Deformations of Skew Group Algebras and Orbifolds

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Skew Group Algebras $R\#G$

R - ring

G - group acting by automorphisms on R,

$$r \mapsto g \cdot r \quad (r \in R,\, g \in G)$$

$R\#G :=$ free R-module with basis G, i.e. $\bigoplus_{g \in G} Rg$,

with multiplication $(rg)(sh) = r(g \cdot s)gh$

Example of Skew Group Algebra

V - repn. of G (fin. dim. as a vector space)
$S(V)$ - symmetric algebra on V
(i.e. polynomials in a basis $v_1, \ldots, v_n$)

Then G acts by automorphisms on $S(V)$;
$S(V)\#G$ is the resulting skew group algebra

Think: $S(V)\#G$ replaces the ring of functions
$S(V)^G$ for the orbifold V/G

Deformations

A - algebra over a field k
$A[[t]] := k[[t]] \otimes_k A$ (just extend scalars to $k[[t]]$)

A formal deformation of A is an (associative) algebra structure on $A[[t]]$ with multiplication

$$a * b = ab + \mu_1(a,b)t + \mu_2(a,b)t^2 + \cdots$$

for some bilinear maps $\mu_i : A \times A \to A$, where ab is the product of a and b in A

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Remarks

(1) $*$ is associative $\implies$ μ_1 is a Hochschild 2-cocycle,
i.e. $a\mu_1(b,c) + \mu_1(a,bc) = \mu_1(a,b)c + \mu_1(ab,c)$

(2) It can be hard to "lift" a μ_1 to a formal deformation

(3) Sometimes one may specialize to $t = t_0 \in k$

Example: Drinfeld's graded Hecke algebras

V - repn. of G as before
Choose $\kappa : V \times V \to \mathbb{C}G$, a skew-symmetric bilin. form

Let $H := T(V)\#G/(vw - wv - \kappa(v,w) \mid v, w \in V\ )$

Let $\deg v = 1,\ \deg g = 0$ for all $v \in V$, $g \in G$, so that H is a filtered algebra

Defn (Drinfeld '86) H is a graded Hecke algebra if its associated graded algebra is $S(V)\#G$

Remark
In this case H arises from a formal deformation of $S(V)\#G$ by specializing to $t = 1$, and

$$\kappa \longleftrightarrow \text{Hochschild 2-cocycle } \mu_1$$

Drinfeld (graded) Hecke algebras for reflection groups

- Lusztig '88: for G a real reflection group, gave different definition in terms of roots, related to affine Hecke algebra

- Cherednik '95: used to prove Macdonald's constant term conjecture for root systems

- Etingof-Ginzburg '02: for G a symplectic reflection group, related to orbifold V/G

- Gordon '03: used to prove Haiman's version of $n!$ conjecture for Weyl groups

In these contexts, graded Hecke algebras have gone by the alternate names: degenerate affine Hecke algebras, rational Cherednik algebras, symplectic reflection algebras

We wish to view the graded Hecke algebras,

$$H = T(V)\#G/\big(vw - wv - \kappa(v,w) \mid v, w \in V\big)$$

(where $\kappa : V \times V \to \mathbb{C}G$),
as special types of deformations of $S(V)\#G$
coming from a particular subspace of $\mathrm{HH}^2(S(V)\#G)$

Structure of Hochschild cohomology of $S(V)\#G$

$$\begin{aligned}
\mathrm{HH}^\bullet(S(V)\#G) &\cong \mathrm{HH}^\bullet(S(V), S(V)\#G)^G \\
&\cong \left(\bigoplus_{g\in G} \mathrm{HH}^\bullet\left(S(V), S(V)g\right)\right)^G \\
&\subset \left(\bigoplus_{g\in G} S(V)g \otimes \bigwedge\nolimits^\bullet(V^*)\right)^G
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Remark

Graded Hecke algebras correspond to Hochschild 2-cocycles of $S(V)\#G$ living in

$$\mathbb{C}G \otimes \textstyle\bigwedge^2(V^*) \subset (S(V)\#G) \otimes \textstyle\bigwedge^2(V^*),$$

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Question

What about the rest of $\mathrm{HH}^2(S(V)\#G)$?
Are there corresponding deformations?

More details (Ginzburg-Kaledin '04, Farinati '05)

$$\mathrm{HH}^i(S(V)\#G)$$
$$\cong \left(\bigoplus_{g\in G} S(V)g \otimes \bigwedge^{\operatorname{codim} V^g}(((V^g)^{\perp})^*) \otimes \bigwedge^{i-\operatorname{codim} V^g}((V^g)^*)\right)^G$$
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Known deformations
(1) Polynomial part of degree ≤ 1: Deformations are filtered algebras with assoc. graded alg. iso. to $S(V)\#G$; Halbout, Oudom, Tang gave conditions for existence
Includes graded Hecke algebras as special case

(2) Polynomial part of degree ≥ 2:
Case $G = 1$. Completely known (Kontsevich '03)
Case $G \neq 1$. Isolated examples known arising from Hopf algebra actions (W. '06, Guccione-Guccione-Valqui)

Noncommutative Poisson structures

Again: A Hochschild 2-cocycle on A is a bilinear map $\mu_1 : A \times A \to A$ for which

$$a\mu_1(b,c) + \mu_1(a,bc) = \mu_1(a,b)c + \mu_1(ab,c)$$

Remark

If μ_1 is the Hochschild 2-cocycle arising from a deformation of A, then $[\mu_1, \mu_1]$ is a coboundary (i.e. it becomes 0 in cohomology) where

$$\begin{aligned}[\nu,\eta](a,b,c) := {} & \nu(\eta(a,b),c) - \nu(a,\eta(b,c)) \\ & + \eta(\nu(a,b),c) - \eta(a,\nu(b,c))\end{aligned}$$

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A noncommutative Poisson structure on A is a $\mu_1 \in \mathrm{HH}^2(A)$ with $[\mu_1, \mu_1] = 0$ as an element of $\mathrm{HH}^3(A)$

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Problem

For an algebra A, find all noncommutative Poisson structures, as a first step towards finding all deformations

Again:

Structure of Hochschild cohomology of $S(V)\#G$

$$\mathrm{HH}^\bullet(S(V)\#G) \subset \left(\bigoplus_{g\in G} S(V)g \otimes \bigwedge\nolimits^\bullet(V^*) \right)^G$$

Cohomology is graded by polynomial degree: This is just the degree of the factor in $S(V)$.

We say a cocycle is constant if it has polynomial deg 0.

Thm $\alpha, \beta \in \mathrm{HH}^\bullet(S(V)\#G)$ constant $\implies [\alpha, \beta] = 0$

Cor $\alpha \in \mathrm{HH}^2(S(V)\#G)$ constant $\implies \alpha$ defines a noncommutative Poisson structure

Thm Each constant $\mu_1 \in \mathrm{HH}^2(S(V)\#G)$ defines a graded Hecke algebra, and vice versa.

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$$\mathrm{HH}^\bullet(S(V)\#G) \subset \left(\bigoplus_{g \in G} S(V) g \otimes \bigwedge\nolimits^\bullet (V^*) \right)^G$$

Cohomology is "graded" by G, i.e. each cocycle is supported on some group elements

The <u>kernel</u> of G on V is $\{g \in G \mid V^g = V\}$.

<u>Thm</u> If $\alpha, \beta \in \mathrm{HH}^2(S(V)\#G)$ are supported <u>off</u> the kernel of G on V, then $[\alpha, \beta] = 0$.

<u>Cor</u> If $G \subseteq \mathrm{GL}(V)$ and 1_G doesn't contribute to $\alpha, \beta \in \mathrm{HH}^2(S(V)\#G)$, then $[\alpha, \beta] = 0$.

<u>Cor</u> Any 2-cocycle supported <u>off</u> the kernel of G on V defines a noncommutative Poisson structure on $S(V)\#G$.

Proofs require formulas for the bracket $[\ ,\ ]$

Problem: Bracket is defined via bar resolution, cohomology is computed via Koszul resolution.

Need chain maps converting between the two.

Such chain maps lead to formulas involving Demazure operators/quantum differentiation; e.g.

$$\partial_{v_1,q}(v_1^{k_1}v_2^{k_2}\cdots v_n^{k_n}) = [k_1]_q v_1^{k_1-1}v_2^{k_2}\cdots v_n^{k_n}$$

where $[k_1]_q = 1 + q + q^2 + \cdots + q^{k_1-1}$
(q is an eigenvalue of a group element acting on v_1)

Chain map

$$(S(V)\# G) \otimes \bigwedge^m (V^*) \xrightarrow{\Upsilon} \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes m}, S(V)\# G)$$

For $\alpha = fg \otimes v_{j_1}^* \wedge \cdots \wedge v_{j_m}^*$,

$$\Upsilon(\alpha)(f_1 \otimes \cdots \otimes f_m) = \left(\prod_{i=1}^{m} (s_1 s_2 \cdots s_{j_i - 1}) \cdot (\partial_{j_i} f_i) \right) fg \ .$$

where $g = s_1 \cdots s_n$ is a product of reflections s_i, specifically

$s_i \cdot v_j = q_i^{\delta_{i,j}} v_j$ for a scalar q_i, $\partial_j := \partial_{v_j, q_j}$

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This map is used in computations of brackets $[\ ,\ ]$

Example: Abelian groups

Diagonalize action: $g \cdot v_i = \chi_i(g) v_i \ \ (g \in G;\, i = 1, \ldots, n)$

$$\text{Let}\ \ \alpha = v_1^{k_1} v_2^{k_2} v_3^{k_3} g \otimes v_1^* \wedge v_2^*,$$
$$\beta = v_1^{l_1} v_2^{l_2} v_3^{l_3} h \otimes v_2^* \wedge v_3^*.$$

Then

$$[\alpha, \beta] = c\ \overline{v_1^{k_1+l_1} v_2^{k_2+l_2-1} v_3^{k_3+l_3} gh \otimes v_1^* \wedge v_2^* \wedge v_3^*},$$

where

$$c = \langle \chi_1^{k_1-1} \chi_2^{k_2-1}, \chi_3^{-k_3} \rangle \langle \chi_1^{l_1} \chi_2^{1-l_2}, \chi_3^{1-l_3} \rangle \cdot$$
$$([k_2]_{\chi_2(h)} \chi_1(h)^{k_1} - [l_2]_{\chi_2(g)} \chi_1(g)^{l_1})$$

Here, $\langle\ ,\ \rangle$ denotes the inner product of characters on G

Orthogonality relations $\implies$ $[\alpha, \beta]$ is usually 0,
but can be nonzero (when g or h acts trivially on V)

Question

Of these noncommutative Poisson structures on $S(V)\#G$, which lift to deformations?

Future directions

(1) Quantum symmetric algebra $S_{\mathbf{q}}(V)$:
generators $v_1, \ldots, v_n$; relations $v_i v_j = q_{ij} v_j v_i$

- Bazlov, Berenstein '09: deformations of $S_{\mathbf{q}}(V)\#G$, called braided Cherednik algebras
- Kirkman, Kuzmanovich, Zhang '10: Shephard-Todd-Chevalley Theorem for G, $S_{\mathbf{q}}(V)$
- Naidu, Shroff, W.: $\mathrm{HH}^{\bullet}(S_{\mathbf{q}}(V)\#G)$, diagonal case

(2) Positive characteristic p dividing $|G|$:

Cohomology is more complicated, specifically there is a spectral sequence converging to $\mathrm{HH}^{\bullet}(S(V)\#G)$ with

$$E_2^{i,j} = \mathrm{H}^i(G, \mathrm{HH}^j(S(V), S(V)\#G))$$