Hill's Lemma and Infinite Jordan–Hölder Theory

New Trends in Noncommutative Algebra

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Ken Goodearl 65 Conference

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University of Washington, Seattle

Jan Trlifaj
Univerzita Karlova, Praha

The Jordan–Hölder Theorem

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Follows by the Schreier–Zassenhaus Lemma (or "Butterfly Lemma")

Lemma

Let M be a module. Any two finite chains of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M \text{ and } 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = M$$

have equivalent refinements.

Transfinite extensions

Definition

Let R be a ring, $\mathcal{C}$ a class of modules, and M a module. A chain of submodules of M, $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$, is a $\mathcal{C}$–filtration of M of length σ provided that

- $M_\alpha \subseteq M_{\alpha+1}$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha < \sigma$,
- $M_0 = 0$, $M_\sigma = M$, and
- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$.

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- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$.

A module M possesing a $\mathcal{C}$–filtration is called $\mathcal{C}$–filtered, or a transfinite extension of the modules in $\mathcal{C}$.

If $\sigma < \omega$, then M is called finitely $\mathcal{C}$–filtered.

Examples

If $\mathcal{C} = \mathrm{simp}\text{-}R$, then $\mathcal{C}$–filtered = semiartinian,
and finitely $\mathcal{C}$–filtered = of finite length.

If $\mathcal{C} = \{R\}$, then $\mathcal{C}$–filtered = free.

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If $\mathcal{C} = \{R\}$, then $\mathcal{C}$–filtered = free.

$\mathcal{C}$–filtered modules include all

- extensions of modules in $\mathcal{C}$,
- direct sums of modules in $\mathcal{C}$.

Notation: $M \in \mathrm{Filt}(\mathcal{C})$.

$\mathcal{C}$ is filtration closed provided that $\mathcal{C} = \mathrm{Filt}(\mathcal{C})$.

Roots of Ext

Definition

Let R be a ring and $\mathcal{C}$ a class of modules.
Then $\mathcal{C}$ is a class of roots of Ext provided $\mathcal{C}$ has the form $\mathcal{C} = {}^{\perp}\mathcal{B}$ for a class of modules $\mathcal{B}$, where

$$ {}^{\perp}\mathcal{B} = \mathrm{KerExt}^1_R(-, \mathcal{B}) = \{M \mid \mathrm{Ext}^1_R(M, B) = 0 \text{ for all } B \in \mathcal{B}\}. $$

Similarly, we define ${}^{\perp_\infty}\mathcal{B} = \bigcap_{i \geq 1} \mathrm{KerExt}^i_R(-, \mathcal{B})$.

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Eklof Lemma

If $\mathcal{C}$ is a class of roots of Ext, then $\mathcal{C} = \mathrm{Filt}(\mathcal{C})$.

Example

Let R be a ring. Then $\mathcal{P}_n$ and $\mathcal{F}_n$ for all $n < \omega$, are classes of roots of Ext.
Let R be an Iwanaga–Gorenstein ring. Then $\mathcal{GP}$ and $\mathcal{GF}$ are classes of roots of Ext.

Deconstructible classes

Let κ be a cardinal and $\mathcal{A}$ a class of modules.
We denote by $\mathcal{A}^{<\kappa}$ the class of all $<\kappa$–presented modules in $\mathcal{A}$.

Definition (Eklof'2006)

Let R be a ring and $\mathcal{A}$ a class of modules.

- Let κ be a cardinal. Then $\mathcal{A}$ is a κ–deconstructible provided that $\mathcal{A} \subseteq \mathit{Filt}(\mathcal{A}^{<\kappa})$.
- $\mathcal{A}$ is deconstructible provided $\mathcal{A}$ is κ–deconstructible for some infinite cardinal κ.

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Example

- Let R be a ring. Then the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ for all $n < \omega$, are deconstructible.
- Let R be an Iwanaga–Gorenstein ring. Then the classes $\mathcal{GP}$ and $\mathcal{GF}$ are deconstructible.

Deconstructible classes of roots of Ext

Lemma

Let $\mathcal{A}$ be a class of modules. Then the following are equivalent:

1. $\mathcal{A} = \mathit{Filt}(\mathcal{S})$ *for a set of modules* $\mathcal{S}$.
2. $\mathcal{A}$ *is a deconstructible class closed under transfinite extensions.*

Lemma

Let $\mathcal{A}$ be a deconstructible class of modules. Then the following are equivalent:

1. $\mathcal{A}$ *is a class of roots of Ext.*
2. $\mathcal{P}_0 \subseteq \mathcal{A}$, *and* $\mathcal{A}$ *is closed under direct summands and transfinite extensions.*

The latter implication (1) $\implies$ (2) holds for any class $\mathcal{A}$ by the Eklof Lemma, but the reverse one fails in general.

Expanding a single $\mathcal{C}$–filtration into a large family

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Expanding a single $\mathcal{C}$–filtration into a large family

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Then there exists a family $\mathcal{H}$ consisting of submodules of M such that

- (H1) $\mathcal{M} \subseteq \mathcal{H}$.
- (H2) $\mathcal{H}$ is closed under arbitrary sums and intersections.
- (H3) P/N has a $\mathcal{C}$–filtration, for all $N \subseteq P$ in $\mathcal{H}$.
- (H4) If $N \in \mathcal{H}$ and S is a subset of M of cardinality $< \kappa$, then there is $P \in \mathcal{H}$ such that $N \cup S \subseteq P$ and P/N is $< \kappa$–presented.

Idea of the proof (Hill'81)

For each $\alpha < \sigma$, take an *arbitrary* $< \kappa$–generated submodule A_α of $M_{\alpha+1}$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$.

Notice that $M_\alpha = \sum_{\alpha<\sigma} A_\alpha$.

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A subset $S \subseteq \sigma$ is called closed in case each $\alpha \in S$ satisfies

$$M_\alpha \cap A_\alpha \subseteq \sum_{\beta<\alpha, \beta\in S} M_\beta.$$

Define $\mathcal{H} = \{\sum_{\alpha\in S} A_\alpha \mid S \text{ closed } \}$.

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Verification of the H–conditions:

All ordinals $\alpha \leq \sigma$ are closed, so $\mathcal{M} \subseteq \mathcal{H}$ and (H1) holds.

Unions and intersections of closed subsets are closed, and (H2) holds.

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The proof gives more: $\mathcal{H}$ forms a complete distributive sublattice of the complete modular lattice of all submodules of M.

Applications of the Hill Lemma

Notation:

Let R be a ring, M a module, κ a regular infinite cardinal, and

$\mathcal{C}$ a class of $< \kappa$–presented modules.

Let $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a $\mathcal{C}$–filtration of M.

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Replacing a $\mathcal{C}$–filtration by a more convenient one

Asume that $\mathcal{C} = \mathcal{A}^{<\kappa}$ for some class, $\mathcal{A}$, of roots of Ext, and $\mathrm{gen}(M) = \lambda \geq \kappa$. Let $\{m_\gamma \mid \gamma < \lambda\}$ be a set of R-generators of M.

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Then M has a $\mathcal{C}$–filtration $\mathcal{M}' = (M'_\beta \mid \beta \leq \lambda)$ such that $\sum_{\gamma<\beta} m_\gamma R \subseteq M'_\beta$ for all $\beta < \lambda$.

Applications of the Hill Lemma II

$\mathcal{C}$–socle length constraints (Enochs'10, Šťovíček'10)

Let Sum($\mathcal{C}$) denote the class of all direct sums of copies of the modules from $\mathcal{C}$. Then there exists a Sum($\mathcal{C}$)–filtration $\mathcal{N}$ of M of length $\leq \kappa$.

Applications of the Hill Lemma II

$\mathcal{C}$–socle length constraints (Enochs'10, Šťovíček'10)

Let Sum($\mathcal{C}$) denote the class of all direct sums of copies of the modules from $\mathcal{C}$. Then there exists a Sum($\mathcal{C}$)–filtration $\mathcal{N}$ of M of length $\leq \kappa$.

Sketch of proof:

Again, let A_α be a $< \kappa$–generated module such that $M_{\alpha+1} = M_\alpha + A_\alpha$.

Let S_α be closed and such that $S_\alpha \subseteq \alpha + 1$, card$(S_\alpha) < \kappa$ and $\alpha \in S_\alpha$.

By induction, we define a "socle level" function $f : \sigma \to \kappa$ by

- $f(\alpha) = 0$ provided that $S_\alpha = \{\alpha\}$, and by
- $f(\alpha) = \sup_{\beta \in S_\alpha, \beta \neq \alpha} f(\beta) + 1$ otherwise.

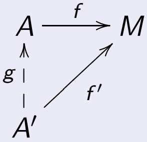
For each $\gamma \leq \kappa$, let $T_\gamma = \{\alpha < \sigma \mid f(\alpha) < \gamma\}$. Then T_γ is closed.

The desired filtration is $\mathcal{N} = (N_\gamma \mid \gamma \leq \kappa)$ where $N_\gamma = \sum_{\beta \in T_\gamma} A_\beta$.

Precovers and covers of modules

Definition

- A class of modules $\mathcal{A}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ has a factorization through f:



  The map f is then called an $\mathcal{A}$–precover of M.

- Let $\mathcal{A}$ be precovering. Assume that for $f = f'$, each factorization g is an automorphism. Then $\mathcal{A}$ is called a covering class.

Precovers and filtrations

Theorem (Enochs'10, Šťovíček'10)

Let $\mathcal{S}$ be a set of modules and $\mathcal{A} = \mathsf{Filt}(\mathcal{S})$. Then $\mathcal{A}$ is precovering. If $\mathcal{A}$ is closed under direct limits, then $\mathcal{A}$ is covering.

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Corollary

- *Each deconstructible class closed under transfinite extensions is precovering.*
- *The classes $\mathcal{P}_n$, and $\mathcal{GP}$ for R Iwanaga–Gorenstein, are precovering.*
- *The classes $\mathcal{F}_n$, and $\mathcal{GF}$ for R Iwanaga–Gorenstein, are covering.*

Applications of the Hill Lemma III

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Shelah's Singular Compactness

Let λ be a singular cardinal $> \kappa$, and M be a module with $\text{gen}(M) = \lambda$. Assume that for each regular cardinal τ with $\kappa < \tau < \lambda$, M is "τ–free"

Applications of the Hill Lemma III

Shelah's Singular Compactness

Let λ be a singular cardinal $> \kappa$, and M be a module with $\text{gen}(M) = \lambda$. Assume that for each regular cardinal τ with $\kappa < \tau < \lambda$, M is "τ–free" that is, there is a set $\mathcal{S}_\tau$ consisting of "free" submodules of M such that

- $\text{gen}(N) < \tau$ for all $N \in \mathcal{S}_\tau$.
- Each subset of M of cardinality $< \tau$ is contained in an element of $\mathcal{S}_\tau$.
- $\mathcal{S}_\tau$ is closed under unions of well–ordered chains of length $< \tau$.

Then M is "free."

Theorem (Shelah'81, Eklof–Mekler'02)

Shelah's Singular Compactness holds when

- *"free" $=$ free,*
- *"free" $=\mathcal{C}$–filtered, where $\mathcal{C}$ is any class of $<\kappa$-presented modules.*

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Shelah's Singular Compactness holds when

- *"free" = free,*
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Often, locally "free" implies "free" for $\text{gen}(M) = \lambda$ singular,

but for λ regular, one needs additional set–theoretic assumptions,

or a more particular algebraic setting.

$\aleph_1$–deconstructibility of the roots of Ext

Theorem

[Eklof–Fuchs–Shelah'90, Šťovíček–T.'07, Angeleri–Šaroch–T.'07, Šaroch–Šťovíček'08]

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2. *$\mathcal{B}$ consists of modules of finite injective dimension, or*

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Then $\mathcal{C}$ is $\aleph_1$–deconstructible whenever either

1. *all modules in $\mathcal{C}$ have finite projective dimension, or*
2. *$\mathcal{B}$ consists of modules of finite injective dimension, or*
3. *$\mathcal{B}$ is closed under products and unions of well–ordered chains, and contains all injective modules.*

Definition (Raynaud–Gruson'71)

$\mathcal{L}$ denotes the class of all Mittag–Leffler modules, i.e., the modules M such that the canonical map

$$M \otimes_R \prod_{i \in I} M_i \to \prod_{i \in I} (M \otimes_R M_i)$$

$$m \otimes_R (m_i)_{i \in I} \mapsto (m \otimes_R m_i)_{i \in I}$$

is monic for each family of left R–modules $(M_i \mid i \in I)$.

$\mathcal{D} = \mathcal{F} \cap \mathcal{L}$ the class of all flat Mittag-Leffler modules.

$\mathcal{P}_0 \subseteq \mathcal{D}$, and $\mathcal{D}$ is closed under direct summands and transfinite extensions.
So $\mathcal{D}$ looks like a deconstructible class of roots of Ext ...

(Goodearl'72) $\mathcal{D}$ = the class of all the modules all of whose finitely generated submodules are projective, in case R is a von Neumann regular ring.

(Azumaya–Facchini'89) $\mathcal{D}$ = the class of all groups all of whose countably generated subgroups are free, in case $R = \mathbb{Z}$.

$\aleph_1$–projective modules

Definition (Shelah'81, Eklof–Mekler'02)

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A module M is $\aleph_1$-projective if M "τ–free" where $\tau = \aleph_1$ and "free" $=$ projective.

That is, there is a set $\mathcal{S}$ consisting of submodules of M such that

- (A1) Each element of $\mathcal{S}$ is a countably generated projective module.
- (A2) Each countable subset of M is contained in an element of $\mathcal{S}$.
- (A3) $\mathcal{S}$ is closed under unions of well–ordered chains of countable length.

Theorem (Herbera–T.'09)

Let R be a ring.

- *A module M is flat Mittag–Leffler, if and only if M is $\aleph_1$–projective.*
- *The class $\mathcal{D}$ is deconstructible, if and only if R is a right perfect ring.*

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Theorem (Bazzoni–Šťovíček'10, Šaroch–T.'10)

Let R be a countable non–right perfect ring.

Then $\mathcal{D}$ is not a class of roots of Ext, and $\mathcal{D}$ is not precovering.

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Open problem

Let R be a von Neumann regular non–artinian right self–injective ring.
Is the class $\mathcal{D}$ ($=$ all non-singular modules) precovering?