Poisson algebras and their deformations: a case study

David Jordan

School of Mathematics and Statistics, University of Sheffield

Context and aims

Poisson algebras	quantized algebras
Poisson prime spectrum	completely prime spectrum
fin dim simple Poisson modules	fin dim simple modules

Aims: to present some results from the Poisson side that can help to predict outcomes on the quantized side for particular algebras, and to illustrate this with a “new” example.

Context and aims

Poisson algebras	quantized algebras
Poisson prime spectrum	completely prime spectrum
fin dim simple Poisson modules	fin dim simple modules

Aims: to present some results from the Poisson side that can help to predict outcomes on the quantized side for particular algebras, and to illustrate this with a “new” example.

Poisson algebras and ideals

- Poisson algebra: commutative finitely generated $\mathbb{C}$-algebra A with $\{-,-\} : A \times A \to A$, such that
  - A is a Lie algebra under $\{-,-\}$,
  - each $\{a,-\}$ is a derivation of A.
- Poisson ideal: ideal I of A with $\{i,a\} \in I$ for all $i \in I, a \in A$.
  A/I is then a Poisson algebra in the obvious way.
- Poisson prime ideal: prime and Poisson ideal; equivalently, Poisson ideal P such that $IJ \subseteq P$, I, J Poisson $\Rightarrow I \subseteq P$ or $J \subseteq P$.
- Poisson maximal: maximal and Poisson $\neq$ maximal Poisson (maximal as a Poisson ideal).

Poisson algebras and ideals

- Poisson algebra: commutative finitely generated $\mathbb{C}$-algebra A with $\{-,-\} : A \times A \to A$, such that
  - A is a Lie algebra under $\{-,-\}$,
  - each $\{a,-\}$ is a derivation of A.
- Poisson ideal: ideal I of A with $\{i,a\} \in I$ for all $i \in I, a \in A$.
  A/I is then a Poisson algebra in the obvious way.
- Poisson prime ideal: prime and Poisson ideal; equivalently, Poisson ideal P such that $IJ \subseteq P$, I, J Poisson $\Rightarrow I \subseteq P$ or $J \subseteq P$.
- Poisson maximal: maximal and Poisson $\neq$ maximal Poisson (maximal as a Poisson ideal).

Poisson algebras and ideals

- Poisson algebra: commutative finitely generated $\mathbb{C}$-algebra A with $\{-,-\} : A \times A \to A$, such that
  - A is a Lie algebra under $\{-,-\}$,
  - each $\{a,-\}$ is a derivation of A.
- Poisson ideal: ideal I of A with $\{i,a\} \in I$ for all $i \in I, a \in A$.
  A/I is then a Poisson algebra in the obvious way.
- Poisson prime ideal: prime and Poisson ideal; equivalently, Poisson ideal P such that $IJ \subseteq P$, I, J Poisson $\Rightarrow I \subseteq P$ or $J \subseteq P$.
- Poisson maximal: maximal and Poisson $\neq$ maximal Poisson (maximal as a Poisson ideal).

Poisson algebras and ideals

- Poisson algebra: commutative finitely generated $\mathbb{C}$-algebra A with $\{-,-\} : A \times A \to A$, such that
  - A is a Lie algebra under $\{-,-\}$,
  - each $\{a,-\}$ is a derivation of A.
- Poisson ideal: ideal I of A with $\{i,a\} \in I$ for all $i \in I, a \in A$.
  A/I is then a Poisson algebra in the obvious way.
- Poisson prime ideal: prime and Poisson ideal; equivalently, Poisson ideal P such that $IJ \subseteq P$, I, J Poisson $\Rightarrow I \subseteq P$ or $J \subseteq P$.
- Poisson maximal: maximal and Poisson $\not\equiv$ maximal Poisson (maximal as a Poisson ideal).

Poisson modules

Let A be a Poisson algebra with Poisson bracket $\{-,-\}$.
An A-module a Poisson module if it is also a Lie module, with

$$\{-,-\}_M : A \times M \to M,$$

and derivation-like compatibility conditions hold for

$$\{-,m\}_M : A \to M$$

and

$$\{a,-\}_M : M \to M.$$

(Farkas, Oh)

Determination of fin. dim. simple Poisson modules (J, 2009)

Let A be a Poisson algebra and let J be a Poisson maximal ideal. Then J/J^2 has a Lie algebra structure given by
$[j_1 + J^2, j_2 + J^2] = \{j_i, j_2\} + J^2$.

Theorem Let A be a Poisson algebra. Up to isomorphism classes, there is a dimension-preserving bijection between finite-dimensional simple Poisson modules and pairs (J, M) where J is a Poisson maximal ideal of A and M is a finite-dimensional simple J/J^2-module.

Determination of fin. dim. simple Poisson modules (J, 2009)

Let A be a Poisson algebra and let J be a Poisson maximal ideal. Then J/J^2 has a Lie algebra structure given by
$[j_1 + J^2, j_2 + J^2] = \{j_i, j_2\} + J^2$.
Theorem Let A be a Poisson algebra. Up to isomorphism classes, there is a dimension-preserving bijection between finite-dimensional simple Poisson modules and pairs (J, M) where J is a Poisson maximal ideal of A and M is a finite-dimensional simple J/J^2-module.

Quantizations and deformations

Let T be a $\mathbb{C}$-algebra with a central non-unit non-zero-divisor t such that $A := T/tT$ is commutative. Then $[-,-]$ in T induces a well-defined Poisson bracket $\{-,-\}$ on A by the rule

$$\{\overline{\alpha}, \overline{\beta}\} = \overline{t^{-1}[\alpha, \beta]}.$$

For this half hour, T is a quantization of the Poisson algebra A and a $\mathbb{C}$-algebra of the form $T_\lambda := T/(t-\lambda)T$, where $\lambda \in \mathbb{C}$ is such that $t - \lambda$ is a non-unit in T, is a deformation of A.

Quantizations and deformations

Let T be a $\mathbb{C}$-algebra with a central non-unit non-zero-divisor t such that $A := T/tT$ is commutative. Then $[-,-]$ in T induces a well-defined Poisson bracket $\{-,-\}$ on A by the rule

$$\{\overline{\alpha}, \overline{\beta}\} = \overline{t^{-1}[\alpha, \beta]}.$$

For this half hour, T is a quantization of the Poisson algebra A and a $\mathbb{C}$-algebra of the form $T_\lambda := T/(t-\lambda)T$, where $\lambda \in \mathbb{C}$ is such that $t - \lambda$ is a non-unit in T, is a deformation of A.

Case study

Let $A = \mathbb{C}[x, y, z]$ with the Poisson bracket

$$\begin{aligned} \{x, y\} &= 2xy - 2, \\ \{y, z\} &= 2yz - 2, \\ \{z, x\} &= 2zx - 2. \end{aligned}$$

Call this F_3; it is the first in a family F_{2n+1}, $n \geq 1$ discussed in Fordy, arXiv:1003.3952v1 (in the context of integrable systems and the Bullough diagram).

Case study

Let $A = \mathbb{C}[x, y, z]$ with the Poisson bracket

$$\begin{aligned} \{x, y\} &= 2xy - 2, \\ \{y, z\} &= 2yz - 2, \\ \{z, x\} &= 2zx - 2. \end{aligned}$$

Call this F_3; it is the first in a family F_{2n+1}, $n \geq 1$ discussed in Fordy, arXiv:1003.3952v1 (in the context of integrable systems and the Bullough diagram).

Deformation

Let $\mathcal{F}_3$ be the $\mathbb{C}$-algebra generated by x, y, z subject to:

$$\begin{aligned} xy - q^2 yx &= 1 - q^2, \\ yz - q^2 zy &= 1 - q^2, \\ zx - q^2 xz &= 1 - q^2, \end{aligned}$$

equivalently

$$\begin{aligned} xy - yx &= (q-1)(q+1)(yx-1), \\ yz - zy &= (q-1)(q+1)(zy-1), \\ zx - xz &= (q-1)(q+1)(xz-1). \end{aligned}$$

This is a deformation of F_3 and contains three quantized Weyl algebras $A_1^{q^2}$ in a cyclic pattern.
$\mathcal{F}_3$ is the first in a family $\mathcal{F}_{2n+1}$, $n \geq 1$.

Deformation

Let $\mathcal{F}_3$ be the $\mathbb{C}$-algebra generated by x, y, z subject to:

$$\begin{aligned} xy - q^2 yx &= 1 - q^2, \\ yz - q^2 zy &= 1 - q^2, \\ zx - q^2 xz &= 1 - q^2, \end{aligned}$$

equivalently

$$\begin{aligned} xy - yx &= (q-1)(q+1)(yx-1), \\ yz - zy &= (q-1)(q+1)(zy-1), \\ zx - xz &= (q-1)(q+1)(xz-1). \end{aligned}$$

This is a deformation of F_3 and contains three quantized Weyl algebras $A_1^{q^2}$ in a cyclic pattern.
$\mathcal{F}_3$ is the first in a family $\mathcal{F}_{2n+1}$, $n \geq 1$.

Deformation

Let $\mathcal{F}_3$ be the $\mathbb{C}$-algebra generated by x, y, z subject to:

$$\begin{aligned} xy - q^2 yx &= 1 - q^2, \\ yz - q^2 zy &= 1 - q^2, \\ zx - q^2 xz &= 1 - q^2, \end{aligned}$$

equivalently

$$\begin{aligned} xy - yx &= (q-1)(q+1)(yx-1), \\ yz - zy &= (q-1)(q+1)(zy-1), \\ zx - xz &= (q-1)(q+1)(xz-1). \end{aligned}$$

This is a deformation of F_3 and contains three quantized Weyl algebras $A_1^{q^2}$ in a cyclic pattern.

$\mathcal{F}_3$ is the first in a family $\mathcal{F}_{2n+1}$, $n \geq 1$.

Deformation

Let $\mathcal{F}_3$ be the $\mathbb{C}$-algebra generated by x, y, z subject to:

$$\begin{aligned} xy - q^2 yx &= 1 - q^2, \\ yz - q^2 zy &= 1 - q^2, \\ zx - q^2 xz &= 1 - q^2, \end{aligned}$$

equivalently

$$\begin{aligned} xy - yx &= (q-1)(q+1)(yx-1), \\ yz - zy &= (q-1)(q+1)(zy-1), \\ zx - xz &= (q-1)(q+1)(xz-1). \end{aligned}$$

This is a deformation of F_3 and contains three quantized Weyl algebras $A_1^{q^2}$ in a cyclic pattern.
$\mathcal{F}_3$ is the first in a family $\mathcal{F}_{2n+1}$, $n \geq 1$.

$\mathcal{F}_3$ and $A_1[t]$

When $q^2 \neq 1$, the relations for $\mathcal{F}_3$ can, by changing generators, be rewritten

$$\begin{aligned} xy - q^2 yx &= 1, \\ yz - q^2 zy &= 1, \\ zx - q^2 xz &= 1. \end{aligned}$$

When $q = 1$, $\mathcal{F}_3$ is a skew polynomial ring over the Weyl algebra A_1 by an inner derivation so it is isomorphic to the polynomial ring $A_1[t]$. So, in an informal sense, $\mathcal{F}_3$ is a deformation of $A_1[t]$.

$\mathcal{F}_3$ and $A_1[t]$

When $q^2 \neq 1$, the relations for $\mathcal{F}_3$ can, by changing generators, be rewritten

$$\begin{aligned} xy - q^2 yx &= 1, \\ yz - q^2 zy &= 1, \\ zx - q^2 xz &= 1. \end{aligned}$$

When $q = 1$, $\mathcal{F}_3$ is a skew polynomial ring over the Weyl algebra A_1 by an inner derivation so it is isomorphic to the polynomial ring $A_1[t]$. So, in an informal sense, $\mathcal{F}_3$ is a deformation of $A_1[t]$.

Questions for $\mathcal{F}_3$

From now on $q \neq \sqrt{1}$.

- What is the prime spectrum of $\mathcal{F}_3$?
- What are the finite-dimensional simple $\mathcal{F}_3$-modules?

We expect these to reflect the Poisson spectrum of F_3 and the finite-dimensional simple Poisson F_3-modules.

For $R = A_1^{q^2}$, the prime spectrum is

- 0;
- $dR = Rd$ $(d = xy - 1)$;
- $dR + (x - \lambda)R$, $0 \neq \lambda \in \mathbb{C}$

and the fin. dim simple modules are 1-dimensional parametrized by $\mathbb{C}^*$.

Questions for $\mathcal{F}_3$

From now on $q \neq \sqrt{1}$.

- What is the prime spectrum of $\mathcal{F}_3$?
- What are the finite-dimensional simple $\mathcal{F}_3$-modules?

We expect these to reflect the Poisson spectrum of F_3 and the finite-dimensional simple Poisson F_3-modules.

For $R = A_1^{q^2}$, the prime spectrum is

- 0;
- $dR = Rd$ $(d = xy - 1)$;
- $dR + (x - \lambda)R$, $0 \neq \lambda \in \mathbb{C}$

and the fin. dim simple modules are 1-dimensional parametrized by $\mathbb{C}^*$.

Questions for $\mathcal{F}_3$

From now on $q \neq \sqrt{1}$.

- What is the prime spectrum of $\mathcal{F}_3$?
- What are the finite-dimensional simple $\mathcal{F}_3$-modules?

We expect these to reflect the Poisson spectrum of F_3 and the finite-dimensional simple Poisson F_3-modules.

For $R = A_1^{q^2}$, the prime spectrum is

- 0;
- $dR = Rd$ $(d = xy - 1)$;
- $dR + (x - \lambda)R$, $0 \neq \lambda \in \mathbb{C}$

and the fin. dim simple modules are 1-dimensional parametrized by $\mathbb{C}^*$.

Questions for $\mathcal{F}_3$

From now on $q \neq \sqrt{1}$.

- What is the prime spectrum of $\mathcal{F}_3$?
- What are the finite-dimensional simple $\mathcal{F}_3$-modules?

We expect these to reflect the Poisson spectrum of F_3 and the finite-dimensional simple Poisson F_3-modules.

For $R = A_1^{q^2}$, the prime spectrum is

- 0;
- $dR = Rd$ $(d = xy - 1)$;
- $dR + (x - \lambda)R$, $0 \neq \lambda \in \mathbb{C}$

and the fin. dim simple modules are 1-dimensional parametrized by $\mathbb{C}^*$.

Folklore: Poisson brackets on $\mathbb{C}[x, y, z]$

Let $\{-,-\}$ be a Poisson bracket on $A = \mathbb{C}[x, y, z]$. There exist a completion $\hat{A}$ of A at a maximal ideal and $a, b \in \hat{A}$ such that

$$\begin{aligned} \{x, y\} &= ba_z \in A \quad (a_z := \partial a/\partial z), \\ \{y, z\} &= ba_x \in A, \\ \{z, x\} &= ba_y \in A. \end{aligned}$$

Examples

- $a = \lambda \log x + \rho \log y + \mu \log z, \ \ b = xyz;$
  $\{x, y\} = \mu xy, \{y, z\} = \lambda yz, \{z, x\} = \rho xz.$
- $a = y^{\alpha+1}, b = xy^{-\alpha}, \alpha \in \mathbb{C};$
  $\{x, y\} = 0, \{y, z\} = y, \{z, x\} = -\alpha x.$
- $a = s/t, \ \ s, t \in A, \ \ t \neq 0, \ \ b = t^2;$
  $\{x, y\} = ts_z - st_z, \{y, z\} = ts_x - st_x, \{z, x\} = ts_y - st_y.$

Call the last type rational. When $t = 1$, so that $\{x, y\} = s_z, \{y, z\} = s_x, \{z, x\} = s_y$, it is exact or Jacobian. In the exact case s is Poisson central, $\{s, -\} = 0$. F_3 is exact with $s = 2(xyz - x - y - z)$.

Examples

- $a = \lambda \log x + \rho \log y + \mu \log z, \;\; b = xyz;$
  $\{x, y\} = \mu xy, \{y, z\} = \lambda yz, \{z, x\} = \rho xz.$
- $a = y^{\alpha+1}, b = xy^{-\alpha}, \alpha \in \mathbb{C};$
  $\{x, y\} = 0, \{y, z\} = y, \{z, x\} = -\alpha x.$
- $a = s/t, \;\; s, t \in A, \;\; t \neq 0, \;\; b = t^2;$
  $\{x, y\} = ts_z - st_z, \{y, z\} = ts_x - st_x, \{z, x\} = ts_y - st_y.$

Call the last type rational. When $t = 1$, so that $\{x, y\} = s_z, \{y, z\} = s_x, \{z, x\} = s_y$, it is exact or Jacobian. In the exact case s is Poisson central, $\{s, -\} = 0$. F_3 is exact with $s = 2(xyz - x - y - z)$.

Examples

- $a = \lambda \log x + \rho \log y + \mu \log z,\ \ b = xyz;$
  $\{x, y\} = \mu xy, \{y, z\} = \lambda yz, \{z, x\} = \rho xz.$
- $a = y^{\alpha+1}, b = xy^{-\alpha}, \alpha \in \mathbb{C};$
  $\{x, y\} = 0, \{y, z\} = y, \{z, x\} = -\alpha x.$
- $a = s/t,\ \ s, t \in A,\ \ t \neq 0,\ \ b = t^2;$
  $\{x, y\} = ts_z - st_z, \{y, z\} = ts_x - st_x, \{z, x\} = ts_y - st_y.$

Call the last type rational. When $t = 1$, so that $\{x, y\} = s_z, \{y, z\} = s_x, \{z, x\} = s_y$, it is exact or Jacobian.
In the exact case s is Poisson central, $\{s, -\} = 0$.
F_3 is exact with $s = 2(xyz - x - y - z)$.

Examples

- $a = \lambda \log x + \rho \log y + \mu \log z, \;\; b = xyz;$
  $\{x, y\} = \mu xy, \{y, z\} = \lambda yz, \{z, x\} = \rho xz.$
- $a = y^{\alpha+1}, b = xy^{-\alpha}, \alpha \in \mathbb{C};$
  $\{x, y\} = 0, \{y, z\} = y, \{z, x\} = -\alpha x.$
- $a = s/t, \;\; s, t \in A, \;\; t \neq 0, \;\; b = t^2;$
  $\{x, y\} = ts_z - st_z, \{y, z\} = ts_x - st_x, \{z, x\} = ts_y - st_y.$

Call the last type rational. When $t = 1$, so that $\{x, y\} = s_z, \{y, z\} = s_x, \{z, x\} = s_y$, it is exact or Jacobian.
In the exact case s is Poisson central, $\{s, -\} = 0$.
F_3 is exact with $s = 2(xyz - x - y - z)$.

Examples

- $a = \lambda \log x + \rho \log y + \mu \log z, \ \ b = xyz;$
  $\{x,y\} = \mu xy, \{y,z\} = \lambda yz, \{z,x\} = \rho xz.$
- $a = y^{\alpha+1}, b = xy^{-\alpha}, \alpha \in \mathbb{C};$
  $\{x,y\} = 0, \{y,z\} = y, \{z,x\} = -\alpha x.$
- $a = s/t, \ \ s,t \in A, \ \ t \neq 0, \ \ b = t^2;$
  $\{x,y\} = ts_z - st_z, \{y,z\} = ts_x - st_x, \{z,x\} = ts_y - st_y.$

Call the last type rational. When $t = 1$, so that $\{x,y\} = s_z, \{y,z\} = s_x, \{z,x\} = s_y$, it is exact or Jacobian. In the exact case s is Poisson central, $\{s,-\} = 0$.
F_3 is exact with $s = 2(xyz - x - y - z)$.

Examples

- $a = \lambda \log x + \rho \log y + \mu \log z, \;\; b = xyz;$
  $\{x, y\} = \mu xy, \{y, z\} = \lambda yz, \{z, x\} = \rho xz.$
- $a = y^{\alpha+1}, b = xy^{-\alpha}, \alpha \in \mathbb{C};$
  $\{x, y\} = 0, \{y, z\} = y, \{z, x\} = -\alpha x.$
- $a = s/t, \;\; s, t \in A, \;\; t \neq 0, \;\; b = t^2;$
  $\{x, y\} = ts_z - st_z, \{y, z\} = ts_x - st_x, \{z, x\} = ts_y - st_y.$

Call the last type rational. When $t = 1$, so that $\{x, y\} = s_z, \{y, z\} = s_x, \{z, x\} = s_y$, it is exact or Jacobian. In the exact case s is Poisson central, $\{s, -\} = 0$. F_3 is exact with $s = 2(xyz - x - y - z)$.

Theorem (J, 201?): Poisson spectrum for rational brackets

Let $s, t \in A \backslash \{0\}$ be coprime and let $a = st^{-1} \in \mathbb{C}(x, y, z)$. Then the Poisson prime ideals for A under the rational bracket determined by a are

- 0;
- those prime ideals P of A such that $\{x, y\}, \{y, z\}, \{z, x\} \in P$ (the Poisson bracket on A/P is 0 for these);
- the height one prime ideals pA, where p is an irreducible factor in A of $\lambda s + \mu t$, for some $\lambda, \mu \in \mathbb{C}$ such that $\lambda s + \mu t$ is a non-unit.

Theorem (J, 201?): Poisson spectrum for rational brackets

Let $s, t \in A\backslash\{0\}$ be coprime and let $a = st^{-1} \in \mathbb{C}(x, y, z)$. Then the Poisson prime ideals for A under the rational bracket determined by a are

- 0;
- those prime ideals P of A such that $\{x, y\}, \{y, z\}, \{z, x\} \in P$ (the Poisson bracket on A/P is 0 for these);
- the height one prime ideals pA, where p is an irreducible factor in A of $\lambda s + \mu t$, for some $\lambda, \mu \in \mathbb{C}$ such that $\lambda s + \mu t$ is a non-unit.

Theorem (J, 201?): Poisson spectrum for rational brackets

Let $s, t \in A \setminus \{0\}$ be coprime and let $a = st^{-1} \in \mathbb{C}(x, y, z)$. Then the Poisson prime ideals for A under the rational bracket determined by a are

- 0;
- those prime ideals P of A such that $\{x, y\}, \{y, z\}, \{z, x\} \in P$ (the Poisson bracket on A/P is 0 for these);
- the height one prime ideals pA, where p is an irreducible factor in A of $\lambda s + \mu t$, for some $\lambda, \mu \in \mathbb{C}$ such that $\lambda s + \mu t$ is a non-unit.

Theorem (J, 201?): Poisson spectrum for rational brackets

Let $s, t \in A \backslash \{0\}$ be coprime and let $a = st^{-1} \in \mathbb{C}(x, y, z)$. Then the Poisson prime ideals for A under the rational bracket determined by a are

- 0;
- those prime ideals P of A such that $\{x, y\}, \{y, z\}, \{z, x\} \in P$ (the Poisson bracket on A/P is 0 for these);
- the height one prime ideals pA, where p is an irreducible factor in A of $\lambda s + \mu t$, for some $\lambda, \mu \in \mathbb{C}$ such that $\lambda s + \mu t$ is a non-unit.

Poisson spectrum of F_3

The Poisson prime ideals of F_3 are:

- 0,
- height one: $(xyz - x - y - z - \lambda)A$, $\lambda \in \mathbb{C}$.
- Two Poisson maximal ideals :
  $J_1 := (x-1)A + (y-1)A + (z-1)A$ and
  $J_2 := (x+1)A + (y+1)A + (z+1)A$.

Finite-dimensional simple Poisson modules for F_3:
Both J_1/J_1^2 and J_2/J_2^2 are isomorphic to $\mathfrak{sl}_2$ so F_3 has two n-dimensional simple Poisson modules for each $n \geq 1$.

Poisson spectrum of F_3

The Poisson prime ideals of F_3 are:

- 0,
- height one: $(xyz - x - y - z - \lambda)A$, $\lambda \in \mathbb{C}$.
- Two Poisson maximal ideals : $J_1 := (x-1)A + (y-1)A + (z-1)A$ and $J_2 := (x+1)A + (y+1)A + (z+1)A$.

Finite-dimensional simple Poisson modules for F_3: Both J_1/J_1^2 and J_2/J_2^2 are isomorphic to $\mathfrak{sl}_2$ so F_3 has two n-dimensional simple Poisson modules for each $n \geq 1$.

Poisson spectrum of F_3

The Poisson prime ideals of F_3 are:

- 0,
- height one: $(xyz - x - y - z - \lambda)A$, $\lambda \in \mathbb{C}$.
- Two Poisson maximal ideals :
  $J_1 := (x-1)A + (y-1)A + (z-1)A$ and
  $J_2 := (x+1)A + (y+1)A + (z+1)A$.

Finite-dimensional simple Poisson modules for F_3:
Both J_1/J_1^2 and J_2/J_2^2 are isomorphic to $\mathfrak{sl}_2$ so F_3 has two n-dimensional simple Poisson modules for each $n \geq 1$.

Poisson spectrum of F_3

The Poisson prime ideals of F_3 are:

- 0,
- height one: $(xyz - x - y - z - \lambda)A$, $\lambda \in \mathbb{C}$.
- Two Poisson maximal ideals :
  $J_1 := (x-1)A + (y-1)A + (z-1)A$ and
  $J_2 := (x+1)A + (y+1)A + (z+1)A$.

Finite-dimensional simple Poisson modules for F_3:
Both J_1/J_1^2 and J_2/J_2^2 are isomorphic to $\mathfrak{sl}_2$ so F_3 has two n-dimensional simple Poisson modules for each $n \geq 1$.

Finite-dimensional simple (left) $\mathcal{F}_3$-modules

($R := A_1^{q^2}$) Every finite-dimensional simple (left) R-module is isomorphic to $R/(R(x-\lambda^{-1}) + R(y-\lambda))$, $\lambda \in \mathbb{C}^*$. Every finite-dimensional simple $\mathcal{F}_3$-module contains one of these.

Let $M_\lambda = \mathcal{F}_3/(\mathcal{F}_3(x-\lambda^{-1}) + \mathcal{F}_3(y-\lambda))$.

- As a $\mathbb{C}[z]$-module, $M_\lambda \simeq \mathbb{C}[z]$.
- If $\lambda \neq \pm q^m$ for all $m \geq 0$ then M_λ is simple.
- If $\lambda = q^{n-1}$, $n \geq 1$ and $f_n(z) = (z - q^{n-1})(z - q^{n-3})\ldots(z - q^{3-n})(z - q^{1-n})$ then $xf_n = q^{-(n+1)} f_n$, $yf_n = q^{n+1} f_n$ and $N_n := \mathcal{F}_3/(\mathcal{F}_3(x - q^{1-n}) + \mathcal{F}_3(y - q^{n-1}) + \mathcal{F}_3 f_n(z))$ is simple n-dimensional.
- If $\lambda = -q^{n-1}$ there is a similar n-dimensional simple N_n'.
- Every fin. dim. $\mathcal{F}_3$-module is isomorphic to N_n or N_n'.

Finite-dimensional simple (left) $\mathcal{F}_3$-modules

($R := A_1^{q^2}$) Every finite-dimensional simple (left) R-module is isomorphic to $R/(R(x-\lambda^{-1}) + R(y-\lambda))$, $\lambda \in \mathbb{C}^*$. Every finite-dimensional simple $\mathcal{F}_3$-module contains one of these.

Let $M_\lambda = \mathcal{F}_3/(\mathcal{F}_3(x-\lambda^{-1}) + \mathcal{F}_3(y-\lambda))$.

- As a $\mathbb{C}[z]$-module, $M_\lambda \simeq \mathbb{C}[z]$.
- If $\lambda \neq \pm q^m$ for all $m \geq 0$ then M_λ is simple.
- If $\lambda = q^{n-1}$, $n \geq 1$ and $f_n(z) = (z-q^{n-1})(z-q^{n-3})\ldots(z-q^{3-n})(z-q^{1-n})$ then $xf_n = q^{-(n+1)}f_n$, $yf_n = q^{n+1}f_n$ and $N_n := \mathcal{F}_3/(\mathcal{F}_3(x-q^{1-n}) + \mathcal{F}_3(y-q^{n-1}) + \mathcal{F}_3 f_n(z))$ is simple n-dimensional.
- If $\lambda = -q^{n-1}$ there is a similar n-dimensional simple N'_n.
- Every fin. dim. $\mathcal{F}_3$-module is isomorphic to N_n or N'_n.

Finite-dimensional simple (left) $\mathcal{F}_3$-modules

($R := A_1^{q^2}$) Every finite-dimensional simple (left) R-module is isomorphic to $R/(R(x-\lambda^{-1}) + R(y-\lambda))$, $\lambda \in \mathbb{C}^*$. Every finite-dimensional simple $\mathcal{F}_3$-module contains one of these.

Let $M_\lambda = \mathcal{F}_3/(\mathcal{F}_3(x-\lambda^{-1}) + \mathcal{F}_3(y-\lambda))$.

- As a $\mathbb{C}[z]$-module, $M_\lambda \simeq \mathbb{C}[z]$.
- If $\lambda \neq \pm q^m$ for all $m \geq 0$ then M_λ is simple.
- If $\lambda = q^{n-1}$, $n \geq 1$ and $f_n(z) = (z-q^{n-1})(z-q^{n-3})\ldots(z-q^{3-n})(z-q^{1-n})$ then $xf_n = q^{-(n+1)}f_n$, $yf_n = q^{n+1}f_n$ and $N_n := \mathcal{F}_3/(\mathcal{F}_3(x-q^{1-n}) + \mathcal{F}_3(y-q^{n-1}) + \mathcal{F}_3 f_n(z))$ is simple n-dimensional.
- If $\lambda = -q^{n-1}$ there is a similar n-dimensional simple N'_n.
- Every fin. dim. $\mathcal{F}_3$-module is isomorphic to N_n or N'_n.

Finite-dimensional simple (left) $\mathcal{F}_3$-modules

($R := A_1^{q^2}$) Every finite-dimensional simple (left) R-module is isomorphic to $R/(R(x-\lambda^{-1}) + R(y-\lambda))$, $\lambda \in \mathbb{C}^*$. Every finite-dimensional simple $\mathcal{F}_3$-module contains one of these.

Let $M_\lambda = \mathcal{F}_3/(\mathcal{F}_3(x-\lambda^{-1}) + \mathcal{F}_3(y-\lambda))$.

- As a $\mathbb{C}[z]$-module, $M_\lambda \simeq \mathbb{C}[z]$.
- If $\lambda \neq \pm q^m$ for all $m \geq 0$ then M_λ is simple.
- If $\lambda = q^{n-1}$, $n \geq 1$ and $f_n(z) = (z-q^{n-1})(z-q^{n-3})\dots(z-q^{3-n})(z-q^{1-n})$ then $xf_n = q^{-(n+1)}f_n$, $yf_n = q^{n+1}f_n$ and $N_n := \mathcal{F}_3/(\mathcal{F}_3(x-q^{1-n}) + \mathcal{F}_3(y-q^{n-1}) + \mathcal{F}_3 f_n(z))$ is simple n-dimensional.
- If $\lambda = -q^{n-1}$ there is a similar n-dimensional simple N'_n.
- Every fin. dim. $\mathcal{F}_3$-module is isomorphic to N_n or N'_n.

Finite-dimensional simple (left) $\mathcal{F}_3$-modules

($R := A_1^{q^2}$) Every finite-dimensional simple (left) R-module is isomorphic to $R/(R(x-\lambda^{-1}) + R(y-\lambda))$, $\lambda \in \mathbb{C}^*$. Every finite-dimensional simple $\mathcal{F}_3$-module contains one of these.

Let $M_\lambda = \mathcal{F}_3/(\mathcal{F}_3(x-\lambda^{-1}) + \mathcal{F}_3(y-\lambda))$.

- As a $\mathbb{C}[z]$-module, $M_\lambda \simeq \mathbb{C}[z]$.
- If $\lambda \neq \pm q^m$ for all $m \geq 0$ then M_λ is simple.
- If $\lambda = q^{n-1}$, $n \geq 1$ and $f_n(z) = (z-q^{n-1})(z-q^{n-3})\ldots(z-q^{3-n})(z-q^{1-n})$ then $xf_n = q^{-(n+1)}f_n$, $yf_n = q^{n+1}f_n$ and $N_n := \mathcal{F}_3/(\mathcal{F}_3(x-q^{1-n}) + \mathcal{F}_3(y-q^{n-1}) + \mathcal{F}_3 f_n(z))$ is simple n-dimensional.
- If $\lambda = -q^{n-1}$ there is a similar n-dimensional simple N_n'.
- Every fin. dim. $\mathcal{F}_3$-module is isomorphic to N_n or N_n'.

Finite-dimensional simple (left) $\mathcal{F}_3$-modules

($R := A_1^{q^2}$) Every finite-dimensional simple (left) R-module is isomorphic to $R/(R(x-\lambda^{-1})+R(y-\lambda))$, $\lambda \in \mathbb{C}^*$. Every finite-dimensional simple $\mathcal{F}_3$-module contains one of these.

Let $M_\lambda = \mathcal{F}_3/(\mathcal{F}_3(x-\lambda^{-1})+\mathcal{F}_3(y-\lambda))$.

- As a $\mathbb{C}[z]$-module, $M_\lambda \simeq \mathbb{C}[z]$.
- If $\lambda \neq \pm q^m$ for all $m \geq 0$ then M_λ is simple.
- If $\lambda = q^{n-1}$, $n \geq 1$ and $f_n(z) = (z-q^{n-1})(z-q^{n-3})\ldots(z-q^{3-n})(z-q^{1-n})$ then $xf_n = q^{-(n+1)}f_n$, $yf_n = q^{n+1}f_n$ and $N_n := \mathcal{F}_3/(\mathcal{F}_3(x-q^{1-n})+\mathcal{F}_3(y-q^{n-1})+\mathcal{F}_3 f_n(z))$ is simple n-dimensional.
- If $\lambda = -q^{n-1}$ there is a similar n-dimensional simple N'_n.
- Every fin. dim. $\mathcal{F}_3$-module is isomorphic to N_n or N'_n.

Finite-dimensional simple (left) $\mathcal{F}_3$-modules

($R := A_1^{q^2}$) Every finite-dimensional simple (left) R-module is isomorphic to $R/(R(x-\lambda^{-1}) + R(y-\lambda))$, $\lambda \in \mathbb{C}^*$. Every finite-dimensional simple $\mathcal{F}_3$-module contains one of these.

Let $M_\lambda = \mathcal{F}_3/(\mathcal{F}_3(x-\lambda^{-1}) + \mathcal{F}_3(y-\lambda))$.

- As a $\mathbb{C}[z]$-module, $M_\lambda \simeq \mathbb{C}[z]$.
- If $\lambda \neq \pm q^m$ for all $m \geq 0$ then M_λ is simple.
- If $\lambda = q^{n-1}$, $n \geq 1$ and $f_n(z) = (z-q^{n-1})(z-q^{n-3})\ldots(z-q^{3-n})(z-q^{1-n})$ then $xf_n = q^{-(n+1)}f_n, yf_n = q^{n+1}f_n$ and $N_n := \mathcal{F}_3/(\mathcal{F}_3(x-q^{1-n}) + \mathcal{F}_3(y-q^{n-1}) + \mathcal{F}_3 f_n(z))$ is simple n-dimensional.
- If $\lambda = -q^{n-1}$ there is a similar n-dimensional simple N'_n.
- Every fin. dim. $\mathcal{F}_3$-module is isomorphic to N_n or N'_n.

Prime spectrum of $\mathcal{F}_3$

- 0;
- $(D-\rho)\mathcal{F}_3$, where $\rho \in \mathbb{C}$ and

$$\begin{aligned} D &= xyz - x - q^2y - z \\ &= yzx - y - q^2z - x \\ &= zxy - z - q^2x - y, \end{aligned}$$

  which is central.
- The annihilators of N_n and N'_n, $n \geq 1$.

Methods:

- localization to get a 3-dimensional quantum torus in x, $d = xy - 1$ and $D = dz - x - q^2y$.
- checking that any prime containing $(xy-1)^n, (yz-1)^n$ and $(zx-1)^n$ for some n has finite codimension.

Prime spectrum of $\mathcal{F}_3$

- 0;
- $(D - \rho)\mathcal{F}_3$, where $\rho \in \mathbb{C}$ and

$$\begin{aligned} D &= xyz - x - q^2y - z \\ &= yzx - y - q^2z - x \\ &= zxy - z - q^2x - y, \end{aligned}$$

  which is central.
- The annihilators of N_n and N'_n, $n \geq 1$.

Methods:

- localization to get a 3-dimensional quantum torus in x, $d = xy - 1$ and $D = dz - x - q^2y$.
- checking that any prime containing $(xy - 1)^n, (yz - 1)^n$ and $(zx - 1)^n$ for some n has finite codimension.

Prime spectrum of $\mathcal{F}_3$

- 0;
- $(D - \rho)\mathcal{F}_3$, where $\rho \in \mathbb{C}$ and

$$\begin{aligned} D &= xyz - x - q^2 y - z \\ &= yzx - y - q^2 z - x \\ &= zxy - z - q^2 x - y, \end{aligned}$$

  which is central.
- The annihilators of N_n and N'_n, $n \geq 1$.

Methods:

- localization to get a 3-dimensional quantum torus in x, $d = xy - 1$ and $D = dz - x - q^2 y$.
- checking that any prime containing $(xy - 1)^n, (yz - 1)^n$ and $(zx - 1)^n$ for some n has finite codimension.

Prime spectrum of $\mathcal{F}_3$

- 0;
- $(D-\rho)\mathcal{F}_3$, where $\rho \in \mathbb{C}$ and

$$\begin{aligned} D &= xyz - x - q^2 y - z \\ &= yzx - y - q^2 z - x \\ &= zxy - z - q^2 x - y, \end{aligned}$$

  which is central.
- The annihilators of N_n and N'_n, $n \geq 1$.

Methods:

- localization to get a 3-dimensional quantum torus in x, $d = xy - 1$ and $D = dz - x - q^2 y$.
- checking that any prime containing $(xy-1)^n, (yz-1)^n$ and $(zx-1)^n$ for some n has finite codimension.

Prime spectrum of $\mathcal{F}_3$

- 0;
- $(D - \rho)\mathcal{F}_3$, where $\rho \in \mathbb{C}$ and

$$\begin{aligned} D &= xyz - x - q^2y - z \\ &= yzx - y - q^2z - x \\ &= zxy - z - q^2x - y, \end{aligned}$$

  which is central.
- The annihilators of N_n and N'_n, $n \geq 1$.

Methods:

- localization to get a 3-dimensional quantum torus in x, $d = xy - 1$ and $D = dz - x - q^2y$.
- checking that any prime containing $(xy - 1)^n, (yz - 1)^n$ and $(zx - 1)^n$ for some n has finite codimension.

Prime spectrum of $\mathcal{F}_3$

- 0;
- $(D-\rho)\mathcal{F}_3$, where $\rho \in \mathbb{C}$ and

$$\begin{aligned} D &= xyz - x - q^2y - z \\ &= yzx - y - q^2z - x \\ &= zxy - z - q^2x - y, \end{aligned}$$

  which is central.
- The annihilators of N_n and N'_n, $n \geq 1$.

Methods:

- localization to get a 3-dimensional quantum torus in x, $d = xy - 1$ and $D = dz - x - q^2y$.
- checking that any prime containing $(xy-1)^n, (yz-1)^n$ and $(zx-1)^n$ for some n has finite codimension.