Extra slides for the talk "Towards the classification of outer actions of finite groups on purely infinite algebras"

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The definition of a C*-algebra

A C*-algebra A is, first, a complex Banach algebra (a complete normed algebra over $\mathbb{C}$, in which $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$). Second, it has, in addition, an adjoint operation, written $a \mapsto a^*$, such that:

- $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ for all $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$ (conjugate linearity).
- $(ab)^* = b^* a^*$ for all $a, b \in A$ (reverses multiplication).
- $(a^*)^* = a$ for all $a \in A$ (involutive).
- $\|a^*\| = \|a\|$ for all $a \in A$.

So far, what we have defined is a Banach *-algebra. Third, the C* relation must be satisfied:

- $\|a^* a\| = \|a\|^2$ for all $a \in A$.

The C* relation looks innocuous but is actually very powerful.

The basic examples of C*-algebras

Let X be a compact Hausdorff space. Let $C(X)$ be the set of all continuous functions from X to $\mathbb{C}$, with the usual (pointwise) vector space operations, pointwise multiplication, $\|f\| = \sup_{x \in X} |f(x)|$, and $f^*(x) = \overline{f(x)}$. Then $C(X)$ is a commutative unital C*-algebra.

Every commutative unital C*-algebra is isomorphic to one of these.

The algebra $M_n = M_n(\mathbb{C})$ for complex $n \times n$ matrices over $\mathbb{C}$ is a C*-algebra, with the usual algebra operations, adjoint being conjugate transpose, and the operator norm

$$\|a\| = \sup\left(\left\{\|a\xi\|_2 \colon \xi \in \mathbb{C}^n \text{ with } \|\xi\|_2 \leq 1\right\}\right).$$

Here $\|\xi\|_2$ is the usual Euclidean norm on $\mathbb{C}^n$.

The adjoint is determined by the equation $\langle a^*\xi, \eta\rangle = \langle \xi, a\eta\rangle$ for all $\xi, \eta \in \mathbb{C}^n$. Also, it turns out that $\|a\|$ is the square root of the largest eigenvalue of a^*a.

The basic examples of C*-algebras (continued)

Let H be a complex Hilbert space. Let $L(H)$ be the set of all bounded linear maps $a \colon H \to H$. "Bounded" means that the operator norm

$$\|a\| = \sup\left(\left\{\|a\xi\| \colon \xi \in H \text{ with } \|\xi\| \leq 1\right\}\right)$$

is finite; equivalently, a is continuous. Then A is a complex algebra in the obvious way, and with this norm it is a Banach algebra. The adjoint operation is determined by $\langle a^*\xi, \eta\rangle = \langle \xi, \eta a\rangle$ for all $\xi, \eta \in H$ (just as for matrices). With this adjoint operation, $L(H)$ is a C*-algebra.

Every C*-algebra is isomorphic to a *-closed ("selfadjoint"), norm closed subalgebra of one of these.

The definition of the Leavitt path algebra of a graph

We consider a countable directed graph $E = (E^0, E^1, r, s)$, in which:

- E^0 is the set of vertices.
- E_1 is the set of (oriented) edges.
- $r\colon E^1 \to E^0$ is the map which assigns to each edge the vertex at which it begins.
- $s\colon E^1 \to E^0$ is the map which assigns to each edge the vertex at which it begins.

Parallel edges are allowed, and edges which start and end at the same vertex are allowed.
Then for any field K (probably any commutative unital ring will do), $L_K(E)$ is the K-algebra (not necessarily unital) on generators p_v for each $v \in E^0$ and y_e and y_e^* for each $e \in E^1$ (with y_e and y_e^* simply taken to be distinct symbols) subject to the relations on the next slide.

Relations for the Leavitt path algebra of a graph

$E = (E^0, E^1, r, s)$, with vertices E^0, edges E^1, range map r, and source map s.

$L_K(E)$ is the K-algebra (not necessarily unital) on generators p_v for each $v \in E^0$ and y_e and y_e^* for each $e \in E^1$ (with y_e and y_e^* simply taken to be distinct symbols) subject to the relations:

- $p_{s(e)} y_e = y_e p_{r(e)} = y_e$ for all edges e.
- $p_{r(e)} y_e^* = y_e^* p_{s(e)} = y_e^*$ for all edges e.
- For edges e and f, we have $y_e^* y_f = 0$ if $e \neq f$ and $y_e^* y_e = p_{r(e)}$.
- Foe every vertex v for which the set $s^{-1}(v)$ of edges which start at v is finite and nonempty, we have $p_v = \sum_{e \in s^{-1}(v)} y_e y_e^*$.

The C*-algebra $C^*(E)$ is gotten to taking $K = \mathbb{C}$, specifying that y_e^* is the adjoint of y_e (rather than just some other symbol), and forming the universal C*-algebra on the given relations.

Notice that y_e is a partial isometry from $p_{s(e)}$ to some subprojection of $p_{r(e)}$.

Examples of graph algebras

Example

For $d = 2, 3, 4, \ldots$, the Cuntz algebra $\mathcal{O}_d$ is the C*-algebra of the graph with one vertex and d edges.

Example

The graph with two vertices and an edge going from one to the other gives $M_2(K)$.

Example

The C*-algebra of the graph with one vertex and one edge is $C(S^1)$, the algebra of continuous complex valued functions on the circle S^1. The Leavitt path algebra over K is the Laurent polynomial algebra $K[x, x^{-1}]$.

The Rokhlin property and vanishing of cohomology

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. An *α-cocycle* is a function $g \mapsto u_g$ from G to the unitary group of A such that $u_{gh} = u_g \alpha_g(u_h)$ for all $g, h \in G$.

The cocycle equation is the obvious sufficient condition for the formula $\beta_g(a) = u_g \alpha_g(a) u_g^*$ to define an action of G on A.

Definition

Let the notation be as in the previous definition. Then u is an *α-coboundary* if there exists a unitary $v \in A$ such that $u_g = v \alpha_g(v^*)$ for all $g \in G$.

The Rokhlin property and vanishing of cohomology (continued)

Coboundary: There is a unitary v such that $u_g = v\alpha_g(v^*)$ for all $g \in G$.

The coboundary equation is the obvious sufficient condition for the action β above to be conjugate to α. Indeed, it implies that $\beta_g = \mathrm{Ad}(v) \circ \alpha_g \circ \mathrm{Ad}(v)^{-1}$ for all $g \in G$.

Proposition (Herman and Jones)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be a Rokhlin action of a finite group G on a unital C*-algebra A. Then every α-cocycle is an α-coboundary.

Izumi's classification of Rokhlin actions

The following is Theorem 4.2 of M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. II*, Adv. Math. **184**(2004), 119–160.

Theorem

Let A be a unital UCT Kirchberg algebra, and let G be a finite group. Let $\alpha, \beta \colon G \to \mathrm{Aut}(A)$ be Rokhlin actions. Then α is conjugate to β if and only if the actions of G they induce on $K_*(A)$ are equal.

Interpreted as a theorem about conjugacy of dynamical systems, the invariant involved includes A, equivalently, it includes $K_*(A)$ and $[1_A] \in K_0(A)$.

There are severe restrictions on the possible actions of G on $K_*(A)$.

The same result holds if "Kirchberg algebra" is replaced by "C*-algebra with tracial rank zero (in the sense of Lin)".

Izumi's classification of actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$

The following is essentially a restatement of part of Theorem 4.8 of M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. I*, Duke Math. J. **122**(2004), 233–280.

Theorem

Let $\alpha, \beta \colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$ be actions which are pointwise outer but strongly approximately inner. Then α is conjugate to β if and only if $K_*^{G,\alpha}(\mathcal{O}_2) \cong K_*^{G,\beta}(\mathcal{O}_2)$ via an isomorphism which sends [1] to [1].

$K_*(\mathcal{O}_2)$ isn't needed in the invariant, since it is zero.

An action $\alpha \colon G \to \mathrm{Aut}(A)$ of a finite abelian group G on a unital C*-algebra A is *trongly approximately inner* if for all $g \in G$, the automorphism α_g is the pointwise norm limit of inner automorphisms $\mathrm{Ad}(u_n)$ using α-invariant unitaries u_n.

$K_*^{G,\alpha}(A)$ (usually written $K_*^G(A)$) is the equivariant K-theory of A (with respect to the group action α).