A residue formula for the fundamental Hochschild cocycle on the Podleś sphere

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Seattle 10.8.2010

The standard Podleś sphere

- Fix $q \in (0,1)$, and let $\mathcal{A} = \mathcal{O}(S_q^2)$ be the $\mathbb{C}$-algebra with generators x_{-1}, x_0, x_1 and and defining relations

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- This induces on $\mathcal{A}$ a module algebra structure over $\mathcal{U} = U_q(\mathfrak{sl}(2))$ which has generators K, K^{-1}, E, F with relations

$$KE = qEK, \quad KF = q^{-1}FK, \quad EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

counit $\varepsilon(1-K) = \varepsilon(E) = \varepsilon(F) = 0$, and coproduct

$$\Delta(K) = K \otimes K, \ \Delta(E) = E \otimes K + K^{-1} \otimes E, \ \Delta(F) = F \otimes K + K^{-1} \otimes F$$

Poincaré duality and the fundamental class

- $\mathcal{A} = \mathcal{O}(S_q^2)$ is a twisted Calabi-Yau algebra of dimension 2, so

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- The above isomorphism is given by the cap product with the fundamental Hochschild homology class

$$\omega \in H_2(\mathcal{A}, \mathcal{A}_\sigma)$$

  that therefore corresponds under the isomorphism to

$$1 \in H^0(\mathcal{A}, \mathcal{A}) = \mathrm{Hom}_{\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}}(\mathcal{A}, \mathcal{A}) \simeq Z(\mathcal{A}).$$

Explicit formulas

- Hadfield had computed an explicit formula for ω in the canonical complex $C_\bullet(\mathcal{A}, \mathcal{A}_\sigma) \simeq \mathcal{A}^{\otimes \bullet + 1}$ computing $H_\bullet(\mathcal{A}, \mathcal{A}_\sigma)$:

$$\begin{aligned}
\omega = {} & 2q^{-2}x_1 \otimes (x_{-1} \otimes x_0 - q^2 x_0 \otimes x_{-1}) \\
& +2q^2 x_{-1} \otimes (q^{-2}x_0 \otimes x_1 - x_1 \otimes x_0) \\
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- By the universal coefficient theorem, there is a dual cocycle $\varphi : \mathcal{A}^{\otimes 3} \to \mathbb{C}$ which pairs nontrivially with ω and hence has nontrivial class in $H^2(\mathcal{A},(\mathcal{A}_\sigma)^*) \simeq (H_2(\mathcal{A},\mathcal{A}_\sigma))^*$ and this

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$$\varphi(a_0,a_1,a_2) = \varepsilon(a_0(E \triangleright a_1)(F \triangleright a_2)) = \varepsilon(a_0)E(a_1)F(a_2)$$

where $X(a) := \varepsilon(X \triangleright a)$, $X \in \mathcal{U}, a \in \mathcal{A}$.

Spectral triples

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- In the classical case, $\mathcal{H}$ is the Hilbert space of square-integrable sections of the spinor bundle and $[D, a]$ is given by the Clifford action of the differential of a function a.

Main result

Theorem

Let $q \in (0,1)$, $(\mathcal{A}, \mathcal{H}, D, \gamma)$ *be the spectral triple over the standard Podleś quantum sphere constructed by Dąbrowski and Sitarz. Then*

$$\varphi_{a_0,a_1,a_2}(z) := \mathrm{Tr}(\gamma a_0 [D, a_1][D, a_2] K^{-2} |D|^{-z}), \quad a_0, a_1, a_2 \in \mathcal{A}$$

is a holomorphic function on $\{z \in \mathbb{C} \,|\, \mathrm{Re} z > 2\}$*, where* K *is the standard group-like generator of* $U_q(\mathfrak{sl}(2))$*. This function has a meromorphic continuation to* $\{z \in \mathbb{C} \,|\, \mathrm{Re} z > 1\}$ *with a pole at* $z = 2$ *of order at most 1. Taking the residue*

$$\varphi(a_0, a_1, a_2) := \operatorname*{Res}_{z=2} \varphi_{a_0,a_1,a_2}(z)$$

yields a Hochschild cocycle with nontrivial class in $H^2(\mathcal{A}, ({}_\sigma\mathcal{A})^*)$*, where* $\sigma \in \mathrm{Aut}(\mathcal{A})$ *is given by the action of* K^2*.*

Twisted traces via D

- We show more generally than needed that

$$\tau_\mu(a) := \frac{\operatorname{Res}_{z=2|\mu|} \operatorname{tr}\left(aK^{2\mu}|D|^{-z}\right)}{\operatorname{Res}_{z=2|\mu|} \operatorname{tr}\left(K^{2\mu}|D|^{-z}\right)}$$

defines for all $\mu \in \mathbb{R}$ a twisted trace on $\mathcal{A}$. In the notation of my arabic paper, these traces are given by:

μ	σ	τ_μ
< 0	any	$\int_{[1]} = \varepsilon$
0	id	$\int_{[1]} + \frac{\ln q}{2(q^{-1}-q)\ln(q^{-1}-q)} \int_{[x_0]}$
> 0	$A \mapsto A,\ B \mapsto q^{2\mu} B$	$\int_{[1]} - \frac{1-q^{-2\mu}}{q(1-q^{-2(\mu+1)})} \int_{[x_0]}$

Here σ is the involved twisting automorphism.