Hopf Actions on Calabi-Yau Algebras

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Notation

Notation

- k is a field of characteristic 0;
- algebras will be k-algebras
- graded algebras will mean connected graded algebras generated in degree 1;
- modules are left (graded) modules;
- $A^e := A \otimes A^o$;
- $(\ )^* := \mathrm{Hom}_k(\ \ ,k)$.

Calabi-Yau algebras

Definition (Ginzburg, 2006)

A graded algebra A is called a Calabi-Yau algebra if the following conditions hold.

(1) A is homologically smooth, i.e., A has a finitely generated A^e-projective resolution of finite length;

(2) there exists integers d, l such that

$$\mathrm{Ext}^i_{A^e}(A, A^e) = \begin{cases} A(l) & \text{if } i = d, \\ 0 & \text{if } i \neq d \end{cases} \tag{0.1}$$

as A^e-modules.

Calabi-Yau Categories

Definition (Konsevich, 1998)

A hom-finite triangulated category $\mathcal{C}$ is called n-Calabi-Yau for some integer n if

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \overset{iso}{\cong} \mathrm{Hom}_{\mathcal{C}}(B, A(n))^*$$

for any $A, B \in ob\,\mathcal{C}$, and the "iso" is natural.

Calabi-Yau ⇒ Derived Calabi-Yau

Theorem (Keller, VdB, 2009)

If A is a Calabi-Yau algebra, then $D^b_{f.dim}(A\text{-Mod})$ is a Calabi-Yau category.

AS regular algebras

Definition (Artin-Schelter)

A connected graded algebra A is called left Artin-Schelter(AS, for short) regular (resp. Gorenstein) algebra if the following conditions hold.

(1) A has finite global (resp. ${}_AA$ has finite injective) dimension d;

(2) there exists an integer l such that

$$\mathrm{Ext}^i_A(k, A) = \begin{cases} k(l) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases} \tag{0.2}$$

Twisted Calabi-Yau algebras

Definition

A k-algebra is called a (left) σ-twisted Calabi-Yau algebra of dimension d, for $\sigma \in \mathrm{Aut}(A)$, if $A \in D_{per}(A^e)$ and as A^e-module

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} {}_1A_\sigma, & i = d, \\ 0, & i \neq d. \end{cases}$$

AS regular algebras

"Twisted" graded Calabi-Yau is AS-regular

Any twisted graded Calabi-Yau is AS-regular;
Any p-Koszul AS-regular algebra is "twisted" Calabi-Yau.

Twisted Calabi-Yau algebras

AS-regular Hopf algebra is "twisted" Calabi-Yau (Brown-Zhang)

Let H be a noetherian Hopf algebra with bijective antipode S. If H is AS-regular with global dimension $\mathrm{gldim}(H) = d$, and if $\int_H^l \cong {}_\varepsilon k_\pi$ and ξ is the winding automorphism determined by π, then

$$\mathrm{H}^i(H, H^e) \cong \begin{cases} {}_1H_\sigma, & i = d, \\ 0, & i \neq d, \end{cases}$$

where $\sigma = S^2\xi$.

AS regular algebras

When AS-regular is Calabi-Yau?

Let H be an noetherian Hopf algebra with bijective antipode S. Then H is Calabi-Yau of dimension d if and only if
(i) H is AS-regular with global dimension $\mathrm{gldim}(H) = d$ and $\int_H^l \cong k_H$,
(ii) S^2 is an inner automorphism of H.

Examples of Calabi-Yau algebras-I

(1) $k[x_1, x_2, \cdots, x_n]$: n-Calabi-Yau;

(2) algebras in the classification of 3-dimensional AS-regular algebras of type diag(1, 1) and diag (1, 1, 1);

(3) Sklyanian algebras of dimension 4;

(4) Weyl algebras $A_n(k)$: $2n$-Calabi-Yau (JLMS, 2009);

(5) the preprojective algebras of non-Dynkin quivers;

(6) Sridharan enveloping algebras of abelian Lie algebras.

Examples of Calabi-Yau algebras-II

(7) Let X be a smooth Calabi-Yau variety (i.e., the canonical sheaf is trivial), then $\mathbb{C}[X]$ is a Calabi-Yau algebra of dimension $\dim X$.

(8) Let G be a finite group acting on $X = \mathbb{C}^n$, X/G be the quotient variety, and $Y \xrightarrow{f} X/G$ be the crepant resolution ($f^*\omega_{X/G} = \omega_Y$), then

$$D^b(\mathbb{C}[x_1, x_2, \cdots, x_n]\#G) \cong D^b(CohY).$$

THeorem (Iyama-Reiten, AMJ 2008, Farinati, JA 2005)

$k[x_1, x_2, \cdots, x_n]\#G$ where $G \leq GL(n, k)$ is Calabi-Yau if and only if that $G \leq SL(n, k)$.

Problems-I

$$\{\text{Poly. algebras}\} \subset \{\text{Calabi-Yau algebras}\} \subset \{\text{AS-regular algebras}\}$$

$$G \curvearrowright k[x_1, x_2, \cdots, x_n];$$

$$k[x_1, \cdots, x_n]\#G \text{ is Calabi-Yau} \Leftrightarrow G \subset SL(n, k),$$

$G \curvearrowright$ Calabi-Yau algebra A

$$A\#G \text{ is Calabi-Yau} \Leftrightarrow G \subset ???$$

Problems-II

If H is a (Calabi-Yau) Hopf algebra, and
$H \curvearrowright$ Calabi-Yau algebra A

$$A\#H \text{ is Calabi-Yau} \Leftrightarrow H???$$

Main Results

Theorem (W-Zhu)

Let A be a Koszul Calabi-Yau algebra, G be a finite subgroup of $\mathrm{GrAut}(A)$. Then

$$A\#G \text{ is Calabi-Yau} \Leftrightarrow G \subset SL(A)$$

where $SL(A) := \{\sigma \in \mathrm{GrAut}(A) \mid \mathrm{hdet}(\sigma) = 1\}$.

Theorem (W-Zhu)

Let A be a Koszul Calabi-Yau algebra, H be a Calabi-Yau Hopf algebra, and A be a left graded H-module algebra. Then

$$A\#H \text{ is Calabi-Yau} \Leftrightarrow \text{The homological determinant } \mathrm{hdet}_H \text{ is trivial.}$$

Homological determinant for group actions

Let A be an AS-Gorenstein algebra, and $\sigma \in \mathrm{GrAut}(A)$.

(1) $\mathrm{H}^i_{\mathfrak{m}}(M) = \varinjlim \mathrm{Ext}^i_A(A/A_{\geq}n, M)$ is the i-th local cohomology of M for any graded module M;

(2) $\mathrm{H}^i_{\mathfrak{m}}(A) \cong \begin{cases} 0, & i \neq d, \\ {}_A A^*(l), & i = d; \end{cases}$

(3) there exists a scalar $c \in k^*$ so that the following is commutative.

$$\begin{array}{ccc} \mathrm{H}^d_{\mathfrak{m}}(A) & \xrightarrow{\mathrm{H}^d_{\mathfrak{m}}(\sigma)} & \mathrm{H}^d_{\mathfrak{m}}(A) \\ \cong \downarrow & & \cong \downarrow \\ {}_A A^*(l) & \xrightarrow{c(\sigma^{-1})^*} & {}_A A^*(l). \end{array}$$

Homological determinant for group actions

Definition (Jorgensen & Zhang, Adv. Math. 1998)

$\mathrm{hdet}(\sigma) := c^{-1}$ is called the **homological determinant** of σ.

If $A = k[x_1, x_2, \cdots, x_n]$ and $\sigma \in \mathrm{GrAut}(A)$. Then $\sigma|_V \in GL(n, k)$ where $V = kx_1 \oplus kx_2 \oplus \cdots \oplus kx_n$ and

$$\mathrm{hdet}(\sigma) = \det(\sigma|_V).$$

Remarks

Remarks

(1) $A := k\langle x, y\rangle / \langle xy + yx\rangle$, $\sigma|_{A_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
then $\mathrm{hdet}(\sigma) = 1$, but $\det(\sigma|_{A_1}) = -1$.

(2) Let A be a p-Koszul AS-regular algebra of dimension d and $\sigma \in \mathrm{GrAut}(A)$. If $\sigma|_{A_1} = (c_{ij})$, then $\sigma^\tau \in \mathrm{GrAut}(A^!)$ where $\sigma^\tau|_{A_1^*} = (c_{ij})^{Tr}$ and $\sigma^\tau(u) = \mathrm{hdet}(\sigma)u$ for $u \in \mathrm{Ext}_A^d(k, k)$;

Homological determinant for Hopf actions

Hypothesis:

(1) H is a Hopf algebra,

(2) A is a noetherian connected graded AS-Gorenstein algebra, and

(3) A is a left H-module algebra and each A_i is a left H-submodule for each i, or A is a graded left H-module algebra.

Homological determinant for Hopf actions

Definition (Kirkman, Kuzmanovich & Zhang, J. Alg. 2009)

Let A be an AS-regular algebra of global dimension d. There is a left H-action on $\mathrm{Ext}_A^d(k, A)$ induced by the left $A\#H$-action on A. Let $\mathbf{e}$ be a nonzero element in $\mathrm{Ext}_A^d(k, A)$. Then there is an algebra morphism $\eta : H \to k$ satisfying $h \cdot \mathbf{e} = \eta(h)\mathbf{e}$ for all $h \in H$.

1. The composite map $\eta \circ S : H \to k$ is called the homological determinant of the H-action on A, and denoted it by hdet (or more precisely hdet_A)
2. The homological determinant hdet_A is said to be trivial if $\mathrm{hdet}_A = \epsilon$, the counit of the Hopf algbera H.

VdB Duality

Theorem (Van den Bergh)

Let A be a (graded) k-algebra. The following are equivalent.

(1) There exists an A^e-module U with both ${}_AU$ and U_A are projective, and an integer d such that for any $i \geq 0$ and any k-algebra R,

$$\mathrm{H}^i(A, -) \cong \mathrm{H}_{d-i}(A, U \otimes_A -) : A^e\text{-Mod-}R \to \text{Mod-}R.$$

VdB Duality

(2) A is homologically smooth, i.e., A has a finitely generated A^e-projective resolution of finite length; and

$$\mathrm{Ext}^i_{A^e}(A, A^e) = \begin{cases} 0 & \text{if } i \neq d, \\ U & \text{if } i = d \end{cases} \tag{0.3}$$

with the A^e-module structure induced by the inner action of A^e.

In this case, we say that A satisfies the VdB duailty with dualizing module U; and denote it as $A \in VdB(U, d)$.

Sketch proof of VdB Duality

(1)$\Rightarrow$ (2) Since

$$\operatorname{Ext}^i_{A^e}(A,-) \cong \operatorname{Tor}^{A^e}_{d-i}(A, U \otimes_A -)$$

commutes with directed direct limits, ${}_{A^e}A$ has a finitely generated projective resolution of finite length. And

$$\operatorname{Ext}^i_{A^e}(A,A^e) \cong \operatorname{Tor}^{A^e}_{d-i}(A, U \otimes_A A^e) \cong \begin{cases} 0 & \text{if } i \neq d, \\ U & \text{if } i = d. \end{cases}$$

Sketch proof of VdB Duality

(2)⇒ (1) Let $P. \to {}_{A^e}A \to 0$ be a f.g. projective resolution of ${}_{A^e}A$ of finite length. Then

$$\begin{aligned} &\mathrm{H}^i(A,M) = \mathrm{Ext}^i_{A^e}(A,M) \cong \mathrm{H}^i(\mathrm{RHom}_{A^e}(A,M)) \\ &\cong \mathrm{H}^i(\mathrm{RHom}_{A^e}(A,A^e) \overset{L}{\otimes}_{A^e} M) \cong \mathrm{H}^i(U[-d] \overset{L}{\otimes}_{A^e} M) \\ &= \mathrm{H}^{i-d}(U \overset{L}{\otimes}_{A^e} M) \cong H^{i-d}(A \overset{L}{\otimes}_{A^e} (U \overset{L}{\otimes}_A M)) \\ &\cong \mathrm{H}_{d-i}(A, U \otimes_A M). \end{aligned}$$

Stefan's Theorem

Theorem

Let H be a Hopf algebra and A a left (graded) H-module algebra, and M be an $(A\#H)^e$-module. Then there is a cohomological spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_H^p({}_H k, \mathrm{Ext}_{A^e}^q(A, M)) \Longrightarrow \mathrm{Ext}_{(A\#H)^e}^{p+q}(A\#H, M)$$

and a homological spectral sequence

$$E_{p,q}^2 = \mathrm{Tor}_p^H({}_H k, \mathrm{Tor}_q^{A^e}(A, M)) \Longrightarrow \mathrm{Tor}_{p+q}^{(A\#H)^e}(A\#H, M)$$

which are natural in M.

Stefan's Theorem

Grotehdieck Spectral Sequences

Stefan's Spectral Sequences are Grothendieck Spectral Sequences, for example, if we take

$$F = \mathrm{Hom}_{B^e}(M,-) : B^e\text{-Mod} \to k\text{-Mod}, M \mapsto M^B$$

$$F_1 = \mathrm{Hom}_{A^e}(M,-)\cdot Rest_{A^e} : B^e\text{-Mod} \to H\text{-Mod},$$

$$F_2 = \mathrm{Hom}_H(N,-) : H\text{-Mod} \to k\text{-Mod},$$

where M^A is viewed as a left H-module via

$$h \rightharpoonup m = h_2 m S^{-1} h_1 \quad ((h \rightharpoonup f)(x) = h_3 f(S^{-1} h_2 x) S^{-1} h_1).$$

Then $F = F_2 F_1$ and for each injective B^e-module, $F_1 M$ is F_2-acyclic.

Stefan's Spectral Sequences

Theorem

Note that

$$\mathrm{Ext}^p_H({}_Hk, \mathrm{Ext}^q_{A^e}(A, M)) = \mathrm{Ext}^p_{H^e}(H, \mathrm{Ext}^q_{A^e}(A, M)),$$

and

$$\mathrm{Tor}^H_p({}_Hk, \mathrm{Tor}^{A^e}_q(A, M)) = \mathrm{Tor}^{H^e}_p(H, \mathrm{Tor}^{A^e}_q(A, M))$$

where the left H-modules $\mathrm{Ext}^q_{A^e}(A, M)$ and $\mathrm{Tor}^{A^e}_q(A, M)$ are viewed as H^e-modules via trivial action on the right. Hence

$$E_2^{p,q} = \mathrm{Ext}^p_{H^e}(H, \mathrm{Ext}^q_{A^e}(A, M)) \Longrightarrow \mathrm{Ext}^{p+q}_{(A\#H)^e}(A\#H, M),$$

$$E^2_{p,q} = \mathrm{Tor}^{H^e}(H, \mathrm{Tor}^{A^e}_q(A, M)) \Longrightarrow \mathrm{Tor}^{(A\#H)^e}_{p+q}(A\#H, M).$$

Ideas of the proof from Farina, JA 2005)

Theorem

Let H be a Calabi-Yau Hopf algebra and A be a left graded H-module algebra. If A is a Koszul Calabi-Yau algebra, then $A\#H$ is graded Calabi-Yau if and only if that the homological determinant of the Hopf-action on A is trivial.

Suppose that H is d_1-Calabi-Yau, and A is d_2-Calabi-Yau. Then

$$\mathrm{H}^{d_1+d_2}(A\#H,(A\#H)^e) = \mathrm{H}^{d_1}(H,\mathrm{H}^{d_2}(A,(A\#H)^e))$$
$$\cong H \otimes_{H^e} (U \otimes_{A^e} (A\#H)^e) \cong U \otimes H \cong A \otimes H.$$

They are isomorphic as $(A\#H)^e$-modules!
$\mathrm{Hom}_{A^e}(P,(A\#H)^e)$ is an $H \otimes (A\#H)^e$-module.

Definition

Definition (Bocklandt, Schedler & Wemyss)

Let S be a finite dimensional semi-simple algebra over k and V an S-bimodule.

(1) $\omega \in V^{\otimes n}$ is called a weak potential of degree n if $s\omega = \omega s$ for any $s \in S$;

(2) For $m \leq n$, let Δ_m^ω be the map $(V^*)^{\otimes n} \to V^{\otimes n-m}$, $\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_m \mapsto \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_m(\omega)$.

(3) a weak potential is called a d-superpotential if $\varphi(\omega) = (-1)^{d-1}(\omega)\varphi$ for $\varphi \in V^*$;

(4) an algebra A is said to be given by a superpotential ω, if $A \cong T_S(V)/\langle \mathrm{Im}\, \Delta_m^\omega \rangle$.

Examples

Examples

(1) $k[x_1, \cdots, x_n] = T_k(V)/(x_ix_j - x_jx_i)$ is given by an n-superpotential $\omega = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$.

$$\langle \mathrm{Im}\, \Delta^{\omega}_{n-2} \rangle = (x_ix_j - x_jx_i).$$

(2) $A = k\langle x, y\rangle / \langle x^3 + axy^2 + ay^2x + byxy, y^3 + ayx^2 + ax^2y + bxyx\rangle$ is given by a superpotential $\omega = x^4 + a(x^2y^2 + xy^2x + yx^2y + y^2x^2) + b(xyxy + yxyx) + y^4$.

$$A = k\langle x, y\rangle / \langle \mathrm{Im}\, \Delta^{\omega}_{1} \rangle.$$

Some results

Theorem (Bocklandt, JAPA 2008, Segal, JA 2008)

Every graded 3-dimensional Calabi-Yau algebra is given by a superpotential.

Theorem (Bocklandt, Schedler & Wemyss, JAPA 2010)

Every Koszul Calabi-Yau algebra is given by a superpotential.

p-Koszul Calabi-Yau algebra

Theorem

Let A be a p-Koszul Calabi-Yau algebra. Then A is given by a superpotential.

The proof uses A_∞-algebra theory.

p-Koszul Calabi-Yau algebra

Let $A = T(V)/(R), E(A) = \mathrm{Ext}_A^*({}_Ak, {}_Ak)$ is a graded symmetric Frobenius algebra, which also has an A_∞-structure (m_2, m_p).

$\omega \in V^{\otimes\delta(d)} \cong [(V^*)^{\otimes\delta(d)}]^*$, is the map $\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_{\delta(d)} \mapsto \langle m_p(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_p), \eta_{p+1} \cdot m_p(\eta_{p+2} \otimes \cdots \otimes \eta_{2p+1} \cdots m_p(\cdots \otimes \eta_{\delta(d)}))\rangle$.

Note that $d = \mathrm{pdim}\, A$ is always odd, and $A \cong T(V)/\langle \mathrm{Im}\, \Delta^{\omega}_{\delta(d-1)} \rangle$.

Superpotential for skew group algebra

Theorem

Let A be a p-Koszul Calabi-Yau algebra and G be a finite subgroup of $SL(A)$. If A is given by a superpotential ω, then $A\#G$ is given by the superpotential $\omega\#1$.

$$\psi : T_k(V)\#G/(R\#G) \cong T_{kG}(V \otimes kG)/(R \otimes kG),$$

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n\#g \mapsto (v_1 \otimes 1) \cdots \otimes (v_{n-1} \otimes 1) \otimes (v_n \otimes g).$$

Similar result for Hopf actions.

Example

Example

Take $A = k\langle x, y, z\rangle / \langle yz - zy, zx - xz, xy - yx + z^2\rangle$,

$G = \langle \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \rangle, \epsilon^3 = 1$, primitive.

Example

Example

(1) $A\#G$ is the path algebra kQ with the relations
$y_iz_j = z_iy_j, z_ix_j = x_iz_j, x_iy_j = y_ix_j + z_iz_j$;

(2) A is defined by a superpotential
$xyz - xzy + yzx - yxz + zxy - zyx + z^3$;

(3) $A\#G$ is defined by a superpotential
$\sum x_iy_jz_k - x_iz_jy_k + y_iz_jx_k - y_ix_jz_k + z_ix_jy_k - z_iy_jx_k + z_iz_jz_k$.

Questions

(1) Remove the p-Koszul condition away?

(2) Calabi-Yau property of the fixed ring A^G?

(3) (Auslander) $R := k[x_1, \cdots, x_n]^G$, $\underline{\mathrm{MCM}(R)}$ is a (n-1)-Calabi-Yau category.
Is this true for noncommutative Calabi-Yau algebras?