Towards the classification of outer actions of finite groups on purely infinite algebras

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Introduction

Possible subtitle: Initiating the Elliot classification program for group actions.

Caution: The results are not complete, and even those that are stated have not been carefully checked. Don't quote them yet!

All the theorems I have are about actions on purely infinite simple C*-algebras, but there are related questions for purely infinite simple Leavitt path algebras (graph algebras), as I will try to make clear during the talk.

Technical details for C*-algebraists will mostly be suppressed. Come to my talks in Nottingham in early September and Banff in late September. (Also, by then, I hope to have made further progress.)

Some of the work was done here:

Rough outline

- The algebras: purely infinite simple C*-algebras and Leavitt path algebras.
- Classification of purely infinite simple algebras.
- Examples of finite group actions on purely infinite simple algebras.
- Pointwise outer actions and the Rokhlin property.
- Equivariant versions of the three main ingredients for pointwise outer actions.

Idempotents and projections

Pure infiniteness is defined in terms of idempotents and projections.

Convention

1. An *idempotent* in a ring A is any element $e \in A$ such that $e^2 = e$.
2. A *projection* in a *-algebra A over $\mathbb{C}$ is a selfadjoint idempotent, that is, we require in addition $e^* = e$.

(In a *-algebra over $\mathbb{C}$, $a \mapsto a^*$ is conjugate linear, reverses multiplication, and satisfies $a^{**} = a$.)

C*-algebraists find it very convenient to "normalize" idempotents to projections. In fact, if A is a unital C*-algebra, then:

1. Every idempotent is similar to a projection.
2. If two projections p and q are similar, then they are unitarily equivalent: there is a unitary $u \in A$ (unitary means $u^* = u^{-1}$, as for complex matrices) such that $upu^* = q$.

Infinite idempotents

Definition

Let A be a ring, and let $e, f \in A$ be idempotents. We write $e \sim f$, and say e and f are *Murray-von Neumann equivalent*, if there are $v, w \in A$ such that $wv = e$ and $vw = f$.

For projections in C*-algebras, the condition is equivalent to the existence of $s \in A$ such that $s^*s = e$ and $ss^* = f$.

Definition

Let A be a ring, and let $e, f \in A$ be idempotents. We write $e \leq f$ if $ef = fe = e$.

Definition

Let A be a ring, and let $e \in A$ be an idempotent. We say e is *infinite* if there exists an idempotent $f \in A$ such that $f \sim e$, $f \leq e$, and $f \neq e$.

Infinite idempotents (continued)

e is infinite if there exists f such that

$$f \sim e, \quad f \leq e, \quad \text{and} \quad f \neq e.$$

Example: Let $V = \bigoplus_{n=0}^{\infty} \mathbb{C}$ be the space of all sequences $(\xi_0, \xi_1, \ldots)$ with $\xi_n \in C$ for all n and $\xi_n = 0$ for all but finitely many n. Let $L(V)$ be the algebra of all linear maps from V to V. Then 1 is infinite in $L(V)$.

Indeed, take f to send

$$(\xi_0, \xi_1, \xi_2, \ldots) \quad \text{to} \quad (0, \xi_1, \xi_2, \ldots)$$

(kill the first coordinate), take v to send

$$(\xi_0, \xi_1, \xi_2, \ldots) \quad \text{to} \quad (0, \xi_0, \xi_1, \ldots)$$

(shift everything one space right), and take w to send

$$(\xi_0, \xi_1, \xi_2, \ldots) \quad \text{to} \quad (\xi_1, \xi_2, \xi_3, \ldots)$$

(delete first coordinate and shift one space left).

A more dramatic example of infiniteness

Let $V = \bigoplus_{n=0}^{\infty} \mathbb{C}$ as before. Define $v_1, v_2, w_1, w_2 \in L(V)$ by:

$$v_1(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_0, 0, \xi_1, 0, \xi_2, 0, \ldots),$$

$$v_2(\xi_0, \xi_1, \xi_2, \ldots) = (0, \xi_0, 0, \xi_1, 0, \xi_2, \ldots),$$

and

$$w_1(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_0, \xi_2, \xi_4, \xi_6, \ldots),$$

$$w_2(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_1, \xi_3, \xi_5, \xi_7, \ldots).$$

Then $w_1 v_1 = w_2 v_2 = 1$, and $v_1 w_1$ and $v_2 w_2$ are orthogonal idempotents, each obviously Murray-von Neumann equivalent to 1, such that $v_1 w_1 + v_2 w_2 = 1$.

Purely infinite simple algebras

Definition

A simple C*-algebra A is called *purely infinite* if every nonzero hereditary subalgebra contains an infinite projection.

Definition

A simple ring A is called *purely infinite* if every nonzero left ideal contains an infinite idempotent.

I won't define hereditary subalgebras, but it is known that they are exactly the subalgebras of the form $L \cap L^*$ for *closed* left ideals L. So a simple C*-algebra A is purely infinite if and only if every nonzero closed left ideal contains an infinite projection.

$L_{\mathbb{C}}(2)$ and $\mathcal{O}_2$

Recall $V = \bigoplus_{n=0}^{\infty} \mathbb{C}$, and $v_1, v_2, w_1, w_2 \in L(V)$, given by:

$$v_1(\xi_0, \xi_1, \ldots) = (\xi_0, 0, \xi_1, 0, \ldots) \quad \text{and} \quad v_2(\xi_0, \xi_1, \ldots) = (0, \xi_0, 0, \xi_1, \ldots),$$

and

$$w_1(\xi_0, \xi_1, \ldots) = (\xi_0, \xi_2, \xi_4, \ldots) \quad \text{and} \quad w_2(\xi_0, \xi_1, \ldots) = (\xi_1, \xi_3, \xi_5, \ldots),$$

satisfying

$$w_1 v_1 = w_2 v_2 = 1, \quad w_1 v_2 = w_2 v_1 = 0, \quad \text{and} \quad v_1 w_1 + v_2 w_2 = 1.$$

These generate the *Leavitt algebra* $L_{\mathbb{C}}(2)$. (One can use other fields in place of $\mathbb{C}$.) It turns out to be purely infinite and simple.

If we replace $\bigoplus_{n=0}^{\infty} \mathbb{C}$ by $l^2(\mathbb{Z}_{\geq 0})$ (square summable sequences instead of those that are eventually zero), and use the same formulas, getting operators s_1, s_2, s_1^*, s_2^*, then the generated C*-algebra is the Cuntz algebra $\mathcal{O}_2$, a purely infinite simple C*-algebra.

$L_{\mathbb{C}}(d)$ and $\mathcal{O}_d$

For $d = 2, 3, 4, \ldots$, there is a Leavitt algebra $L_{\mathbb{C}}(d)$ over $\mathbb{C}$, generated by elements

$$v_1, v_2, \ldots, v_d \quad \text{and} \quad w_1, w_2, \ldots, w_d$$

such that

$$w_1 v_1 = w_2 v_2 = \cdots = w_d v_d = 1,$$

$w_j v_k = 0$ for $j \neq k$, and $v_1 w_1, v_2 w_2, \ldots, v_d w_d$ are orthogonal idempotents which add up to 1.

Similarly, there is a Cuntz algebra $\mathcal{O}_d$ generated as a C*-algebra by $s_1, s_2, \ldots, s_d$ satisfying

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1.$$

(The other Leavitt algebra relations are then automatic.)
For $d = \infty$, one gets $L_{\mathbb{C}}(\infty)$ and $\mathcal{O}_\infty$ by omitting the condition that the $v_j w_j$ or $s_j s_j^*$ add up to 1, but keeping the requirement that they be orthogonal.

$L_{\mathbb{C}}(d)$, $\mathcal{O}_d$, and generalizations

Relations for $L_d(\mathbb{C})$ for $d < \infty$:

$$w_1 v_1 = w_2 v_2 = \cdots = w_d v_d = 1,$$

$w_j v_k = 0$ for $j \neq k$, and $v_1 w_1, v_2 w_2, \ldots, v_d w_d$ are orthogonal idempotents which add up to 1.

For $d = 2, 3, 4, \ldots, \infty$, the algebras $L_{\mathbb{C}}(d)$ and $\mathcal{O}_d$ are purely infinite and simple; in fact, they are in some sense the basic examples.

In both the algebraic and C*-algebraic contexts, there are generalizations, to Cuntz-Krieger algebras, and to algebras made from relations defined by directed graphs, giving *Leavitt path algebras* and *graph C*-algebras*.

In general, graph algebras need be neither simple nor purely infinite. But there are known criteria for when they are, the same for the algebraic and C*-algebraic settings, and many purely infinite simple algebras arise this way.

Classifiable purely infinite simple C*-algebras

Definition

A *Kirchberg algebra* is a purely infinite simple separable nuclear C*-algebra. A *UCT Kirchberg algebra* is a Kirchberg algebra which satisfies the Universal Coefficient Theorem.

"Separable" means that it has a countable dense subset. "Nuclear" is a more subtle way of saying "not too large". The Universal Coefficient Theorem is a technical statement about K-theory; no counterexamples are known to the conjecture that all nuclear C*-algebras satisfy it.

The graph C*-algebra of every countable graph is nuclear, separable, and satisfies the UCT. Thus, if such an algebra is purely infinite and simple, then it is a UCT Kirchberg algebra.

In particular, $\mathcal{O}_d$ is a UCT Kirchberg algebra for $d \in \{2, 3, 4, \ldots, \infty\}$.

The classification theorem for purely infinite simple C*-algebras

Here is the classification theorem for purely infinite simple C*-algebras, by now about 15 years old:

Theorem

Let A and B be two UCT Kirchberg algebras, either both unital or both nonunital. Suppose $K_0(A) \cong K_0(B)$ (with $[1_A] \mapsto [1_B]$ in the unital case) and $K_1(A) \cong K_1(B)$. Then $A \cong B$.

Here, $K_0(A)$ is the usual algebraic K_0-group: the Grothendieck group made from finitely generated projective modules over A, or, equivalently, from Murray-von Neumann equivalence classes of projections in matrix algebras over A. (C*-algebraists generally talk about projections rather than finitely generated projective modules.)

$K_1(A)$ is the topological K_1-group of A. When A is unital and purely infinite simple (even properly infinite is enough), it happens to be isomorphic to the algebraic K_1-group of A.

Classification for Leavitt path algebras?

There are many parallels between graph C*-algebras and Leavitt path algebras, and there are by now about 15 published papers (authors include Ken Goodearl) on Leavitt path algebras. Several talks at this conference have been about these algebras.

There are no known counterexamples to either direction of the suggestion that for countable graphs E and F, one has $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ if and only if $C^*(E) \cong C^*(F)$. There is work of Abrams, Áhn, Pardo, and Tomforde (in several papers) which verifies some isomorphisms predicted by combining this suggestion with the C*-algebra classification theorem above ($K_*(A) \cong K_*(B)$ implies $A \cong B$).

For example, for any field K, one has $M_n(L_K(d)) \cong L_K(d)$ if and only if n and $d - 1$ are relatively prime. The C* analog was first obtained in full generality as a consequence of the classification theorem, which makes heavy use of analysis.

Classification for Leavitt path algebras? (continued)

However, there are definite limits on the possibilities for classification of purely infinite simple rings. For example, let M_{2^∞} be the 2^∞ UHF algebra, the C* direct limit of the system (with unital maps)

$$\mathbb{C} \longrightarrow M_2(\mathbb{C}) \longrightarrow M_4(\mathbb{C}) \longrightarrow M_8(\mathbb{C}) \longrightarrow \cdots .$$

Let K be the algebra of compact operators on $l^2(\mathbb{Z}_{\geq 0})$ (the C* analog of the infinite matrices with only finitely many nonzero entries). Then $K \otimes M_{2^\infty} \otimes \mathcal{O}_2 \cong K \otimes \mathcal{O}_2$, but the algebraic analog of this statement is false. The right hand side is a graph C*-algebra, and the left hand side is a tensor product of two graph C*-algebras.

The nonisomorphism is easy to see: if one cuts down by nonzero idempotents, in the algebraic analog of $K \otimes \mathcal{O}_2$ the resulting corners are finitely generated and in the algebraic analog of $K \otimes M_{2^\infty} \otimes \mathcal{O}_2$ they are not.

Classification for Leavitt path algebras? (continued)

We have

$$K \otimes M_{2^\infty} \otimes \mathcal{O}_2 \cong K \otimes \mathcal{O}_2,$$

but the algebraic analog of this statement is false. The right hand side is a graph C*-algebra, and the left hand side is a tensor product of two graph C*-algebras, but its algebraic analog might not be a Leavitt path algebra.

By Elliott's Theorem (superseded by one of Kirchberg's absorption theorems; see below),

$$\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2.$$

Open question: Is $L_{\mathbb{C}}(2) \otimes L_{\mathbb{C}}(2) \cong L_{\mathbb{C}}(2)$? All known proofs of Elliott's Theorem make serious use of analysis, and most operator algebraists believe $L_{\mathbb{C}}(2) \otimes L_{\mathbb{C}}(2) \not\cong L_{\mathbb{C}}(2)$.

Three main ingredients

Here are three main ingredients for the analysis part of the proof of the classification theorem. The first two are Kirchberg's absorption theorems, and the third is almost immediate.

Recall that $\mathcal{O}_2$ is generated by elements s_1 and s_2 such that

$$s_1^* s_1 = s_2^* s_2 = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* = 1,$$

and that $\mathcal{O}_\infty$ is generated by elements $s_1, s_2, \ldots$ such that $s_j^* s_j = 1$ for all j and $s_1 s_1^*$, $s_2 s_2^*$, ... are orthogonal projections.

Theorem

Let A be a simple separable unital nuclear C*-algebra. Then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

Theorem

Let A be a Kirchberg algebra. Then $\mathcal{O}_\infty \otimes A \cong A$.

(In fact, there is an isomorphism from A to $\mathcal{O}_\infty \otimes A$ which is asymptotically unitarily equivalent to the map $a \mapsto 1 \otimes a$.)

Three main ingredients (continued)

The first two ingredients:

- $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for A simple separable unital nuclear.
- $\mathcal{O}_\infty \otimes A \cong A$ for A a Kirchberg algebra.

Here is the third:

Theorem

Let A be a purely infinite simple C*-algebra, and let $p \in A$ be a nonzero projection such that $[p] = 0$ in $K_0(A)$. Then there exists a unital homomorphism $\mathcal{O}_2 \to pAp$.

There are many such projections p. In fact, if $q \in A$ is a nonzero projection, then qAq is a nonzero hereditary subalgebra, so contains an infinite projection. It easily follows that q itself is infinite. Thus there is e such that $e \leq q$, $e \neq q$, and $e \sim q$. Then $p = q - e$ satisfies $[p] = 0$ in $K_0(A)$.

How the ingredients are used

For the C*-algebraists, here is what the ingredients are used for.

Theorem

Let A be a unital Kirchberg algebra, and let D be any unital C*-algebra. Let φ and ψ be two homotopic full asymptotic morphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$. Then φ and ψ are asymptotically unitarily equivalent.

To obtain the classification theorem from this result, one uses the Universal Coefficient Theorem, the Elliott approximate intertwining argument, and some additional more algebraic topological material.

Generalizing to actions of finite groups

We want to generalize the classification theory to actions of finite groups on Kirchberg algebras. The objective is a theorem to the effect that if two UCT actions $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ of a finite group G on Kirchberg algebras A and B have the same K-theoretic invariants, than α and β are conjugate, that is, there exists an isomorphism $\varphi \colon A \to B$ such that $\beta_g = \varphi \circ \alpha_g \circ \varphi^{-1}$ for all $g \in G$.

As we explain later, we have to restrict to pointwise outer actions. Izumi has already done this for actions with the Rokhlin property. Moreover, with the current state of knowledge for the Universal Coefficient Theorem, at the final step we must assume the group is cyclic of prime order.

Before doing this, we give some examples of actions of finite groups on UCT Kirchberg algebras, often restricting to the group $\mathbb{Z}_2$.

Permuting and multiplying the generators

Recall that $\mathcal{O}_d$ is generated as a C*-algebra by $s_1, s_2, \ldots, s_d$ satisfying

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1,$$

and $L_{\mathbb{C}}(d)$ is generated by $v_1, v_2, \ldots, v_d$ and $w_1, w_2, \ldots, w_d$ such that $w_1 v_1 = w_2 v_2 = \cdots = w_d v_d = 1$, $w_j v_k = 0$ for $j \neq k$, and $v_1 w_1, v_2 w_2, \ldots, v_d w_d$ are orthogonal idempotents which add up to 1.

Let S_d be the symmetric group. Then there is an action $\alpha \colon S_d \to \mathrm{Aut}(\mathcal{O}_d)$ which permutes the generators: $\alpha_\sigma(s_j) = s_{\sigma(j)}$ for $\sigma \in S_d$ and $j = 1, 2, \ldots, m$. One can restrict to any subgroup of S_d.

This action also makes sense algebraically: $\alpha_\sigma(v_j) = v_{\sigma(j)}$ and $\alpha_\sigma(w_j) = w_{\sigma(j)}$ for $\sigma \in S_d$ and $j = 1, 2, \ldots, m$.

Here is an action of $\mathbb{Z}_2$ on $\mathcal{O}_d$: Let the nontrivial element send s_j to $\varepsilon_j s_j$ with $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d \in \{1, -1\}$. The algebraic version and the geneneralization to other cyclic groups work in the obvious way.

Quasifree actions

Here is a generalization. Let $\rho \colon G \to L(\mathbb{C}^d)$ be a unitary representation of G. Write

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\alpha \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ such that

$$\alpha_g(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j$$

for $j = 1, 2, \ldots, d$. (To check this: A computation shows that the proposed elements $\alpha_g(s_k)$ satisfy the correct relations, so α_g exists, and another computation shows that $\alpha_g \circ \alpha_h = \alpha_{gh}$.)

The permutation action on the previous page comes from the permutation representation of S_d on $\mathbb{C}^d$, and the multiplication action on the previous page comes from the representation sending the nontrivial element of $\mathbb{Z}_2$ to $\mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)$.

Quasifree actions also make sense on Leavitt algebras

The quasifree actions on the previous page come from actions on the corresponding Leavitt algebras. Recall that v_j plays the role of s_j and w_j plays the role of s_j^*. Define

$$\beta_g(v_k) = \sum_{j=1}^{d} \rho_{j,k}(g) v_j.$$

To find the formula for $\beta_g(w_k)$, take the adjoint of the expression for $\alpha_g(s_k)$ and substitute w_j for s_j^*.

In fact, I presume that there are corresponding formulas for actions on $L_K(d)$ (using inverse instead of adjoint) that work for representations whose values are merely invertible instead of unitary, and over an arbitrary field K.

An action of $\mathbb{Z}_2$ on $M_3 \otimes \mathcal{O}_4$

Let s_1, s_2, s_3, and s_4 be the standard generating isometries of $\mathcal{O}_4$. Let $e_{j,k}$, for $1 \leq j,\ k \leq 3$, be the standard matrix units of M_3, satisfying $e_{j,k}e_{k,l} = e_{j,l}$ etc. Then there is an automorphism φ of $M_3 \otimes \mathcal{O}_4$ of order 2 determined by:

$$\begin{aligned}
e_{1,1} \otimes 1 &\mapsto (e_{2,2} + e_{3,3}) \otimes 1 \\
e_{2,2} \otimes 1 &\mapsto e_{1,1} \otimes (s_1 s_1^* + s_2 s_2^*) \\
e_{3,3} \otimes 1 &\mapsto e_{1,1} \otimes (s_3 s_3^* + s_4 s_4^*) \\
e_{1,2} \otimes 1 &\mapsto e_{2,1} \otimes s_1^* + e_{3,1} \otimes s_2^* \\
e_{1,3} \otimes 1 &\mapsto e_{2,1} \otimes s_3^* + e_{3,1} \otimes s_4^* \\
e_{1,1} \otimes s_1 &\mapsto e_{2,2} \otimes s_1 + e_{2,3} \otimes s_2 \\
e_{1,1} \otimes s_2 &\mapsto e_{2,2} \otimes s_3 + e_{2,3} \otimes s_4 \\
e_{1,1} \otimes s_3 &\mapsto e_{3,2} \otimes s_1 + e_{3,3} \otimes s_2 \\
e_{1,1} \otimes s_4 &\mapsto e_{3,2} \otimes s_3 + e_{3,3} \otimes s_4
\end{aligned}$$

(The proof can be done by computations. It works on $M_3 \otimes L_K(4)$ for any K.)

Permuting tensor factors

Fix m. Let the symmetric group S_m, or any subgroup, act on $\mathcal{O}_d^{\otimes m}$ by

$$\alpha_\sigma(a_1 \otimes a_2 \otimes \cdots \otimes a_m) = a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(m)}$$

for $\sigma \in S_m$ and $a_1, a_2, \ldots, a_m \in \mathcal{O}_d$.

The tensor product $\mathcal{O}_d^{\otimes m}$ is a UCT Kirchberg algebra. In the special case $d = 2$, we have $\mathcal{O}_2^{\otimes m} \cong \mathcal{O}_2$, so we have an action on $\mathcal{O}_2$.

These actions exist in the algebraic context as well. The algebra $L_{\mathbb{C}}(d)^{\otimes m}$ is purely infinite and simple. However, $L_{\mathbb{C}}(2)^{\otimes m}$ is probably not isomorphic to $L_{\mathbb{C}}(2)$.

What kind of actions can we classify?

Recall the three main ingredients for classification without the group:

1. $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for A simple separable unital nuclear.
2. $\mathcal{O}_\infty \otimes A \cong A$ for A a Kirchberg algebra.
3. If A is purely infinite and $p \in A$ is a nonzero projection such that $[p] = 0$ in $K_0(A)$, then there is a unital homomorphism $\mathcal{O}_2 \to pAp$.

We want equivariant versions of these. If we allow arbitrary actions, taking the trivial action on A in (2) forces one to use the trivial action on $\mathcal{O}_\infty$ and taking a nontrivial action on A in (1) forces one to use a nontrivial action on $\mathcal{O}_2$. These choices make (3) impossible.

The right condition on the action is pointwise outerness.

It seems plausible that one can deal with more general actions, by including invariants coming from group cohomology and settling for cocycle conjugacy instead of conjugacy. This has been done for actions on II_1 factors, but here is left for a future project.

The Universal Coefficient Theorem

Manuel Koehler has proved a Universal Coefficient Theorem for equivariant KK-theory when the group is cyclic of prime order. It is more complicated than the usual Universal Coefficient Theorem in KK-theory, involving both ordinary and equivariant K-theory as well as a third group, together with various operations. (In spirit, it is similar to Jeff Boersema's Universal Coefficient Theorem for real C*-algebras.)

The Universal Coefficient Theorem gets used at the end of the reasoning. Without it, we should still be able to prove that KK^G-equivalence implies conjugacy. However, until Koehler's Universal Coefficient Theorem is generalized, the final form of the classification theorem will only be proved for actions of cyclic groups of prime order.

The other more topological ingredients for the proof of the classification theorem have mostly already been generalized to the equivariant case, leaving, at the start of this project, the equivariant versions of the three main ingredients above as the most likely source of difficulty. Those have now been proved.

Pointwise outer actions

Definition

Let A be a unital C*-algebra. An automorphism $\alpha \in \mathrm{Aut}(A)$ is *inner* if there is a unitary $u \in A$ such that $\alpha = \mathrm{Ad}(u)$, that is, $\alpha(a) = uau^*$ for all $a \in A$. If α is not inner, it is *outer*.

Recall that a unitary u satisfies $u^* = u^{-1}$. We use unitaries rather than arbitrary invertible elements to preserve the adjoint operation.

Definition

Let A be a unital C*-algebra, and let G be a group. An action $\alpha \colon G \to \mathrm{Aut}(A)$ is said to be *pointwise outer* if, for every $g \in G \setminus \{1\}$, the automorphism α_g is outer.

Pointwise outer actions (continued)

Definition

Let A be a unital C*-algebra, and let G be a group. An action $\alpha \colon G \to \mathrm{Aut}(A)$ is said to be *pointwise outer* if, for every $g \in G \setminus \{1\}$, the automorphism α_g is outer.

An action $\alpha \colon G \to \mathrm{Aut}(A)$ is inner if there exists a homomorphism $g \mapsto u_g$ from G to the unitary group of A such that $\alpha_g(a) = u_g a u_g^*$ for all $g \in G$ and $a \in A$. With this terminology, there are actions which are not inner but for which α_g is inner for all $g \in G$. (Example omitted; it has $G = \mathbb{Z}_2^2$.)

All this makes sense purely algebraically, but, without an adjoint operation, one should use invertible elements in place of unitaries.

Pointwise outer actions (continued)

A quasifree action is known to be pointwise outer provided the group representation it is made from is injective.

The action of $\mathbb{Z}_2$ on $M_3 \otimes \mathcal{O}_4$ with the complicated formulas is pointwise outer. (It is nontrivial on K-theory.)

Actions obtained by permuting tensor factors are known in some cases to be pointwise outer; probably this is always true.

It turns out that we also need a stronger condition, the Rokhlin property.

The Rokhlin property

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there exist orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\sum_{g \in G} e_g = 1$.
2. For all $g, h \in G$, we have $\alpha_g(e_h) = e_{gh}$.
3. For all $g \in G$ and all $a \in F$, we have $\|e_g a - a e_g\| < \varepsilon$.

Note to C*-algebraists: The condition in (2) is usually taken to be $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$. The two versions are equivalent, as I discovered while working on this project.

To make an algebraic version of the Rokhlin property, replace projections by idempotents, and in (3) ask that $e_g a = a e_g$.

The Rokhlin property (continued)

The conditions in the Rokhlin property:

1. $\sum_{g \in G} e_g = 1$.
2. For all $g, h \in G$, we have $\alpha_g(e_h) = e_{gh}$.
3. For all $g \in G$ and all $a \in F$, we have $\|e_g a - a e_g\| < \varepsilon$.

Here is the trivial example of an action with the Rokhlin property. Fix a unital C*-algebra (or algebra) B, set $A = \bigoplus_{g \in G} B$, and let G act by permuting the summands. Take e_g in the definition to be the identity of the g summand.

Nontrivial examples do exist (see below), but in the algebraic case not if A is finitely generated. (Take the finite set F to be a generating set, and conclude that e_g commutes with every element of A.)

It is not hard to show that the Rokhlin property implies pointwise outerness. The converse is false. The quasifree action of $\mathbb{Z}_2$ on $\mathcal{O}_2$ generated by $s_1 \mapsto -s_1$ and $s_2 \mapsto -s_2$ is pointwise outer, but turns out not to have the Rokhlin property.

A nontrivial example of the Rokhlin property

We take $G = \mathbb{Z}_2$. Let M_{2^∞} be the 2^∞ UHF algebra, the (C* or algebraic) direct limit of the system (written slightly differently than before)

$$\mathbb{C} \longrightarrow M_2(\mathbb{C}) \longrightarrow M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \longrightarrow M_2(\mathbb{C})^{\otimes 3} \longrightarrow M_2(\mathbb{C})^{\otimes 4} \longrightarrow \cdots,$$

with maps $a \mapsto a \otimes 1_{M_2(\mathbb{C})}$ at each stage. The nontrivial group element acts on $M_2(\mathbb{C})^{\otimes n}$ by acting on each factor of M_2 as $\mathrm{Ad}\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$. Call this automorphism α_n, and let α be the corresponding automorphism of M_{2^∞}.

Let $F \subset A$ be finite. We need orthogonal projections (idempotents) e_0, e_1 such that $e_0 + e_1 = 1$, $\alpha(e_0) = e_1$, and e_0 and e_1 approximately (in the C* case) or exactly (in the algebraic case) commute with all the elements of F.

In the algebraic case, we have $F \subset M_2(\mathbb{C})^{\otimes n}$ for some n. In the C* case, this is approximately true, and we are allowed to perturb F slightly, so we can assume it is exactly true. Now take

$$e_0 = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})^{\otimes(n+1)} \subset M_{2^\infty} \quad \text{and} \quad e_1 = 1 - e_0.$$

A nontrivial example of the Rokhlin property (continued)

We have $F \subset M_2(\mathbb{C})^{\otimes n} \subset M_{2^\infty}$,

$$e_0 = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})^{\otimes(n+1)} \subset M_{2^\infty}$$

and

$$e_1 = 1 - e_0 = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and α acts on $M_2(\mathbb{C})^{\otimes(n+1)} \subset M_{2^\infty}$ via

$$\alpha_{n+1} = \mathrm{Ad}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes(n+1)} \right).$$

We have $e_0 + e_1 = 1$ by definition. One checks immediately that $\alpha_{n+1}(e_0) = e_1$ (and also $\alpha_{n+1}(e_1) = e_0$, but this follows from $\alpha_{n+1}^2 = \mathrm{id}$). Moreover, e_0 and e_1 commute with everything in $M_2(\mathbb{C})^{\otimes n}$ and hence with everything in F.

Thus, our action has the Rokhlin property.

Actions on purely infinite simple algebras with the Rokhlin property

The construction above works for any finite group, using the infinite tensor product of copies of conjugation by the regular representation of the group.

The algebra M_{2^∞} is simple but not purely infinite. However:

Lemma

Let $\alpha\colon G \to \mathrm{Aut}(A)$ have the Rokhlin property, and let $\beta\colon G \to \mathrm{Aut}(B)$ be any action on a unital algebra. Then $g \mapsto \alpha_g \otimes \beta_g \colon G \to Aut(A \otimes B)$ has the Rokhlin property.

In the algebraic case, form the tensor product over $\mathbb{C}$ (or whichever field is being used). In the C* case, use the minimal tensor product.

So we can tensor the action above with any action on any unital purely infinite simple (C*-) algebra.

Proof of the lemma

The lemma from the previous slide:

Lemma

Let $\alpha\colon G \to \mathrm{Aut}(A)$ have the Rokhlin property, and let $\beta\colon G \to \mathrm{Aut}(B)$ be any action on a unital algebra. Then $g \mapsto \alpha_g \otimes \beta_g \colon G \to Aut(A \otimes B)$ has the Rokhlin property.

Proof.

In the algebraic case, let $F \subset A \otimes B$ be a finite set. Choose a finite set $S \subset A$ such that all elements of F have the form $\sum_{k=1}^{n} a_j \otimes b_j$ with all $a_j \in S$ for all j. Choose Rokhlin idempotents for α and S, say f_g for $g \in G$. Thus, $\sum_{g \in G} f_g = 1$, $\alpha_g(f_h) = f_{gh}$, and f_g commutes with all elements of S. Then take $e_g \in A \otimes B$ to be $e_g = f_g \otimes 1$.

In the C* case, approximate F by a finite set whose elements are all finite sums of elementary tensors. Then proceed essentially as above. □

Classification of actions with the Rokhlin property

Izumi has given a K-theoretic classification of Rokhlin actions of finite groups on Kirchberg algebras. However, such actions are rare, while pointwise outer actions are common.

For example, if $\alpha\colon G \to \mathrm{Aut}(A)$ has the Rokhlin property, then there exists a projection $e_1 \in A$ (a Rokhlin projection for $1 \in G$) such that in $K_0(A)$ we have

$$\sum_{g \in G} (\alpha_g)_*([e_1]) = [1].$$

If $A = \mathcal{O}_d$ (including $d = \infty$), every automorphism of A is trivial on $K_0(A)$, so we get $\mathrm{card}(G) \cdot [e_1] = [1]$. Since for d finite, $K_0(\mathcal{O}_d) \cong \mathbb{Z}_{d-1}$, generated by $[1]$, this can happen only if $\mathrm{card}(G)$ is relatively prime to d. Since $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, generated by $[1]$, this can never happen if $d = \infty$.

The action on $\mathcal{O}_2$

Recall the first main ingredient: $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for A simple separable unital nuclear. We need an action $\zeta\colon G \to \mathrm{Aut}(\mathcal{O}_2)$ such that this isomorphism holds equivariantly whenever A is purely infinite simple and the action on it is pointwise outer. Since a tensor product action has the Rokhlin property if one factor does, our ζ had better have the Rokhlin property.

Start as follows. Set $d = \mathrm{card}(G)$. Take D to be the d^∞ UHF algebra, the C* direct limit of

$$\mathbb{C} \longrightarrow M_d(\mathbb{C}) \longrightarrow M_d(\mathbb{C}) \otimes M_d(\mathbb{C}) \longrightarrow M_d(\mathbb{C})^{\otimes 3} \longrightarrow M_d(\mathbb{C})^{\otimes 4} \longrightarrow \cdots,$$

with maps $a \mapsto a \otimes 1_{M_d(\mathbb{C})}$ at each stage, and with the action being conjugation by the regular representation in each tensor factor.

Now take ζ to be the tensor product of this action with the trivial action on $\mathcal{O}_2$. Since $\mathcal{O}_2 \otimes D \cong \mathcal{O}_2$, this is a Rokhlin action on $\mathcal{O}_2$.

The absorption theorem for the action on $\mathcal{O}_2$

We have an action $\zeta \colon G \to \mathrm{Aut}(\mathcal{O}_2)$ which has the Rokhlin property.

Izumi's classification tells us that there is, up to conjugacy, only one Rokhlin action on $\mathcal{O}_2$.

So let $\alpha \colon G \to \mathrm{Aut}(A)$ be any action on a simple separable unital nuclear C*-algebra A. (It need not even be pointwise outer.) Then the tensor product action $\zeta \otimes \alpha \colon G \to \mathrm{Aut}(\mathcal{O}_2 \otimes A)$ has the Rokhlin property, and $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$, so Izumi tells us that $\zeta \otimes \alpha$ is conjugate to ζ, as desired.

What happens algebraically

Recall that $\mathcal{O}_2$ is generated by s_1 and s_2 such that $s_1^* s_1 = s_2^* s_2 = 1$ and $s_1 s_1^* + s_2 s_2^* = 1$. Moreover, $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ (Elliott's Theorem).

Algebraically, first, if (as expected) $L_{\mathbb{C}}(2) \otimes L_{\mathbb{C}}(2) \not\cong L_{\mathbb{C}}(2)$, there is no hope of an equivariant absorption theorem.

Take $G = \mathbb{Z}_2$. The following are all Rokhlin actions of G on $\mathcal{O}_2$:

1. The action from above, on $\mathcal{O}_2 \otimes M_{2^\infty}$.
2. The flip action on $\mathcal{O}_2 \otimes \mathcal{O}_2$, generated by $a \otimes b \mapsto b \otimes a$.
3. The quasifree action on $\mathcal{O}_2$ generated by $s_1 \mapsto s_1$ and $s_2 \mapsto -s_2$.

(We omit proofs.) So they are all conjugate.

In (1), the algebraic version has the Rokhlin property, but the algebra is not isomorphic to $L_{\mathbb{C}}(2)$. (It is not finitely generated but $L_{\mathbb{C}}(2)$ is.)

In (2) and (3), the algebra is finitely generated, so no action has has the Rokhlin property. Moreover, these two actions can't be conjugate to each other if $L_{\mathbb{C}}(2) \otimes L_{\mathbb{C}}(2) \not\cong L_{\mathbb{C}}(2)$.

The Rokhlin property and vanishing of cohomology

Izumi considers the Rokhlin property to be a tool for proving vanishing lemmas for group cohomology. For example, if $\alpha, \beta \colon G \to \mathrm{Aut}(A)$ are Rokhlin actions of a finite group G on a unital C*-algebra A, and α and β are cocycle conjugate, then they are conjugate. (Details omitted.)

Probably the Rokhlin property is stronger than needed for this purpose.

I do not know whether this result, or other cohomology vanishing results, follows in the algebraic situation from the algebraic version of the Rokhlin property. Nor do I know whether they hold, say, for the algebraic version of the quasifree action on $\mathcal{O}_2$ generated by $s_1 \mapsto s_1$ and $s_2 \mapsto -s_2$.

The action on $\mathcal{O}_\infty$

Recall the second main ingredient: $\mathcal{O}_\infty \otimes A \cong A$ for A a Kirchberg algebra. We need an action $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ such that this isomorphism holds equivariantly whenever A is unital and the action on A is pointwise outer.

Rokhlin actions on $\mathcal{O}_\infty$ do not exist. (If they did, we would only be able to absorb Rokhlin actions on Kirchberg algebra, and we would then only be able to classify Rokhlin actions.)

We do, however, want an action that is somehow close to having the Rokhlin property.

Let G be a finite group, set $d = \mathrm{card}(G)$, and let $m \in \mathbb{Z}_{>0}$. Label the generators of $\mathcal{O}_{md}$ as $s_{g,j}$ for $g \in G$ and $j = 1, 2, \ldots, m$. Thus, they satisfy the relations

$$s_{g,j}^* s_{g,j} = 1 \quad \text{and} \quad \sum_{g \in G} \sum_{j=1}^{m} s_{g,j} s_{g,j}^* = 1.$$

Define an action $\iota^{(m)} \colon G \to \mathrm{Aut}(\mathcal{O}_{md})$ by $\iota_g^{(m)}(s_{h,j}) = s_{gh,j}$. This action is known to have the Rokhlin property.

The action on $\mathcal{O}_\infty$ (continued)

$d = \mathrm{card}(G)$, and $\mathcal{O}_{md}$ is generated by $s_{g,j}$ for $g \in G$ and $j = 1, 2, \ldots, m$, with

$$s_{g,j}^* s_{g,j} = 1 \quad \text{and} \quad \sum_{g \in G} \sum_{j=1}^{m} s_{g,j} s_{g,j}^* = 1.$$

The action is $\iota_g^{(m)}(s_{h,j}) = s_{gh,j}$.

(This is the quasifree action from the direct sum of m copies of the regular representation.)

We want to let $m \to \infty$. We take $\mathcal{O}_\infty$ to be generated by $s_{g,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$, with $s_{g,j}^* s_{g,j} = 1$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$, and such that the projections $s_{g,j} s_{g,j}^*$ are mutually orthogonal.

The action $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ is given by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$.

The absorption theorem for the action on $\mathcal{O}_\infty$

This action does what we want.

Theorem

Let G be a finite group, let A be a unital Kirchberg algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a pointwise outer action of G on A. Equip $\mathcal{O}_\infty$ with the action ι above. Then there is an equivariant isomorphism $\psi \colon A \to \mathcal{O}_\infty \otimes A$ which is equivariantly approximately unitarily equivalent to the map $\varphi(a) = 1 \otimes a$.

(For equivariant approximate unitary equivalence, we require that the unitaries in the approximate unitary equivalence be G-invariant.)

The proof is long and uses equivariant versions of ideas from a number of papers. We need to apply the methods used for the nonequivariant version of this absorption theorem. In particular, we need to show that the two maps $\mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ given by $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$ are equivariantly approximately unitarily equivalent. ("The half flip is approximately inner.")

The absorption theorem for the action on $\mathcal{O}_\infty$ (continued)

We need to know that the maps $\mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ given by

$$a \mapsto a \otimes 1 \quad \text{and} \quad a \mapsto 1 \otimes a$$

are equivariantly approximately unitarily equivalent.

There is no difficulty with getting this for $\mathcal{O}_2$. It is known without the group. Also, if actions $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ are given, one of which has the Rokhlin property, and two equivariant unital homomorphisms $\varphi, \psi \colon A \to B$ are approximately unitarily equivalent, then φ and ψ are in fact equivariantly approximately unitarily equivalent.

If $A = \mathcal{O}_\infty$ and B is a unital Kirchberg algebra with a pointwise outer action, then any two equivariant unital homomorphisms from A to B are the same K-theoretically. (Proving this requires calculating $K_*^G(\mathcal{O}_\infty)$.) They are therefore expected to be equivariantly approximately unitarily equivalent.

The absorption theorem for the action on $\mathcal{O}_\infty$ (continued)

Suppose two equivariant unital homomorphisms $\varphi, \psi \colon \mathcal{O}_\infty \to B$ are given, and B is a Kirchberg algebra with a pointwise outer action of G. We want to show that φ and ψ are equivariantly approximately unitarily equivalent.

Recall that the action: $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ is given by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$. Further recall the action $\iota^{(m)} \colon G \to \mathrm{Aut}(\mathcal{O}_{md})$ by $\iota_g^{(m)}(s_{h,j}) = s_{gh,j}$ for $g, h \in G$ and $j = 1, 2, \ldots, m$. (Caution: We have used the same names for generators of different algebras.)

Adapting some trickery from the case without the group, one can show that it suffices to consider the case in which $[1_B] = 0$ in $K_0^G(B)$.

In this case, one can show that there are equivariant unital homomorphisms $\sigma_m, \tau_m \colon \mathcal{O}_{(m+1)d} \to B$ which "agree" with φ and ψ on the generators $s_{g,j}$ of the same name, for $g \in G$ and $j = 1, 2, \ldots, m$ (not $m + 1$!), and such that σ_m and τ_m have the same class in $KK^0(\mathcal{O}_{(m+1)d}, B)$ (note: ordinary KK-theory, ignoring the group).

The absorption theorem for the action on $\mathcal{O}_\infty$ (continued)

Equivariant unital homomorphisms $\varphi, \psi \colon \mathcal{O}_\infty \to B$ are given, and $[1_B] = 0$ in $K_0^G(B)$. We want to show that φ and ψ are equivariantly approximately unitarily equivalent.

For each m there are equivariant unital homomorphisms $\sigma_m, \tau_m \colon \mathcal{O}_{(m+1)d} \to B$ which "agree" with φ and ψ on the generators $s_{g,j}$ of the same name, but only through $j = m$, not for $j = m + 1$. Moreover, σ_m and τ_m have the same class in nonequivariant KK-theory.

Nonequivariant classification (or, rather, a result used in its proof) implies that σ_m and τ_m are approximately unitarily equivalent (disregarding the action of G). Since the action of G on $\mathcal{O}_{(m+1)d}$ has the Rokhlin property, it follows that they are equivariantly approximately unitarily equivalent.

We conclude: For every m and every $\varepsilon > 0$, there is a G-invariant unitary $u \in B$ such that for $g \in G$ and $j = 1, 2, \ldots, m$ we have $\| u \sigma_m(s_{g,j}) u^* - \tau_m(s_{g,j}) \| < \varepsilon$. That is, $\| u \varphi(s_{g,j}) u^* - \psi(s_{g,j}) \| < \varepsilon$. Since $\varepsilon > 0$ and $m \in \mathbb{Z}_{>0}$ are arbitrary, this says that φ is equivariantly approximately unitarily equivalent to ψ.