Quantised coordinate rings and total positivity

Stéphane Launois (University of Kent)

New Trends in Noncommutative Algebra

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http://www.kent.ac.uk/ims/personal/sl261/index.htm

Quantised coordinate rings

Prime spectrum of quantum matrices

Poisson geometry

Symplectic leaves in Poisson matrix varieties

Total Positivity

Cells in totally nonnegative matrices

The quantum world

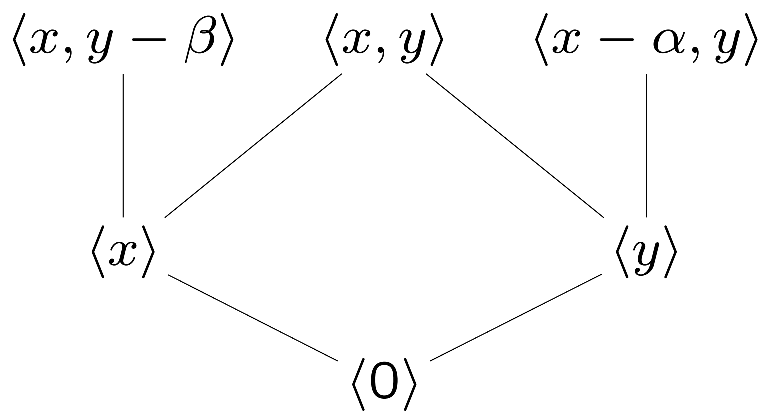
Quantum plane.

Let $q \in \mathbb{C}^*$, $q^N \neq 1$. $A := \mathbb{C}\langle x, y \mid xy = qyx \rangle$.
A is a noetherian domain.

There is an action of the torus $H := (\mathbb{C}^*)^2$ on A

$$(h, g).x = hx \text{ and } (h, g).y = gy$$

Here is the picture of the prime spectrum of A.



$(\alpha, \beta \neq 0)$

Quantum affine n-space.

$$T := \mathbb{C}\langle t_1, \ldots, t_n \mid t_i t_j = \lambda_{ij} t_j t_i,\ i < j\rangle.$$

There is an action of the torus $H := (\mathbb{C}^*)^n$ on T by automorphisms

$$(h_1, \ldots, h_n).t_i = h_i t_i$$

For each $w \subseteq \{1, \ldots, n\}$, we set $J_w := \langle t_i \rangle_{i \in w}$.

Then H-Spec$(T) = \{J_w\}$.

Quantum 2×2 matrices

The coordinate ring of quantum 2×2 matrices

$$\mathcal{O}_q(\mathcal{M}_2(\mathbb{C})) := \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is generated by four indeterminates a, b, c, d subject to the following rules:

$$ab = qba, \qquad cd = qdc$$

$$ac = qca, \qquad bd = qdb$$

$$bc = cb, \qquad ad - da = (q - q^{-1})cb.$$

The **quantum determinant** $ad - qbc$ is a central element

The algebra of $m \times p$ quantum matrices.

$$R = O_q(\mathcal{M}_{m,p}(\mathbb{C})) := \mathbb{C}\begin{bmatrix} Y_{1,1} & \cdots & Y_{1,p} \\ \vdots & \cdots & \vdots \\ Y_{m,1} & \cdots & Y_{m,p} \end{bmatrix},$$

where each 2×2 sub-matrix is a copy of $O_q(\mathcal{M}_2(\mathbb{C}))$.

$O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ is an iterated Ore extension with the indeterminates $Y_{i,\alpha}$ adjoined in the lexicographic order and so is a noetherian integral domain.

In the square case ($m = p = n$)

$$D = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} Y_{1,\sigma(1)} \cdots Y_{n,\sigma(n)}$$

is the **quantum determinant**, a central element.

Quantum minors of $R = \mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

They are the quantum determinants of square sub-matrices of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

More precisely, if $I \subseteq [\![1, m]\!]$ and $\Lambda \subseteq [\![1, p]\!]$ with $\mid I \mid = \mid \Lambda \mid$, the **quantum minor** associated with the rows I and columns Λ is

$$[I \mid \Lambda] := D_q(\mathcal{O}_q(M_{I,\Lambda}(\mathbb{C}))).$$

For example, $[12|23] = Y_{1,2}Y_{2,3} - qY_{1,3}Y_{2,2}$ is the quantum minor of R associated with the rows $1, 2$, and the columns $2, 3$.

- **The prime spectrum of** $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$

We now assume that $q \in \mathbb{C}^*$ **is not a root of unity**, and we set $R := \mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

- **Goodearl-Letzter** Prime ideals of R are completely prime.

The torus $\mathcal{H} := (\mathbb{C}^*)^{m+p}$ acts by automorphisms on R via :

$$(a_1, \ldots, a_m, b_1, \ldots, b_p).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha}.$$

This action of $\mathcal{H}$ on R induces an action of $\mathcal{H}$ on Spec(R). We denote by $\mathcal{H}$-Spec(R) the set of those prime ideals in R which are $\mathcal{H}$-invariant.

- **Goodearl-Letzter** R has at most 2^{mp} $\mathcal{H}$-primes.

Stratification Theorem (Goodearl-Letzter) :

If $J \in \mathcal{H}\text{-Spec}(R)$, then we set

$$\text{Spec}_J(R) := \{P \in \text{Spec}(R) \mid \bigcap_{h \in \mathcal{H}} h.P = J\}.$$

1. $\text{Spec}(R) = \bigsqcup_{J \in \mathcal{H}\text{-Spec}(R)} \text{Spec}_J(R)$

2. For all $J \in \mathcal{H}\text{-Spec}(R)$, $\text{Spec}_J(R)$ is homeomorphic to the prime spectrum of a (commutative) Laurent polynomial ring in $n(J)$ indeterminates over $\mathbb{C}$.

3. The primitive ideals of R are precisely the primes maximal in their $\mathcal{H}$-strata.

An observation

Recall that in $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$

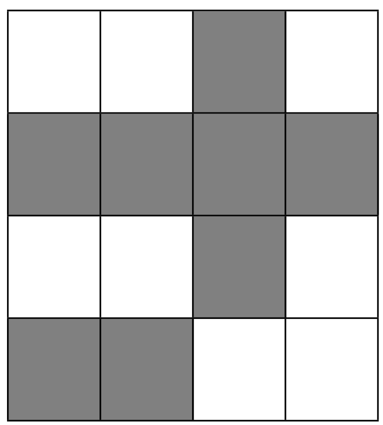
$$ad - da = (q - q^{-1})bc.$$

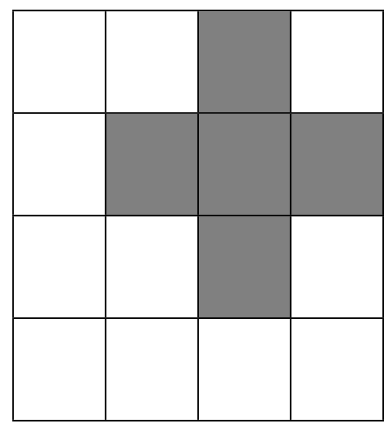
As a result, if P is a prime ideal and $d \in P$ then this forces $bc \in P$ so either $b \in P$ or $c \in P$.

Thus there is no prime ideal P of $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ for which d is the only quantum minor in P.

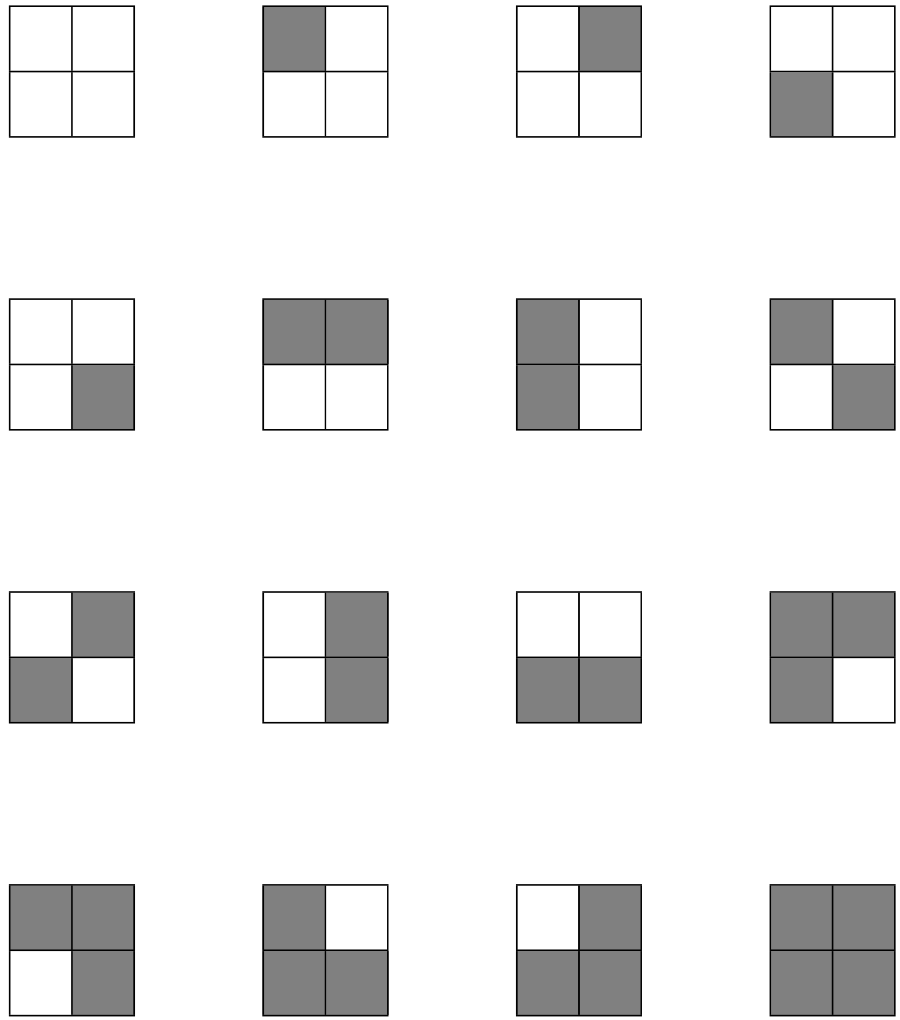
Cauchon diagrams

A **Cauchon diagram** on an $m \times p$ array is an $m \times p$ array of squares filled either black or white such that if a square is coloured black then either each square to the left is coloured black, or each square above is coloured black. Here are an example and a non-example





2×2 Cauchon Diagrams



Parametrisation of $\mathcal{H}$-Spec($\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$)

• **Cauchon (2003)** There is a bijection between Cauchon diagrams on an $m \times p$ array and $\mathcal{H} -$ Spec($\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$).

• **L.** The height of a $\mathcal{H}$-prime is given by the number of black boxes in the corresponding Cauchon diagram.

n	$C_n :=\mid \mathcal{H}$-Spec($O_q(\mathcal{M}_n(\mathbb{C}))$) $\mid$
2	14
3	230
4	6902

Generators of $\mathcal{H}$-primes

Conjecture (Goodearl-Lenagan): *$\mathcal{H}$-primes are generated by quantum minors.*

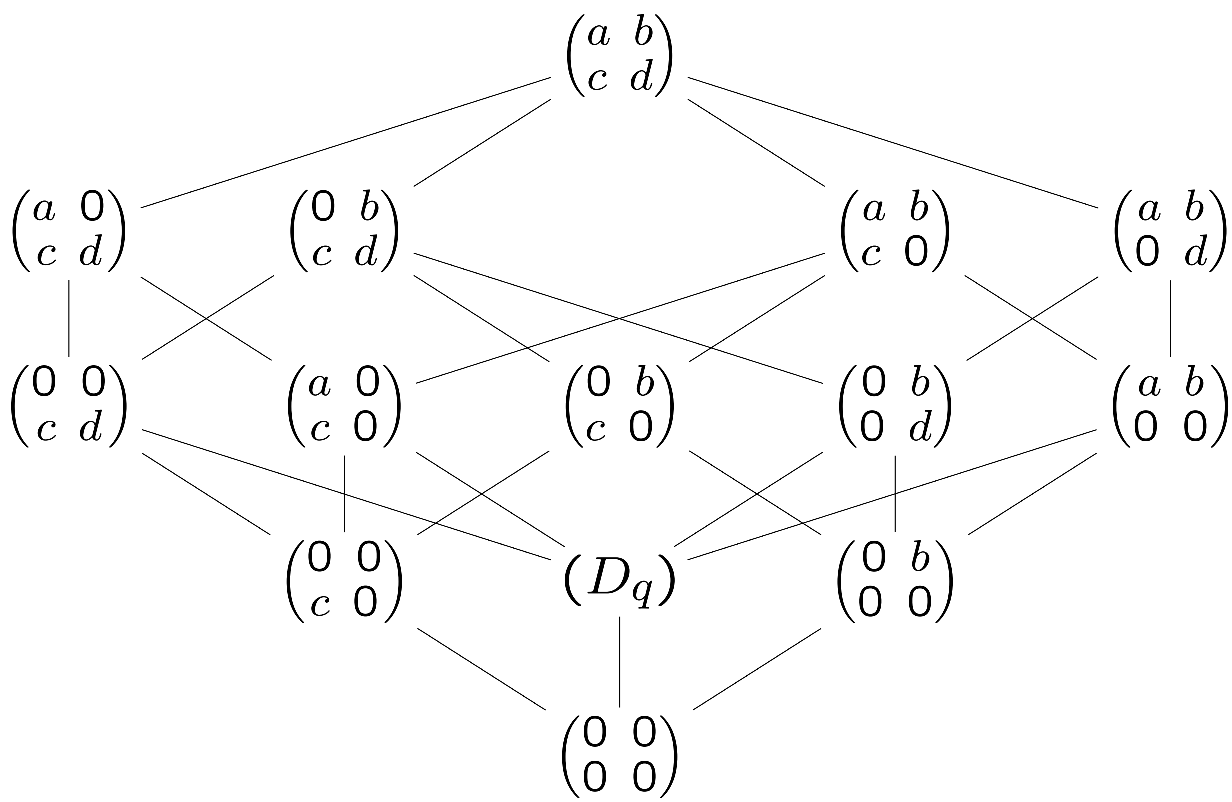
The conjecture is true for $O_q(\mathcal{M}_2(\mathbb{C}))$ and $O_q(\mathcal{M}_3(\mathbb{C}))$ (Goodearl-Lenagan).

- **L. (2004)** Assume that q is transcendental. Then $\mathcal{H}$-primes of R are generated by quantum minors.

- **Yakimov (201?)** Also, assuming q transcendental, Yakimov gives explicit generating sets of quantum minors.

2×2 **quantum matrices**

The following 14 $\mathcal{H}$-invariant ideals are all prime and these are the only $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$.



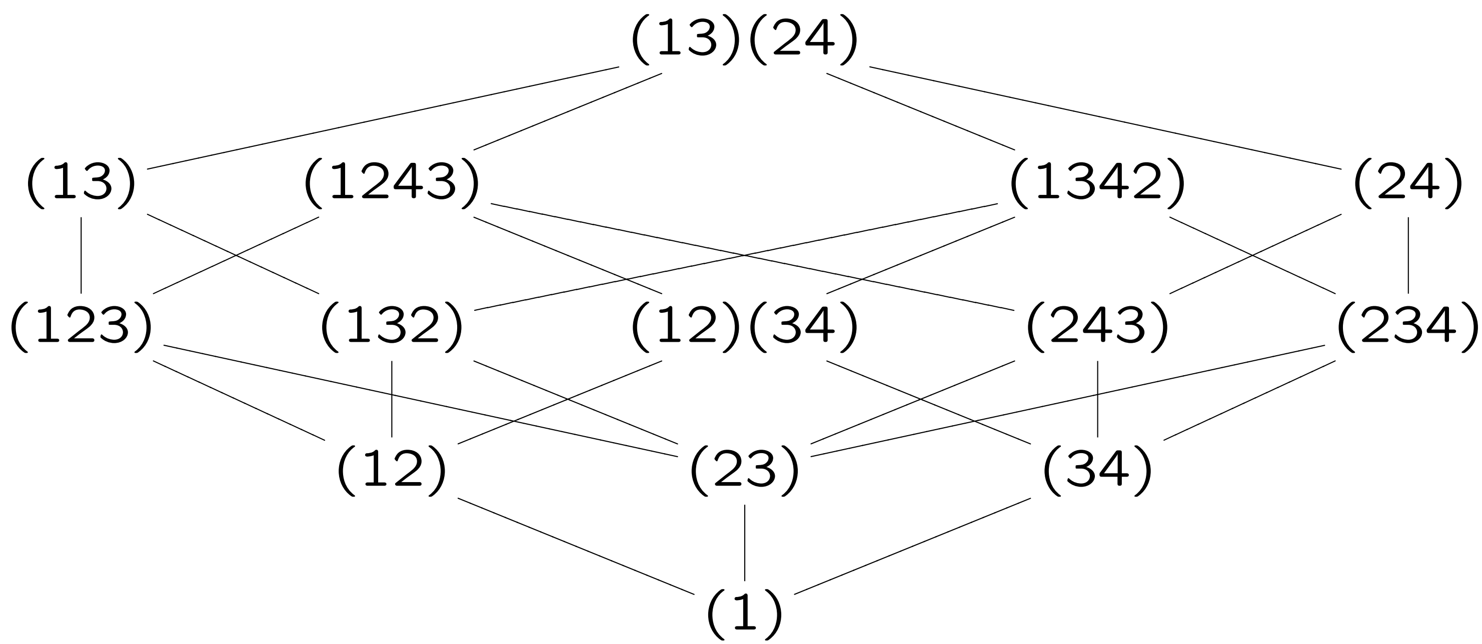
Restricted permutations

$$\mathcal{S} = \{w \in S_{m+p} \mid -p \leq w(i) - i \leq m \text{ for all } i = 1, 2, \ldots, m+p\}.$$

In the 2×2 case, this subposet of the Bruhat poset of S_4 is

$$\mathcal{S} = \{w \in S_4 \mid -2 \leq w(i) - i \leq 2 \text{ for all } i = 1, 2, 3, 4\}.$$

and is shown below.



The poset $\mathcal{H}$-Spec$(\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C})))$

Set

$$\mathcal{S} := \{\sigma \in S_{m+p} \mid -p \leq \sigma(i) - i \leq m, \ \forall i \in [\![1, m+p]\!]\}$$

and

$$w_0 := \begin{bmatrix} 1 & 2 & \dots & p & p+1 & p+2 & \dots & p+m \\ m+1 & m+2 & \dots & m+p & 1 & 2 & \dots & m \end{bmatrix}.$$

Then

$$\mathcal{S} = \{w \in S_{m+p} \mid w \leq w_0\}$$

and

L. (2007) We have a poset isomorphism

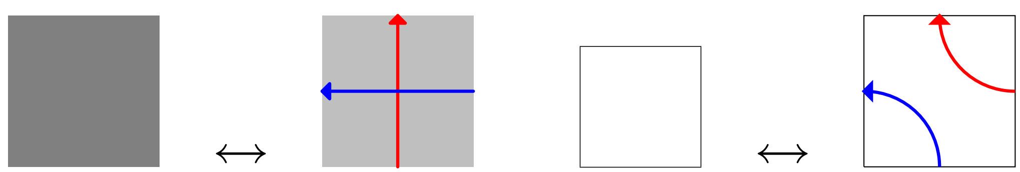
$$\mathcal{H}\text{-Spec}(\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))) \simeq \mathcal{S}.$$

Pipe dreams

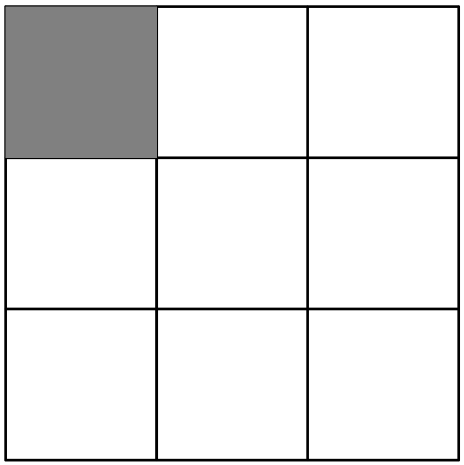
Previous results imply the existence of a bijection between the set of $m \times p$ Cauchon diagrams and the set $\mathcal{S}$ of restricted permutations.

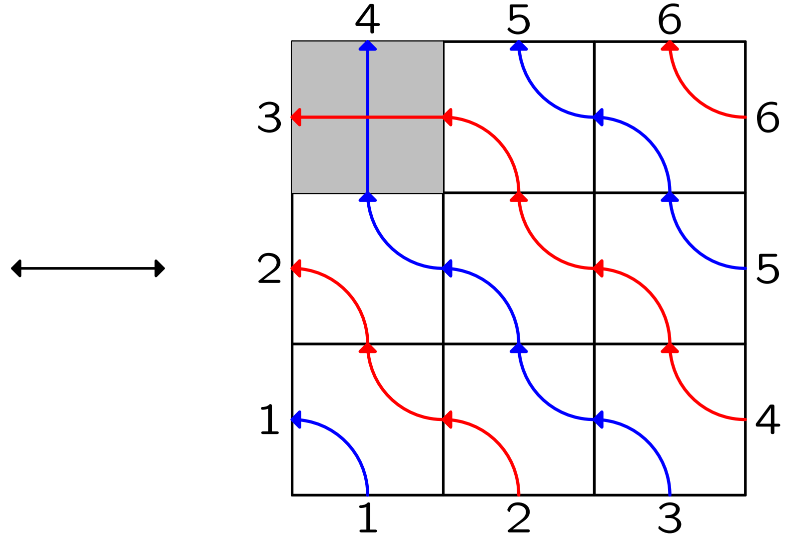
This is no coincidence, and the connection between the two posets can be illuminated by using *Pipe Dreams*.

The procedure to produce a restricted permutation from a Cauchon diagram goes as follows. Given a Cauchon diagram, replace each black box by a cross, and each white box by an elbow joint, that is:



Pipe dreams: an example





So the restricted permutation associated to this Cauchon diagram is (3 4).

Observe that the all black diagram produces the restricted permutation w_0.

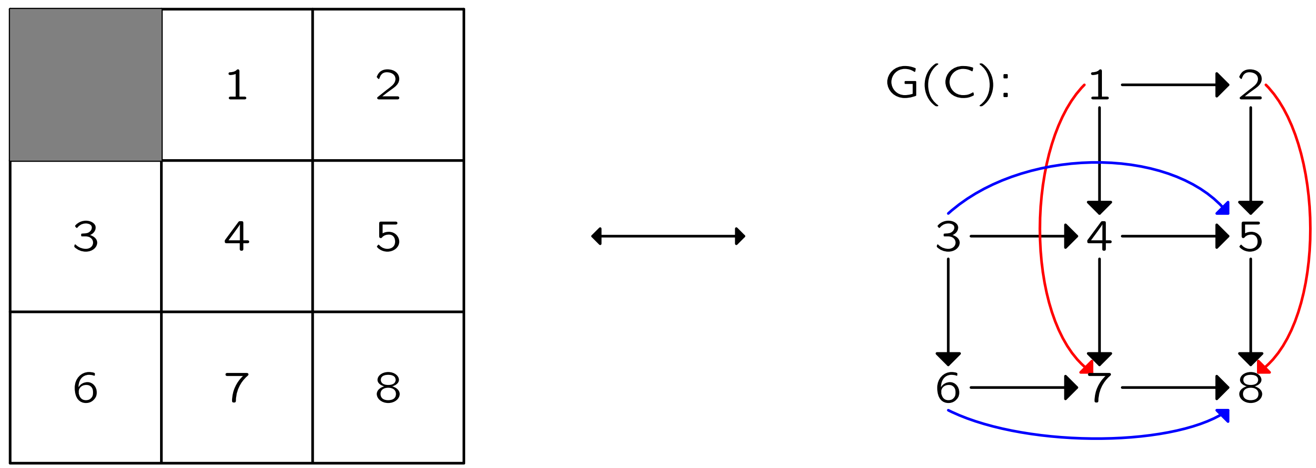
Direct graph associated to a diagram

To a Cauchon diagram C, one can associate a direct graph $G(C)$ and and skew-symmetric matrix A_C as follows.

The vertices of $G(C)$ are the white boxes of C labeled 1 to N. We draw an arrow between two vertices in the same column (going from North to South) or on the same row (going from West to East).

A_C is the $N \times N$ skew-symmetric matrix whose coefficient a_{ij} $(i < j)$ is the number of arrows going from the vertex labeled i to the vertex labeled j.

An example



$$\rightsquigarrow \quad A_C = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}$$

Dimension of strata

Bell-L. (2010). The $\mathcal{H}$-stratum of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$ associated to C is a Laurent polynomial ring over $\mathbb{C}$ in $\dim\ker A_C$ indeterminates.

Problem: this matrix is huge. It can be $mp \times mp$ whereas we have proved that the dimension of a stratum is always less than or equal to $\min(m, p)$.

Bell-Casteels-L. (201?). Let C be an $m \times p$ Cauchon diagram and w be the corresponding restricted permutation.

$$\dim\ker A_C = \dim\ker(P_w + P_{w_0}),$$

where P_σ denotes the permutation-matrix associated to σ.

Explicit bijections

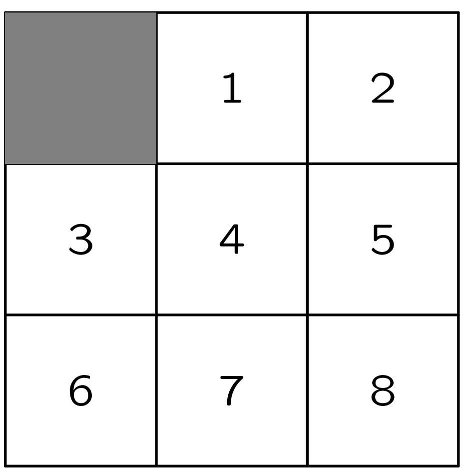
We define two bijections

$$\phi : \mathrm{ker}\, A_C \rightarrow \mathrm{ker}(P_w + P_{w_0})$$

and

$$\psi : \mathrm{ker}(P_w + P_{w_0}) \rightarrow \mathrm{ker}\, A_C.$$

To avoid technicalities, we explain their construction on an example. Consider the following Cauchon diagram



Recall that $A_C = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}$

So we have

$$\ker A_C = \mathrm{Vect}(u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix})$$

Recall that in this case we have $w = (3\,4)$ and $w_0 = (1\,4)(2\,5)(3\,6)$.

So $P_w + P_{w_0} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$

So we have

$$\ker(P_w + P_{w_0}) = \text{Vect}(\alpha = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix})$$

Bijection 1: Image of u

1. Put the coordinates of u in C

	4	5	6
3		1	0
2	0	0	1
1	-1	0	0

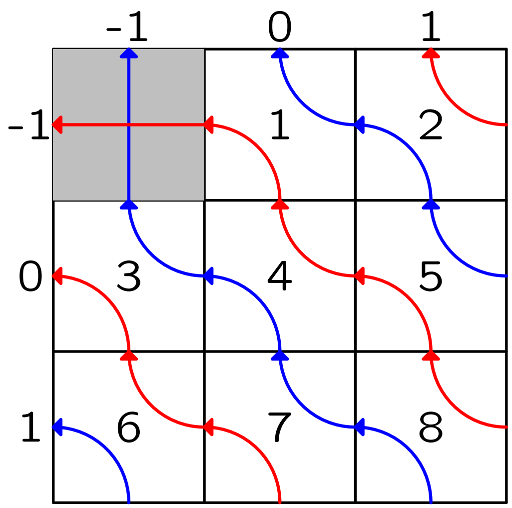
2. The image of u is the vector $(y_1, \ldots, y_6)$ with

$y_1 = -(-1+0+0) = 1,\ y_2 = -(0+0+1) = -1,\ y_3 = -(1+0) = -1$

$$y_4 = 0 + (-1) = -1,\ \ y_5 = 1 + 0 + 0,\ \ y_6 = 0 + 1 + 0.$$

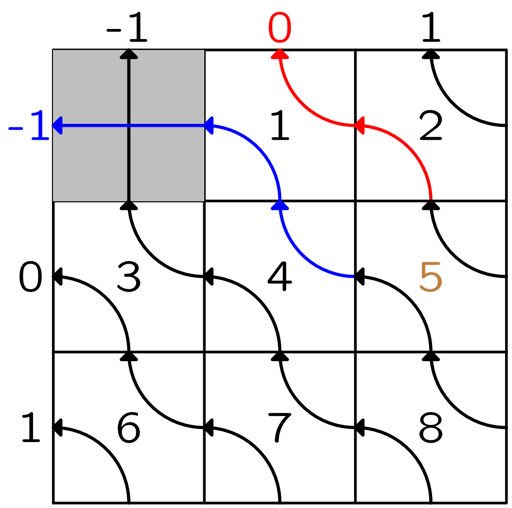
One can check that $\phi(u) = \alpha - \beta \in \ker(P_w + P_{w_0})$.

Bijection 2: image of α



The image of α is the vector $(x_1, \ldots, x_8)$.

Bijection 2: image of α



The image of α is the vector $(x_1, \ldots, x_8)$.

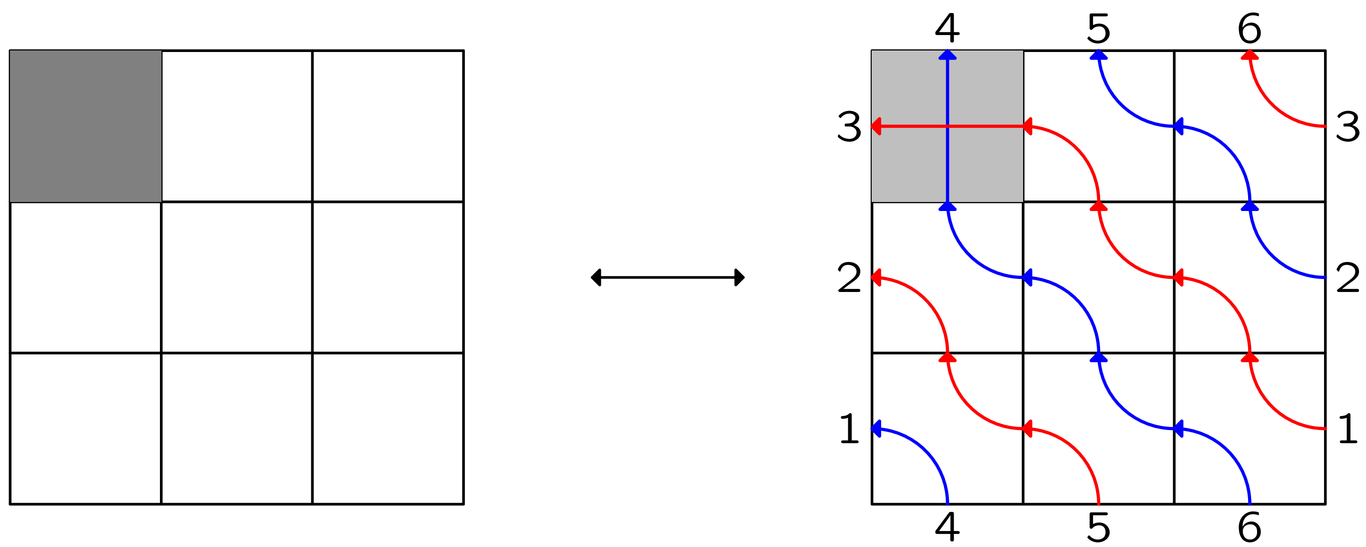
$$x_5 = -1 - 0 = -1$$

One can easily check that

$$\psi(\alpha) = (-1, 0, 1, 0, -1, 1, 1, 0) = -(u + v) \in \ker A_C.$$

Moreover $\phi \circ \psi = -2\text{id}$ and $\psi \circ \phi = -2\text{id}$.

Dimension of strata: toric permutation



So the toric permutation associated to this Cauchon diagram is (1 3 6 4)(2 5).

Bell-Casteels-L. (201?) The dimension of the stratum associated to C is equal to the number of odd cycles in the decomposition of the corresponding toric permutation.

The nonnegative world

- A matrix is **totally positive** if each of its minors is positive.
- A matrix is **totally nonnegative** if each of its minors is nonnegative.

Examples

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

¿ How much work is involved in checking if a matrix is totally positive?

Eg. $n = 4$:

$$\#\text{minors} = \sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n} - 1 \approx \frac{4^n}{\sqrt{\pi n}}$$

by using Stirling's approximation

$$n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$$

2×2 **case**

The matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has **five** minors: $a, b, c, d, \Delta = ad - bc$.

If $b, c, d, \Delta = ad - bc > 0$ then

$$a = \frac{\Delta + bc}{d} > 0$$

so it is sufficient to check **four** minors.

Gasca and Pena: optimal test for total positivity and algorithm for total nonnegativity.

Let $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ be the set of totally nonnegative $m \times p$ real matrices.

Let Z be a subset of minors. The **cell** S_Z^o is the set of matrices in $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ for which the minors in Z are zero (and those not in Z are nonzero).

Some cells may be empty. The space $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ is partitioned by the non-empty cells.

A trivial example In $\mathcal{M}_{2,1}^{\mathsf{tnn}}$ every cell is non-empty. There are 4 cells:

$$S^\circ_{\{\emptyset\}} = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x, y > 0\} \quad S^\circ_{\{[1,1]\}} = \{\begin{pmatrix} 0 \\ y \end{pmatrix} \mid y > 0\}$$

$$S^\circ_{\{[2,1]\}} = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x > 0\} \quad S^\circ_{\{[1,1],[2,1]\}} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

Example In $\mathcal{M}_2^{\text{tnn}}$ the cell $S^{\circ}_{\{[2,2]\}}$ is empty.

For, suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is tnn and $d = 0$.

Then $a, b, c \geq 0$ and also $ad - bc \geq 0$.

Thus, $-bc \geq 0$ and hence $bc = 0$ so that $b = 0$ or $c = 0$.

(This is exactly the same reasoning as in the the proof that a prime in $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ that contains d must contain either b or c!)

Exercise There are 14 non-empty cells in $\mathcal{M}_2^{\text{tnn}}$.

Postnikov (arXiv:math/0609764) defines **Le-diagrams**: an $m \times p$ array with entries either 0 or 1 is said to be a **Le-diagram** if it satisfies the following rule: if there is a 0 in a given square then either each square to the left is also filled with 0 or each square above is also filled with 0.

An example and a non-example of a Le-diagram on a 5×5 array

1	1	0	1	0
0	0	0	1	0
1	1	1	1	0
0	0	0	1	0
1	1	1	1	0

1	1	0	1	0
0	0	1	0	1
1	1	1	0	1
0	0	1	1	1
1	1	1	1	1

• **Postnikov (arXiv:math/0609764)** There is a bijection between Le-diagrams on an $m \times p$ array and non-empty cells S_Z° in $\mathcal{M}_{m,p}^{\mathsf{tnn}}$.

For 2×2 **matrices**, this says that there is a bijection between Le-diagrams on 2×2 arrays and non-empty cells in $\mathcal{M}_2^{\mathsf{tnn}}$.

2×2 Le-diagrams

1	1
1	1

0	1
1	1

1	0
1	1

1	1
0	1

1	1
1	0

0	0
1	1

0	1
0	1

0	1
1	0

1	0
0	1

1	0
1	0

1	1
0	0

0	0
0	1

0	0
1	0

0	1
0	0

1	0
0	0

0	0
0	0

The Link

Totally nonnegative cells

Totally nonnegative cells are defined by the vanishing of families of minors. Some of the TNN cells are empty.

We denote by S_Z^0 the TNN cell associated to the family of minors Z.

A family of minors is *admissible* if the corresponding TNN cell is non-empty.

Question: what are the admissible families of minors?

Generators of $\mathcal{H}$-primes in quantum matrices.

Assume that q is transcendental.

Then $\mathcal{H}$-primes of $\mathcal{O}_q(\mathcal{M}(m,p))$ are generated by quantum minors.

Question: which families of quantum minors?

Conjecture

Let Z_q be a family of quantum minors, and Z be the corresponding family of minors.

$\langle Z_q \rangle$ is a $\mathcal{H}$-prime ideal iff the cell S_Z^0 is non-empty.

An algorithm to rule them all

Deleting derivations algorithm:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a - bd^{-1}c & b \\ c & d \end{pmatrix}$$

Restoration algorithm:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a + bd^{-1}c & b \\ c & d \end{pmatrix}$$

An algorithm to rule them all

If $M = (x_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$, then we set

$$f_{j,\beta}(M) = (x'_{i,\alpha}) \in \mathcal{M}_{m,p}(K),$$

where

$$x'_{i,\alpha} := \begin{cases} x_{i,\alpha} + x_{i,\beta} x_{j,\beta}^{-1} x_{j,\alpha} & \text{if } x_{j,\beta} \neq 0,\ i < j \text{ and } \alpha < \beta \\ x_{i,\alpha} & \text{otherwise.} \end{cases}$$

We set $M^{(j,\beta)} := f_{j,\beta} \circ \cdots \circ f_{1,2} \circ f_{1,1}(M)$.

An example

Set $M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then

$$M^{(2,2)} = M^{(2,1)} = M^{(1,3)} = M^{(1,2)} = M^{(1,1)} = M,$$

$$M^{(3,1)} = M^{(2,3)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M^{(3,2)} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$M^{(3,3)} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Exercise. Is this matrix TNN?

TNN Matrices and restoration algorithm

Goodearl-L.-Lenagan (201?)

• If the entries of M are nonnegative and its zeros form a Cauchon diagram, then $M^{(m,p)}$ is TNN.

• Let M be a matrix with real entries. We can apply the deleting derivation algorithm to M. Let N denote the resulting matrix.

Then M is TNN iff the matrix N is nonnegative and its zeros form a Cauchon diagram.

Main Result

Goodearl-L.-Lenagan (201?) Let $\mathcal{F}$ be a family of minors in the coordinate ring of $\mathcal{M}_{m,p}(\mathbb{C})$, and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Then the following are equivalent:

1. The totally nonnegative cell associated to $\mathcal{F}$ is non-empty.

2. $\mathcal{F}_q$ is the set of quantum minors that belong to torus-invariant prime in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.