If $A = \mathcal{O}_X$, X smooth affine curve, then

$$\Pi^0(A) \cong \mathcal{O}_{T^*X} \quad \text{and} \quad \Pi^1(A) \cong \mathcal{D}_X \;.$$

More precisely, $\Pi^1(\mathcal{O}_X) \cong \mathcal{D}_X(\Omega_X^{1/2})$, where $\Omega_X^{1/2}$, where $\Omega_X^{1/2}$ is a square root of the canonical bundle on X.

Framing a curve: For a line bundle $\mathcal{I}$, define $\mathcal{O}_X[\mathcal{I}]$ to be the infinitesimal extension

$$0 \to \mathcal{I} \to \mathcal{O}_X[\mathcal{I}] \to \mathcal{O}_X \times \mathbb{C} \to 0 \ .$$

Thus $\mathcal{O}_X[\mathcal{I}]$ is a 'noncommutative' thickening of $\mathrm{Spec}(X \times \mathbb{C}) = X \bigsqcup \mathrm{pt}$. It is convenient to write $\mathcal{O}_X[\mathcal{I}]$ in the matrix form

$$\mathcal{O}_X[\mathcal{I}] := \begin{pmatrix} \mathcal{O}_X & \mathcal{I} \\ 0 & \mathbb{C} \end{pmatrix}$$

Note that $\mathcal{O}_X[\mathcal{I}]$ has two canonical idempotents

$$e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

Lemma 1. *For two line bundles $\mathcal{I}$ and $\mathcal{J}$ on X, the algebras $\mathcal{O}_X[\mathcal{I}]$ and $\mathcal{O}_X[\mathcal{J}]$ are Morita equivalent.*

Let $B := \mathcal{O}_X[\mathcal{I}]$, $S := \mathbb{C}\, e \oplus \mathbb{C}\, e_\infty \subset B$. For $\boldsymbol{\lambda} = \lambda e + \lambda_\infty e_\infty \in S$ and $\boldsymbol{n} = (n,\, n_\infty) \in \mathbb{N}^2$, define

$$\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I}) := \mathrm{Rep}_S(\Pi_S^{\boldsymbol{\lambda}}(B), \boldsymbol{n})//\mathrm{GL}_S(\boldsymbol{n})$$

where $\mathrm{GL}_S(\boldsymbol{n}) := [\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n_\infty, \mathbb{C})]/\mathbb{C}^*$.

By definition, $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I})$ is an affine scheme, whose (closed) points are in bijection with isomorphism classes of semisimple $\Pi_S^{\boldsymbol{\lambda}}(B)$-modules of dimension vector $\boldsymbol{n}$. Note that $\overline{\mathcal{C}}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I})$ depends only on the class $[\mathcal{I}] \in \mathrm{Pic}(X)$. More generally, for any line bundles $\mathcal{I}$ and $\mathcal{J}$ on X, we have

$$\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I}) \cong \mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{J})$$

however, there is no natural choice for such an isomorphism.

Definition. For $\boldsymbol{\lambda} = (1, -n)$ and $\boldsymbol{n} = (n, 1)$ with $n \in \mathbb{N}$, we let

$$\mathcal{C}_n(X, \mathcal{I}) := \mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I})$$

and call it the *n-th Calogero-Moser space* associated to $(X, \mathcal{I})$.

Theorem 3 (Chalykh and B., 2008). *For each $n \geq 0$ and $[\mathcal{I}\,] \in \mathrm{Pic}(X)$, $\mathcal{C}_n(X, \mathcal{I})$ is a smooth irreducible affine variety of dimension $2n$.*

Remark: In case $X = \mathbb{A}^1$, this theorem was first proved by Wilson (1998).

In general, the varieties $\mathcal{C}_n(X, \mathcal{I})$ can be described quite explicitly.

Lemma 2. *There is an isomorphism of B-bimodules*

$$\mathbb{D}\mathrm{er}_S(B) \cong \begin{pmatrix} \mathbb{D}\mathrm{er}(\mathcal{O}) & \mathrm{Der}(\mathcal{O}, \mathcal{I} \otimes \mathcal{O}) \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} \mathcal{I} \otimes \mathcal{I}^* & \mathcal{I} \otimes \mathcal{O} \\ \mathcal{I}^* & \mathcal{O} \end{pmatrix},$$

with $\Delta_{B,S}$ *corresponding to the element*

$$\left[\begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\sum_i \mathrm{v}_i \otimes \mathrm{w}_i & 0 \\ 0 & 1 \end{pmatrix} \right],$$

where $(\mathrm{v}_i, \mathrm{w}_i)$ *is a pair of dual bases for* $\mathcal{I}$ *and* $\mathcal{I}^*$*.*

As a consequence, we get

Proposition 2. *The algebra* $\Pi_S^{\boldsymbol{\lambda}}(B)$ *is generated by the following elements*

$$\hat{a} := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\ , \ \hat{\mathrm{v}}_i := \begin{pmatrix} 0 & \mathrm{v}_i \\ 0 & 0 \end{pmatrix}\ , \ \hat{d} := \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}\ , \ \hat{\mathrm{w}}_i := \begin{pmatrix} 0 & 0 \\ \mathrm{w}_i & 0 \end{pmatrix}\ ,$$

where $\hat{a}\,,\,\hat{\mathrm{v}}_i \in B$ *and* $\hat{d}\,,\,\hat{\mathrm{w}}_i \in \mathbb{D}\mathrm{er}_S(B)$ *with* $d \in \mathbb{D}\mathrm{er}(\mathcal{O})$ *. Apart from the obvious relations induced by matrix multiplication, these elements satisfy*

$$\hat{\Delta} - \sum_{i=1}^{N} \hat{\mathrm{v}}_i \cdot \hat{\mathrm{w}}_i = \lambda\, e\ , \quad \sum_{i=1}^{N} \hat{\mathrm{w}}_i \cdot \hat{\mathrm{v}}_i = \lambda_\infty\, e_\infty\ , \qquad (1)$$

where '$\cdot$' denotes the action of B *on the bimodule* $\mathbb{D}\mathrm{er}_S(B)$*.*

With above Proposition, we can describe $\mathcal{C}_n(X,\mathcal{I})$ as the space of equivalence classes of linear maps (matrices)

$$\{\,\bar{a},\ \bar{d} \in \mathrm{End}(\mathbb{C}^n)\,,\ \bar{\mathrm{v}}_i \in \mathrm{Hom}(\mathbb{C},\mathbb{C}^n),\ \bar{\mathrm{w}}_i \in \mathrm{Hom}(\mathbb{C}^n,\mathbb{C})\,\}\ ,$$

satisfying the relations

$$\bar{\Delta} - \sum_{i=1}^{N} \bar{\mathrm{v}}_i\,\bar{\mathrm{w}}_i = \mathrm{Id}_n\ , \qquad \sum_{i=1}^{N} \bar{\mathrm{w}}_i\,\bar{\mathrm{v}}_i = -n\ .$$

In addition, $\bar{a}$ and $\bar{d}$ should also obey the internal relations of the algebra $\mathcal{O}$ and the bimodule $\mathbb{D}\mathrm{er}(\mathcal{O})$. Giving a matrix presentation of $\mathcal{C}_n(X,\mathcal{I})$ thus boils down to describing $\mathcal{O}$ and $\mathbb{D}\mathrm{er}(\mathcal{O})$ in terms of generators and relations. This can be easily done in practice.

Example 4. Let $X = \mathbb{A}^1$. Choosing a global coordinate, we identify $\mathcal{O}_X \cong \mathbb{C}[x]$. All line bundles on X are trivial, so we only need to consider $\mathcal{I} = \mathcal{O}_X$. The n-th Calogero-Moser variety $\mathcal{C}_n := \mathcal{C}_n(X, \mathcal{I})$ is isomorphic to the space of equivalence classes of matrices

$$\{\bar{X} \in \mathrm{End}(\mathbb{C}^n),\, \bar{Y} \in \mathrm{End}(\mathbb{C}^n),\, \bar{\mathrm{v}} \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n),\, \bar{\mathrm{w}} \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})\}\ ,$$

satisfying the relation

$$\bar{Y}\bar{X} - \bar{X}\bar{Y} = \mathrm{Id}_n + \bar{\mathrm{v}}\bar{\mathrm{w}}$$

modulo the natural action of $\mathrm{GL}_n(\mathbb{C})$:

$$(\bar{X},\, \bar{Y},\, \bar{\mathrm{v}},\, \bar{\mathrm{w}}) \ \mapsto\ (g\bar{X}g^{-1},\, g\bar{Y}g^{-1},\, g\bar{\mathrm{v}},\, \bar{\mathrm{w}}g^{-1}), \quad g \in \mathrm{GL}_n(\mathbb{C})\ .$$

Example 5 (Plane curves). Let X be a smooth curve on $\mathbb{C}^2$ defined by $F(x,y)=0$, with $F(x,y) := \sum_{r,s} a_{rs} x^r y^s \in \mathbb{C}[x,y]$. The bimodule $\mathbb{D}\mathrm{er}(\mathcal{O}_X)$ is generated by the canonical derivation $\Delta = \Delta_X$ and the element z defined by

$$z(x) = \sum_{r,s} a_{rs} \sum_{k=0}^{s-1} x^r y^k \otimes y^{s-k-1} \,, \quad z(y) = -\sum_{r,s} a_{rs} \sum_{l=0}^{r-1} x^l \otimes x^{r-l-1} y^s \,.$$

These generators satisfy the following commutation relations

$$[z,\, x] = \sum_{r,s} a_{rs} \sum_{k=0}^{s-1} y^{s-k-1} \Delta y^k x^r \,,$$

$$[z,\, y] = -\sum_{r,s} a_{rs} \sum_{l=0}^{r-1} y^s x^{r-l-1} \Delta x^l \,.$$

By Proposition, the algebra $\Pi^{\boldsymbol{\lambda}}(B)$ is then generated by the elements $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{\mathrm{v}}_i$, $\hat{\mathrm{w}}_i$ and $\hat{\Delta}$, subject to the above relations.

We can explicitly describe generic points of the varieties $\mathcal{C}_n(X, \mathcal{I})$ (for simplicity, we take $\mathcal{I} \cong \mathcal{O}_X$). Choose n distinct points $p_i = (x_i,\, y_i) \in X$ and define the matrices $(\bar{X}, \bar{Y}, \bar{Z}, \bar{\mathrm{v}}, \bar{\mathrm{w}})$ by

$$\bar{X} = \mathrm{diag}(x_1, \ldots, x_n)\,, \quad \bar{Y} = \mathrm{diag}(y_1, \ldots, y_n)\,, \quad \bar{\mathrm{v}}^t = -\bar{\mathrm{w}} = (1,\, \ldots\, ,\, 1)\,,$$

$$\bar{Z}_{ii} = \alpha_i \quad , \quad \bar{Z}_{ij} = \frac{F(x_i,\, y_j)}{(x_i - x_j)(y_i - y_j)} \quad (i \neq j)\,,$$

where $\alpha_1, \ldots, \alpha_n$ are arbitrary scalars. Then

$$\hat{x} \mapsto \bar{X}\ , \quad \hat{y} \mapsto \bar{Y}\ , \quad \hat{z} \mapsto \bar{Z}\ , \quad \hat{\mathrm{v}} \mapsto \bar{\mathrm{v}}\ , \quad \hat{\mathrm{w}} \mapsto \bar{\mathrm{w}}\ , \quad \hat{\Delta} \mapsto \mathrm{Id}_n + \bar{\mathrm{v}}\,\bar{\mathrm{w}}$$

defines a representation of $\Pi^{\boldsymbol{\lambda}}_S(B)$ on $\boldsymbol{V} = \mathbb{C}^n \oplus \mathbb{C}$. The equivalence classes of such representations correspond to generic points of $\mathcal{C}_n(X,\, \mathcal{O})$.

4. Recollement of triangulated categories

Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. The *recollement situation* is described by the diagram of six additive functors:

$$\begin{tikzcd}
\mathcal{D}' \arrow[r, "i_*"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{D}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

satisfying

(a) $(i^*, i_*, i^!)$ is an adjoint triple, with $i^*i_* \simeq \mathrm{Id} \simeq i^!i_*$.

(b) $(j_!, j^*, j_*)$ is an adjoint triple, with $j^*j_* \simeq \mathrm{Id} \simeq j^*j_!$.

(c) there are (functorial) exact triangles in $\mathcal{D}$:

$$j_! j^* V \to V \to i_* i^* V \to \quad , \qquad i_* i^! V \to V \to j_* j^* V \to$$

(d) $j^* i_* = 0$, $i^* j_! = 0$, $i^! j_* = 0$.

The ‘recollement’ conditions were originally designed to abstract a natural structure on the derived category $\mathcal{D}(\mathcal{S}h_X)$ of abelian sheaves arising from the stratification of a topological space into a closed subspace and its open complement (Beilinson-Bernstein-Deligne, 1982). It is remarkable that such a stratification may occur in purely algebraic and algebro-geometric settings.

Lemma 3. *There is a canonical surjective algebra map* $\Pi^{\boldsymbol{\lambda}}_S(B) \to \Pi^1(\mathcal{O}_X)$, *with* $\mathrm{Ker}(i) = \langle\, e_\infty \,\rangle$.

Indeed, the map $\Pi^{\boldsymbol{\lambda}}_S(B) \to \Pi^1(\mathcal{O}_X)$ is induced by the natural extension $\mathcal{O}_X[\mathcal{I}] \to \mathcal{O}_X$ in view of functoriality of the Π^λ-construction. Identifying $\Pi^1(\mathcal{O}_X) \cong \mathcal{D}_X$ (see Example 3 above), we get

$$i :\ \Pi^{\boldsymbol{\lambda}}_S(B) \to \mathcal{D}_X\ .$$

Remark. The map i is an example of a *noncommutative* Hamiltonian reduction.

We now relate the module categories of $\mathcal{D}_X$ and $\Pi^{\boldsymbol{\lambda}}_S(B)$.

Write $\Pi^{\boldsymbol{\lambda}} := \Pi_S^{\boldsymbol{\lambda}}(B)$ and set $U^{\boldsymbol{\lambda}} := e_\infty \Pi^{\boldsymbol{\lambda}} e_\infty$. There are six natural functors $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ between the module categories of $\Pi^{\boldsymbol{\lambda}}$, $\mathcal{D}_X$ and $U^{\boldsymbol{\lambda}}$:

$i_* : \mathtt{Mod}(\mathcal{D}_X) \to \mathtt{Mod}(\Pi^{\boldsymbol{\lambda}})$ is just the restriction functor for the algebra map $i : \Pi^{\boldsymbol{\lambda}} \to \mathcal{D}_X$. This is fully faithful and has both the right adjoint $i^! := \mathrm{Hom}_\Pi(\mathcal{D}_X, —)$ and the left adjoint $i^* := \mathcal{D}_X \otimes_\Pi —$, with adjunction maps $i^* i_* \simeq \mathrm{Id} \simeq i^! i_*$ being isomorphisms.

$j^* : \mathtt{Mod}(\Pi^{\boldsymbol{\lambda}}) \to \mathtt{Mod}(U^{\boldsymbol{\lambda}})$ is defined by $j^* \boldsymbol{V} := e_\infty \boldsymbol{V}$. Since $e_\infty \in \Pi^{\boldsymbol{\lambda}}$ is an idempotent, j^* is exact and has the right and the left adjoint functors: $j_* := \mathrm{Hom}_U(e_\infty \Pi^{\boldsymbol{\lambda}}, —)$ and $j_! := \Pi^{\boldsymbol{\lambda}} e_\infty \otimes_U —$ respectively, satisfying $j^* j_* \simeq \mathrm{Id} \simeq j^* j_!$.

Proposition 3 (Chalykh-B., 2008). *The functors* $(i^*, i_*, i^!)$ *and* $(j_!, j^*, j_*)$ *induce the six functors at the level of (bounded) derived categories, which satisfy the recollement axioms*

$$\begin{tikzcd} \mathfrak{D}^b(\texttt{Mod}\,\mathcal{D}_X) \arrow[r, "i_*"] & \mathfrak{D}^b(\texttt{Mod}\,\Pi^{\boldsymbol{\lambda}}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathfrak{D}^b(\texttt{Mod}\,U^{\boldsymbol{\lambda}}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} .$$

Theorem 4 (Chalykh-B., 2008). *Let* X *be a smooth affine irreducible curve over* $\mathbb{C}$.

(*a*) *For each* $n \geq 0$ *and* $[\mathcal{I}] \in \mathrm{Pic}(X)$, *the functor* $L_1 i^*$ *induces an injective map*

$$\omega_n : \mathcal{C}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}] \ ,$$

which is equivariant under the action of the group $\mathrm{Pic}(\mathcal{D}_X)$ *in the* *following sense:*

$$\begin{tikzcd}
\mathcal{C}_n(X,\mathcal{I}) \arrow[r, "\bar{f}_{\mathcal{P}}"] \arrow[d, "\omega_n"'] & \mathcal{C}_n(X,\mathcal{J}) \arrow[d, "\omega_n"] \\
\gamma^{-1}[\mathcal{I}] \arrow[r, "{[\mathcal{P}]}"] & \gamma^{-1}[\mathcal{J}]
\end{tikzcd}$$

where $[\mathcal{P}] \in \mathrm{Pic}(\mathcal{D}_X)$ *is such that* $[\mathcal{P}] \cdot [\mathcal{I}] = [\mathcal{J}]$.

(b) *Amalgamating the maps* ω_n *for all* $n \geq 0$ *yields a bijective correspondence*

$$\omega : \coprod_{n \geq 0} \mathcal{C}_n(X,\mathcal{I}) \xrightarrow{\sim} \gamma^{-1}[\mathcal{I}] \ .$$

Remark: The noncommutative Picard group $\mathrm{Pic}(\mathcal{D}_X)$ acts naturally on $K_0(\mathcal{D}_X)$ and hence on $K_0(X)$ via the Quillen isomorphism. This last action on $K_0(X)$ restricts to $\mathrm{Pic}(X)$ via the splitting $K_0(X) \cong \mathbb{Z} \oplus \mathrm{Pic}(X)$. It is not hard to show that the resulting action of $\mathrm{Pic}(\mathcal{D}_X)$ on $\mathrm{Pic}(X)$ is transitive.

Despite the complete naturality of all contructions, the proof of the above Theorem is fairly long and subtle. By analogy with classic recollement situations (holonomic $\mathcal{D}_X$-modules, perverse sheaves), V. Ginzburg suggested that one should first prove an analog of Beilinson's glueing theorem. For this, one needs to develop an appropriate formalism of nearby and vanishing cycles functors in this noncommutative setting...

5. Classification of twisted differential operators

Assumption: For simplicity, we assume that $X \neq \mathbb{A}^1$ and $\mathrm{Aut}(X)$ is trivial.

Recall that

$$\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X,\mathcal{I}) := \mathrm{Rep}_S(\Pi_S^{\boldsymbol{\lambda}}(B),\boldsymbol{n})//\mathrm{GL}_S(\boldsymbol{n})$$

Let $\pi : \mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X,\mathcal{I}) \to \mathrm{Rep}_S(B,\, \boldsymbol{n})//\mathrm{GL}_S(\boldsymbol{n})$ be the canonical projection.

For $\varrho : \Pi_S^{\boldsymbol{\lambda}} \to \mathrm{End}(\boldsymbol{V})$, we define

$$\delta_\varrho \in \mathrm{Hom}_{B^{\mathrm{e}}}(\mathbb{D}\mathrm{er}(B), \mathrm{End}(\boldsymbol{V})) \cong \mathrm{End}(\boldsymbol{V}) \otimes_{B^{\mathrm{e}}} \Omega^1 B\ .$$

For $\varrho : B \to \mathrm{End}(V)$, we define

$$\varrho_* :\ \Omega^1 X \cong \mathrm{HH}_1(\mathcal{O}_X) \cong \mathrm{HH}_1(B) \to \mathrm{HH}_1(B, \mathrm{End}(\boldsymbol{V}))\ .$$

Theorem 5. *Let* $[M]$ *and* $[N]$ *be two classes in* $\gamma^{-1}(\mathcal{I}) \subset \mathfrak{I}(\mathcal{D})$. *Let* $[\boldsymbol{V}_N]$ *and* $[\boldsymbol{V}_M]$ *be the corresponding (classes) of representations of* $\Pi_S^{\boldsymbol{\lambda}}(B)$. *Then*

$$\mathrm{End}_{\mathcal{D}}(M) \cong \mathrm{End}_{\mathcal{D}}(N)$$

if and only if

(1) $\pi[\boldsymbol{V}_N] = \pi[\boldsymbol{V}_M]$,

(2) $\delta_M - \delta_N \in \mathrm{Im}(\rho_*)$.