Structure of torus invariant prime ideals of quantum Schubert cells

New Trends in Noncommutative Algebra
In honor of Ken Goodearl's 65th birthday

Milen Yakimov

Louisiana State University and University of California at Santa Barbara

Quantum groups

The quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators

$$X_i^{\pm}, K_i^{\pm 1},\ i = 1, \dots, r,$$

subject to the relations

$$K_i^{-1}K_i = K_iK_i^{-1} = 1,\ K_iK_j = K_jK_i,\ K_iX_j^{\pm}K_i^{-1} = q^{\pm c_{ij}}X_j^{\pm},$$
$$X_i^+X_j^- - X_j^-X_i^+ = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-c_{ij}} \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_q (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-c_{ij}-k} = 0,\ i \neq j.$$

Here r=rank of $\mathfrak{g}$, Cartan matrix (c_{ij}), $q \in \mathbb{C}$ is transcendental, $q_i = q^{d_i}$.

Quantum groups

The quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators

$$X_i^{\pm}, K_i^{\pm 1}, \; i = 1, \dots, r,$$

subject to the relations

$$K_i^{-1} K_i = K_i K_i^{-1} = 1, \; K_i K_j = K_j K_i, \; K_i X_j^{\pm} K_i^{-1} = q^{\pm c_{ij}} X_j^{\pm},$$

$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-c_{ij}} \begin{bmatrix} 1 - c_{ij} \\ k \end{bmatrix}_q (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-c_{ij}-k} = 0, \; i \neq j.$$

Here r=rank of $\mathfrak{g}$, Cartan matrix (c_{ij}), $q \in \mathbb{C}$ is transcendental, $q_i = q^{d_i}$.

It is a Hopf algebra. Its finite dimensional weight irreps are parametrized by the set of dominant integral weights P_+, $\lambda \in P_+ \mapsto V(\lambda)$.

There is a natural action of the related Braid group on $\mathcal{U}_q(\mathfrak{g})$ and $V(\lambda)$, $w \in W \mapsto T_w$.

Quantum groups

The quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators

$$X_i^{\pm}, K_i^{\pm 1},\ i = 1, \ldots, r,$$

subject to the relations

$$K_i^{-1} K_i = K_i K_i^{-1} = 1,\ K_i K_j = K_j K_i,\ K_i X_j^{\pm} K_i^{-1} = q^{\pm c_{ij}} X_j^{\pm},$$

$$X_i^{+} X_j^{-} - X_j^{-} X_i^{+} = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-c_{ij}} \begin{bmatrix} 1 - c_{ij} \\ k \end{bmatrix}_q (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-c_{ij}-k} = 0,\ i \neq j.$$

Here r=rank of $\mathfrak{g}$, Cartan matrix (c_{ij}), $q \in \mathbb{C}$ is transcendental, $q_i = q^{d_i}$.

It is a Hopf algebra. Its finite dimensional weight irreps are parametrized by the set of dominant integral weights P_+, $\lambda \in P_+ \mapsto V(\lambda)$.

There is a natural action of the related Braid group on $\mathcal{U}_q(\mathfrak{g})$ and $V(\lambda)$, $w \in W \mapsto T_w$.

$\mathcal{U}_{\pm}$ the subalg. generated by $X_i^{\pm}$, $H = \langle K_1, \ldots, K_r \rangle$ the group of group-like elements.

DKP algebras

Fix $w \in W$. De Concini, Kac and Procesi defined a family of subalgebras $\mathcal{U}^w_\pm \subset \mathcal{U}_\pm$ which are deformations of $\mathcal{U}(\mathfrak{n}_+ \cap \mathrm{Ad}_w(\mathfrak{n}_-))$.

For a reduced expression $w = s_{i_1} \dots s_{i_k}$ define the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}).$$

Let $\mathcal{U}^w_\pm$ be the subalgebras of $\mathcal{U}_q(\mathfrak{g})$, generated by the root vectors

$$X^\pm_{\beta_1} = X^\pm_{i_1}, X^\pm_{\beta_2} = T_{s_{i_1}}(X^\pm_{i_2}), \dots, X^\pm_{\beta_k} = T_{s_{i_1} \dots s_{i_{k-1}}}(X^\pm_{i_k}).$$

DKP algebras

Fix $w \in W$. De Concini, Kac and Procesi defined a family of subalgebras $\mathcal{U}^w_\pm \subset \mathcal{U}_\pm$ which are deformations of $\mathcal{U}(\mathfrak{n}_+ \cap \mathrm{Ad}_w(\mathfrak{n}_-))$.

For a reduced expression $w = s_{i_1} \dots s_{i_k}$ define the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}).$$

Let $\mathcal{U}^w_\pm$ be the subalgebras of $\mathcal{U}_q(\mathfrak{g})$, generated by the root vectors

$$X^\pm_{\beta_1} = X^\pm_{i_1}, X^\pm_{\beta_2} = T_{s_{i_1}}(X^\pm_{i_2}), \dots, X^\pm_{\beta_k} = T_{s_{i_1} \dots s_{i_{k-1}}}(X^\pm_{i_k}).$$

Theorem [De Concini-Kac-Procesi]. The definition of the algebras $\mathcal{U}^w_\pm$ does not depend on the choice of a reduced decomposition of w. The algebras $\mathcal{U}^w_\pm$ have the PBW bases

$$(X^\pm_{\beta_k})^{n_k} \dots (X^\pm_{\beta_1})^{n_1}, \;\; n_1, \dots, n_k \in \mathbb{N}.$$

DKP algebras

Fix $w \in W$. De Concini, Kac and Procesi defined a family of subalgebras $\mathcal{U}^w_\pm \subset \mathcal{U}_\pm$ which are deformations of $\mathcal{U}(\mathfrak{n}_+ \cap \mathrm{Ad}_w(\mathfrak{n}_-))$.

For a reduced expression $w = s_{i_1} \dots s_{i_k}$ define the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}).$$

Let $\mathcal{U}^w_\pm$ be the subalgebras of $\mathcal{U}_q(\mathfrak{g})$, generated by the root vectors

$$X^\pm_{\beta_1} = X^\pm_{i_1}, X^\pm_{\beta_2} = T_{s_{i_1}}(X^\pm_{i_2}), \dots, X^\pm_{\beta_k} = T_{s_{i_1} \dots s_{i_{k-1}}}(X^\pm_{i_k}).$$

Theorem [De Concini-Kac-Procesi]. The definition of the algebras $\mathcal{U}^w_\pm$ does not depend on the choice of a reduced decomposition of w. The algebras $\mathcal{U}^w_\pm$ have the PBW bases

$$(X^\pm_{\beta_k})^{n_k} \dots (X^\pm_{\beta_1})^{n_1}, \ \ n_1, \dots, n_k \in \mathbb{N}.$$

Theorem [Heckenberger–Schneider]. All right coideal subalgebras of $\mathcal{U}_q(\mathfrak{b}_+)$ containing H are of the form $\mathcal{U}^w_+ \mathbb{C}[H]$.

An Example

Let $\mathfrak{g} = \mathfrak{sl}_{m+n}$ and $w = c^m$ where c is the Coxeter element $(12\ldots m+n)$. Think of $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$. Then $\mathcal{U}_-^w(\mathfrak{g})$ is isomorphic to the algebra of quantum matrices $R_q[M_{m,n}]$. The latter is the $\mathbb{C}$-algebra generated by x_{ij}, $1 \le i \le m$, $1 \le j \le n$, with relations

$$x_{ij}x_{lj} = qx_{lj}x_{ij}, \quad \text{for } i < l,$$

$$x_{ij}x_{ik} = qx_{ik}x_{ij}, \quad \text{for } j < k,$$

$$x_{ij}x_{lk} = x_{lk}x_{ij}, \quad \text{for } i < l, j > k,$$

$$x_{ij}x_{lk} - x_{lk}x_{ij} = (q - q^{-1})x_{ik}x_{lj}, \quad \text{for } i < l, j < k,$$

An Example

Let $\mathfrak{g} = \mathfrak{sl}_{m+n}$ and $w = c^m$ where c is the Coxeter element $(12\ldots m+n)$. Think of $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$. Then $\mathcal{U}^w_-(\mathfrak{g})$ is isomorphic to the algebra of quantum matrices $R_q[M_{m,n}]$. The latter is the $\mathbb{C}$-algebra generated by x_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, with relations

$$x_{ij}x_{lj} = qx_{lj}x_{ij}, \quad \text{for } i < l,$$

$$x_{ij}x_{ik} = qx_{ik}x_{ij}, \quad \text{for } j < k,$$

$$x_{ij}x_{lk} = x_{lk}x_{ij}, \quad \text{for } i < l, j > k,$$

$$x_{ij}x_{lk} - x_{lk}x_{ij} = (q - q^{-1})x_{ik}x_{lj}, \quad \text{for } i < l, j < k,$$

Theorem of Goodearl–Letzter: A partition of $\mathrm{Spec}\,\mathcal{U}^w_-$ into strata indexed by H-invariant primes of $\mathcal{U}^w_-$, each stratum is isomorphic to the spectrum of a (commutative) Laurent polynomial ring.

Plan. 1. Describe $H - \mathrm{Spec}\,\mathcal{U}^w_-$ as a poset. 2. Describe explicit generating sets for the H-primes of $\mathcal{U}^w_-$. 3. Prove the Goodearl–Lenagan conjecture on existence polynormal generating sequences for H-primes of $R_q[M_{m,n}]$ (and $\mathcal{U}^w_-$). 4. Prove that $\mathrm{Spec}\,\mathcal{U}^w_-$ is normally separated. 5. Prove a dimension formula for the H-strata of $\mathrm{Spec}\,\mathcal{U}^w_-$. All based on another realization of $\mathcal{U}^w_-$ in which the H-invariant primes are explicitly described.

Relations to Poisson geometry

Let A be a an associative algebra over $\mathbb{C}$ with a $\mathbb{Z}_{\geq 0}$ filtration:

$$A_0 \subset A_1 \subset \ldots \subset A, \quad A = \cup_k A_k, \quad A_k.A_l \subset A_{k+l}.$$

If the associated graded $\mathrm{gr}A$ is commutative, then it inherits a canonical structure of a Poisson algebra:

$$\{a_k + A_{k-1}, a_l + A_{l-1}\} = a_k a_l - a_l a_k + A_{k+l-2}, \quad a_k \in A_k, a_l \in A_l,$$

note that $a_k a_l - a_l a_k \in A_{k+l-1}$. If in addition $\mathrm{gr}A$ has no nilpotent elements, then one obtains a canonical Poisson structure on the affine variety $\mathrm{Spec}(\mathrm{gr}A)$.

Relations to Poisson geometry

Let A be a an associative algebra over $\mathbb{C}$ with a $\mathbb{Z}_{\geq 0}$ filtration:

$$A_0 \subset A_1 \subset \ldots \subset A, \quad A = \cup_k A_k, \quad A_k.A_l \subset A_{k+l}.$$

If the associated graded $\mathrm{gr}A$ is commutative, then it inherits a canonical structure of a Poisson algebra:

$$\{a_k + A_{k-1}, a_l + A_{l-1}\} = a_k a_l - a_l a_k + A_{k+l-2}, \quad a_k \in A_k, a_l \in A_l,$$

note that $a_k a_l - a_l a_k \in A_{k+l-1}$. If in addition $\mathrm{gr}A$ has no nilpotent elements, then one obtains a canonical Poisson structure on the affine variety $\mathrm{Spec}(\mathrm{gr}A)$.

Example. $\mathcal{U}(\mathfrak{g})$, $\mathrm{gr}\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$, linear Poisson str. on $\mathfrak{g}^*$, symplectic foliation given by coadjoint orbits.

Relations to Poisson geometry

Let A be a an associative algebra over $\mathbb{C}$ with a $\mathbb{Z}_{\geq 0}$ filtration:

$$A_0 \subset A_1 \subset \ldots \subset A, \quad A = \cup_k A_k, \quad A_k . A_l \subset A_{k+l}.$$

If the associated graded $\mathrm{gr} A$ is commutative, then it inherits a canonical structure of a Poisson algebra:

$$\{a_k + A_{k-1}, a_l + A_{l-1}\} = a_k a_l - a_l a_k + A_{k+l-2}, \quad a_k \in A_k, a_l \in A_l,$$

note that $a_k a_l - a_l a_k \in A_{k+l-1}$. If in addition $\mathrm{gr} A$ has no nilpotent elements, then one obtains a canonical Poisson structure on the affine variety $\mathrm{Spec}(\mathrm{gr} A)$.

Example. $\mathcal{U}(\mathfrak{g})$, $\mathrm{gr}\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$, linear Poisson str. on $\mathfrak{g}^*$, symplectic foliation given by coadjoint orbits.

Orbit method. Prove that $\mathrm{Prim} A$ and the quotient space of the symplectic foliation of the Poisson structure on $\mathrm{Spec}(\mathrm{gr} A)$ are homeomorphic.

Group Poisson structures

For $w \in W$ we will put a quadratic Poisson structure π_w on the Schubert cell $X_w \subset G/B_+$.

Conjecture. $\mathrm{Prim}\mathcal{U}_-^w$ and the quotient space of the symplectic foliation of (X_w, π_w) are homeomorphic.

Group Poisson structures

For $w \in W$ we will put a quadratic Poisson structure π_w on the Schubert cell $X_w \subset G/B_+$.

Conjecture. $\mathrm{Prim}\mathcal{U}_-^w$ and the quotient space of the symplectic foliation of (X_w, π_w) are homeomorphic.

Fix a pair of opposite Borel subgroups $B_\pm$ of G, $T = B_+ \cap B_-$ – a maximal torus of G.

- Let Δ_+ be the set of all positive roots of $\mathfrak{g} = \mathrm{Lie}\, G$,
- Fix two dual sets of root vectors, $\{e_\alpha\}_{\alpha\in\Delta_+}$, $\{f_\alpha\}_{\alpha\in\Delta_+}$, normalized by $\langle e_\alpha, f_\alpha\rangle = 1$, where $\langle ., . \rangle$ is the Killing form on $\mathfrak{g}$.

Define

$$\pi_G = \sum_{\alpha\in\Delta_+} L_{e_\alpha} \wedge L_{f_\alpha} - \sum_{\alpha\in\Delta_+} R_{e_\alpha} \wedge R_{f_\alpha}$$

called the standard Poisson structure on G. (Here L and R denote left and right invariant vector fields on G.)

Group Poisson structures

For $w \in W$ we will put a quadratic Poisson structure π_w on the Schubert cell $X_w \subset G/B_+$.

Conjecture. $\mathrm{Prim}\mathcal{U}_-^w$ and the quotient space of the symplectic foliation of (X_w, π_w) are homeomorphic.

Fix a pair of opposite Borel subgroups $B_\pm$ of G, $T = B_+ \cap B_-$ – a maximal torus of G.

- Let Δ_+ be the set of all positive roots of $\mathfrak{g} = \mathrm{Lie}\, G$,
- Fix two dual sets of root vectors, $\{e_\alpha\}_{\alpha\in\Delta_+}$, $\{f_\alpha\}_{\alpha\in\Delta_+}$, normalized by $\langle e_\alpha, f_\alpha\rangle = 1$, where $\langle ., .\rangle$ is the Killing form on $\mathfrak{g}$.

Define

$$\pi_G = \sum_{\alpha\in\Delta_+} L_{e_\alpha} \wedge L_{f_\alpha} - \sum_{\alpha\in\Delta_+} R_{e_\alpha} \wedge R_{f_\alpha}$$

called the standard Poisson structure on G. (Here L and R denote left and right invariant vector fields on G.)

Example. $(SL_n(\mathbb{C}), \pi_{SL_n})$ embeds in $M_{n\times n}$ with

$$\sum_{i,k=1}^{n} \sum_{j,l=1}^{n} (\mathrm{sign}(k-i) + \mathrm{sign}(l-j)) x_{il} x_{kj} \frac{\partial}{\partial x_{ij}} \wedge \frac{\partial}{\partial x_{kl}}.$$

Poisson structures on flag varieties

Fix a parabolic subgroup $P \supset B_+$ of G. Under the map $p \colon G \to G/P$ the Poisson structures π_G can be pushed forward to a well defined Poisson structure $\pi_{G/P} = p_*(\pi_G)$ on G/P.

Poisson structures on flag varieties

Fix a parabolic subgroup $P \supset B_+$ of G. Under the map $p\colon G \to G/P$ the Poisson structures π_G can be pushed forward to a well defined Poisson structure $\pi_{G/P} = p_*(\pi_G)$ on G/P.

Special case: $P = B_+$. The T-orbits of symplectic leaves of $(G/B_+, \pi_{G/B})$ are the open Richardson varieties

$$R_{y_-,y_+} = B_- y_- \cdot B_+ \cap B_+ y_+ \cdot B_+ \subset G/B_+, \quad y_\pm \in W, y_- \leq y_+.$$

Poisson structures on flag varieties

Fix a parabolic subgroup $P \supset B_+$ of G. Under the map $p\colon G \to G/P$ the Poisson structures π_G can be pushed forward to a well defined Poisson structure $\pi_{G/P} = p_*(\pi_G)$ on G/P.

Special case: $P = B_+$. The T-orbits of symplectic leaves of $(G/B_+, \pi_{G/B})$ are the open Richardson varieties

$$R_{y_-,y_+} = B_- y_- \cdot B_+ \cap B_+ y_+ \cdot B_+ \subset G/B_+, \quad y_\pm \in W, y_- \leq y_+.$$

Theorem. [Brown, Goodearl, Y.] The T–orbits of symplectic leaves of $(G/P, \pi_{G/P})$ are precisely the sets

$$S_P(y_-, y_+) = q(B_- y_- \cdot B_+ \cap B_+ y_+ \cdot B_+), \quad y_- \in W, y_+ \in W^{W_P}, y_- \leq y_+$$

where W^{W_P} is the set of min length repr. of the cosets W/W_P and $q\colon G/B_+ \to G/P$ is the canonical projection. (This is the Lusztig stratification of G/P.) One has

$$\overline{S_P(y_-, y_+)} = \sqcup \{ S_P(y'_-, y'_+) \,|y'_- \in W, y'_+ \in W^{W_P}, y'_- \leq y'_+, \\ \exists z \in W_P, y_- \leq y'_- z, y_+ \geq y'_+ z \}$$

Note that $q\colon B_+ y_+ \cdot B_+ \to B_+ y_+ \cdot P$ is an isom. of (Poisson) affine spaces for $y_+ \in W^{W_P}$.

Poisson side I

The codimension of a symplectic leaf in an open Richardson variety R_{y_-,y_+} is

$$\dim \ker(1 + y_+^{-1} y_-) = \dim E_{-1}(y_+^{-1} y_-).$$

The trancendence degree of the center of the Poisson field of rational functions on R_{y_-,y_+} is given by the same number.

Poisson side I

The codimension of a symplectic leaf in an open Richardson variety R_{y_-,y_+} is

$$\dim \ker(1 + y_+^{-1} y_-) = \dim E_{-1}(y_+^{-1} y_-).$$

The trancendence degree of the center of the Poisson field of rational functions on R_{y_-,y_+} is given by the same number.

We will interpret DKP algebras as quantized algebras of functions on Schubert cells $(B_+ w \cdot B_+, \pi|_{B_+ w \cdot B_+})$, where $\pi := \pi_{G/B_+}$. First restrict the Poisson structure π to the translated open Schubert cell $wB_- \cdot B_+$. Note that $B_+ w \cdot B_+ \subset wB_- \cdot B_+$.

Poisson side I

The codimension of a symplectic leaf in an open Richardson variety R_{y_-,y_+} is

$$\dim \ker(1 + y_+^{-1} y_-) = \dim E_{-1}(y_+^{-1} y_-).$$

The trancendence degree of the center of the Poisson field of rational functions on R_{y_-,y_+} is given by the same number.

We will interpret DKP algebras as quantized algebras of functions on Schubert cells $(B_+ w \cdot B_+, \pi|_{B_+ w \cdot B_+})$, where $\pi := \pi_{G/B_+}$. First restrict the Poisson structure π to the translated open Schubert cell $wB_- \cdot B_+$. Note that $B_+ w \cdot B_+ \subset wB_- \cdot B_+$.

Theorem. The T-orbits of symplectic leaves of the translated open Schubert cell $(wB_- \cdot B_+, \pi)$ are

$$S(y_-, y_+) = wB_- \cdot B_+ \cap R_{y_-,y_+} = wB_- \cdot B_+ \cap B_- y_- \cdot B_+ \cap B_+ y_+ \cdot B_+$$

parametrized by pairs $(y_-, y_+) \in W \times W$ such that $y_- \leq w \leq y_+$.

Poisson side I

The codimension of a symplectic leaf in an open Richardson variety R_{y_-,y_+} is

$$\dim \ker(1 + y_+^{-1} y_-) = \dim E_{-1}(y_+^{-1} y_-).$$

The trancendence degree of the center of the Poisson field of rational functions on R_{y_-,y_+} is given by the same number.

We will interpret DKP algebras as quantized algebras of functions on Schubert cells $(B_+ w \cdot B_+, \pi|_{B_+ w \cdot B_+})$, where $\pi := \pi_{G/B_+}$. First restrict the Poisson structure π to the translated open Schubert cell $wB_- \cdot B_+$. Note that $B_+ w \cdot B_+ \subset wB_- \cdot B_+$.

Theorem. The T-orbits of symplectic leaves of the translated open Schubert cell $(wB_- \cdot B_+, \pi)$ are

$$S(y_-, y_+) = wB_- \cdot B_+ \cap R_{y_-,y_+} = wB_- \cdot B_+ \cap B_- y_- \cdot B_+ \cap B_+ y_+ \cdot B_+$$

parametrized by pairs $(y_-, y_+) \in W \times W$ such that $y_- \leq w \leq y_+$.

Identify

$$\mathbb{C}[wB_- \cdot B_+] \cong \mathbb{C}[wB_- B_+]^{B_+} = \{c^\lambda_{\xi, v_\lambda} / c^\lambda_w \mid \lambda \in P_+, \xi \in V(\lambda)^*\},$$

$c^\lambda_{\xi,v}$ denotes the matrix coefficient of $v \in V(\lambda)$ and $\xi \in V(\lambda)^*$: for $g \in G$, $c^\lambda_{\xi,v}(g) = \langle \xi, gv \rangle$. Moreover v_λ is a h.w.v. of $V(\lambda)$, ξ_λ is a dual vector and $c^\lambda_w = c^\lambda_{w\xi_\lambda, v_\lambda}$.

Poisson side II

Denote $\mathfrak{n}_\pm = \mathrm{Lie}\, U_\pm$. For $y \in W$, define the ideals

$$Q(y)^\pm_w = \{c^\lambda_{\xi, v_\lambda}/c^\lambda_w \mid \lambda \in P_+, \xi \in (\mathcal{U}(\mathfrak{n}_\pm) y v_\lambda)^\perp \subset V(\lambda)^*\} = \mathcal{V}(\overline{wB_- \cdot B_+ \cap B_\pm y \cdot B_+})$$

of $\mathbb{C}[wB_- \cdot B_+]$. Scheme theoretic intersections of dual Schubert varieties are reduced (Ramanathan):

Poisson side II

Denote $\mathfrak{n}_\pm = \mathrm{Lie}\, U_\pm$. For $y \in W$, define the ideals

$$Q(y)^\pm_w = \{c^\lambda_{\xi, v_\lambda}/c^\lambda_w \mid \lambda \in P_+, \xi \in (\mathcal{U}(\mathfrak{n}_\pm) y v_\lambda)^\perp \subset V(\lambda)^*\} = \mathcal{V}(\overline{wB_- \cdot B_+ \cap B_\pm y \cdot B_+})$$

of $\mathbb{C}[wB_- \cdot B_+]$. Scheme theoretic intersections of dual Schubert varieties are reduced (Ramanathan):

Proposition. The vanishing ideal of the Zariski closure of $S_w(y_-, y_+)$ in $wB_- \cdot B_+$ is

$$\mathcal{V}(\overline{S_w(y_-, y_+)}) = Q(y_-)^-_w + Q(y_+)^+_w$$
$$= \{c^\lambda_{\xi, v_\lambda}/c^\lambda_w \mid \lambda \in P_+, \xi \in (\mathcal{U}(\mathfrak{n}_-) y_- v_\lambda \cap \mathcal{U}(\mathfrak{n}_+) y_+ v_\lambda)^\perp \subset V(\lambda)^*\}.$$

Poisson side II

Denote $\mathfrak{n}_\pm = \mathrm{Lie}\, U_\pm$. For $y \in W$, define the ideals

$$Q(y)^\pm_w = \{c^\lambda_{\xi, v_\lambda}/c^\lambda_w \mid \lambda \in P_+, \xi \in (\mathcal{U}(\mathfrak{n}_\pm) y v_\lambda)^\perp \subset V(\lambda)^*\} = \mathcal{V}(\overline{wB_- \cdot B_+ \cap B_\pm y \cdot B_+})$$

of $\mathbb{C}[wB_- \cdot B_+]$. Scheme theoretic intersections of dual Schubert varieties are reduced (Ramanathan):

Proposition. The vanishing ideal of the Zariski closure of $S_w(y_-, y_+)$ in $wB_- \cdot B_+$ is

$$\mathcal{V}(\overline{S_w(y_-, y_+)}) = Q(y_-)^-_w + Q(y_+)^+_w$$
$$= \{c^\lambda_{\xi, v_\lambda}/c^\lambda_w \mid \lambda \in P_+, \xi \in (\mathcal{U}(\mathfrak{n}_-) y_- v_\lambda \cap \mathcal{U}(\mathfrak{n}_+) y_+ v_\lambda)^\perp \subset V(\lambda)^*\}.$$

Schubert varieties are linearly defined (Kempf-Ramanathan):

$$\oplus_{\lambda \in P_+} H^0(G/B_+, \mathcal{L}_\lambda) \to \oplus_{\lambda \in P_+} H^0(X_y, \mathcal{L}_\lambda)$$

is surjective and its kernel is generated by elements in deg 1. So the ideal of $\overline{S_w(y_-, y_+)}$ $\subset wB_- \cdot B_+$ is generated by

$$\bigcup_j \{c^{\omega_j}_{\xi, v_{\omega_j}}/c^{\omega_j}_w \mid \xi \in (\mathcal{U}(\mathfrak{n}_-) y_- v_{\omega_j} \cap \mathcal{U}(\mathfrak{n}_+) y_+ v_{\omega_j})^\perp\}$$

Poisson str. on Schubert cells

Denote $U_+^w = U_+ \cap wU_- w^{-1}$, identify $j_w \colon U_+^w \cong B_+ w \cdot B_+$. Set $\pi_w = (j_w^{-1})_* (\pi|_{B_+ w \cdot B_+})$.

Demazure modules $V_w(\lambda) = \mathcal{U}(\mathfrak{b}_+) w v_\lambda = \mathcal{U}(\mathfrak{n}_+^w) w v_\lambda$. Then $\eta \in V_w(\lambda)^* \mapsto d_\eta^{w,\lambda} \in \mathbb{C}[U_+^w]$, $d_\eta^{w,\lambda}(u) = \langle \eta, u\dot{w} v_\lambda \rangle$, $u \in U_+^w$. One has

$$\mathbb{C}[U_+^w] = \{ d_\eta^{w,\lambda} \mid \lambda \in P_+, \eta \in V_w(\lambda)^* \}.$$

Poisson str. on Schubert cells

Denote $U_+^w = U_+ \cap wU_- w^{-1}$, identify $j_w : U_+^w \cong B_+ w \cdot B_+$. Set $\pi_w = (j_w^{-1})_* (\pi|_{B_+ w \cdot B_+})$.

Demazure modules $V_w(\lambda) = \mathcal{U}(\mathfrak{b}_+) w v_\lambda = \mathcal{U}(\mathfrak{n}_+^w) w v_\lambda$. Then $\eta \in V_w(\lambda)^* \mapsto d_\eta^{w,\lambda} \in \mathbb{C}[U_+^w]$, $d_\eta^{w,\lambda}(u) = \langle \eta, u \dot{w} v_\lambda \rangle$, $u \in U_+^w$. One has

$$\mathbb{C}[U_+^w] = \{ d_\eta^{w,\lambda} \mid \lambda \in P_+, \eta \in V_w(\lambda)^* \}.$$

Theorem. (1) The T-orbits of symplectic leaves of the Schubert cells (U_+^w, π_w) are

$$S_w(y) = j_w^{-1}(R_{y,w}) = U_+^w \cap B_- y B_+ w^{-1},$$

parametrized by $y \in W^{\leq w}$.

(2) The vanishing ideal of $\overline{S_w(y)}$ is:

$$\mathcal{V}(\overline{S_w(y)}) = \{ d_\eta^{w,\lambda} \mid \eta \in (\mathcal{U}(\mathfrak{n}_+) w v_\lambda \cap \mathcal{U}(\mathfrak{n}_-) y v_\lambda)^\perp \subset V_w(\lambda)^* \}.$$

(3) $\overline{S_w(y)}$ is generated by the above sets for $\lambda = \omega_1, \ldots, \omega_r$.

Quantum Schubert cells

Define the quantized coordinate ring $R_q[U_+^w]$ of the Schubert cell $B_+ w \cdot B_+$ as the subset of $(\mathcal{U}_+)^*$ consisting of all matrix coefficients $d_\eta^{w,\lambda}(x) := \langle \eta, x T_w v_\lambda \rangle$ for $\eta \in V_w(\lambda)^*$.
Multiplication:

$$d_{\eta_1}^{w,\lambda_1} d_{\eta_2}^{w,\lambda_2} = q^{\langle \lambda_2, \lambda_1 - w^{-1}\mu_1 \rangle} d_\eta^{w,\lambda_1+\lambda_2},$$

where $\eta = \eta_1 \otimes \eta_2|_{\mathcal{U}_+(T_w v_{\lambda_1} \otimes T_w v_{\lambda_2})} \in V_w(\lambda_1 + \lambda_2)^*$

for $\eta_1 \in V_w(\lambda_1)^*$ of weight μ_1 and $\eta_2 \in V_w(\lambda_2)^*$.

Quantum Schubert cells

Define the quantized coordinate ring $R_q[U_+^w]$ of the Schubert cell $B_+ w \cdot B_+$ as the subset of $(\mathcal{U}_+)^*$ consisting of all matrix coefficients $d_\eta^{w,\lambda}(x) := \langle \eta, x T_w v_\lambda \rangle$ for $\eta \in V_w(\lambda)^*$.
Multiplication:

$$d_{\eta_1}^{w,\lambda_1} d_{\eta_2}^{w,\lambda_2} = q^{\langle \lambda_2, \lambda_1 - w^{-1}\mu_1 \rangle} d_\eta^{w,\lambda_1+\lambda_2},$$

$$\text{where } \; \eta = \eta_1 \otimes \eta_2|_{\mathcal{U}_+(T_w v_{\lambda_1} \otimes T_w v_{\lambda_2})} \in V_w(\lambda_1 + \lambda_2)^*$$

for $\eta_1 \in V_w(\lambda_1)^*$ of weight μ_1 and $\eta_2 \in V_w(\lambda_2)^*$.

Motivation: the q-term comes from $R_q[G]$ via a Joseph-Gorelik algebra.

Quantum Schubert cells

Define the quantized coordinate ring $R_q[U_+^w]$ of the Schubert cell $B_+ w \cdot B_+$ as the subset of $(\mathcal{U}_+)^*$ consisting of all matrix coefficients $d_\eta^{w,\lambda}(x) := \langle \eta, x T_w v_\lambda \rangle$ for $\eta \in V_w(\lambda)^*$.
Multiplication:

$$d_{\eta_1}^{w,\lambda_1} d_{\eta_2}^{w,\lambda_2} = q^{\langle \lambda_2, \lambda_1 - w^{-1}\mu_1 \rangle} d_\eta^{w,\lambda_1+\lambda_2},$$

where $\eta = \eta_1 \otimes \eta_2|_{\mathcal{U}_+(T_w v_{\lambda_1} \otimes T_w v_{\lambda_2})} \in V_w(\lambda_1 + \lambda_2)^*$

for $\eta_1 \in V_w(\lambda_1)^*$ of weight μ_1 and $\eta_2 \in V_w(\lambda_2)^*$.

Motivation: the q-term comes from $R_q[G]$ via a Joseph-Gorelik algebra.

The universal R-matrix associated to w is given by

$$\mathcal{R}^w = \prod_{j=k,\ldots,1} \exp_{q_{i_j}} \left((1 - q_{i_j})^{-2} X_{\beta_j}^+ \otimes X_{\beta_j}^- \right), \quad \exp_{q_i}(y) = \sum_{n=0}^{\infty} q_i^{n(n+1)/2} \frac{y^n}{[n]_{q_i}!}.$$

Quantum Schubert cells

Define the quantized coordinate ring $R_q[U_+^w]$ of the Schubert cell $B_+ w \cdot B_+$ as the subset of $(\mathcal{U}_+)^*$ consisting of all matrix coefficients $d_\eta^{w,\lambda}(x) := \langle \eta, x T_w v_\lambda \rangle$ for $\eta \in V_w(\lambda)^*$.
Multiplication:

$$d_{\eta_1}^{w,\lambda_1} d_{\eta_2}^{w,\lambda_2} = q^{\langle \lambda_2, \lambda_1 - w^{-1}\mu_1 \rangle} d_\eta^{w,\lambda_1+\lambda_2},$$

$$\text{where} \;\; \eta = \eta_1 \otimes \eta_2|_{\mathcal{U}_+(T_w v_{\lambda_1} \otimes T_w v_{\lambda_2})} \in V_w(\lambda_1 + \lambda_2)^*$$

for $\eta_1 \in V_w(\lambda_1)^*$ of weight μ_1 and $\eta_2 \in V_w(\lambda_2)^*$.

Motivation: the q-term comes from $R_q[G]$ via a Joseph-Gorelik algebra.

The universal R-matrix associated to w is given by

$$\mathcal{R}^w = \prod_{j=k,\dots,1} \exp_{q_{i_j}} \left((1 - q_{i_j})^{-2} X_{\beta_j}^+ \otimes X_{\beta_j}^- \right), \quad \exp_{q_i}(y) = \sum_{n=0}^{\infty} q_i^{n(n+1)/2} \frac{y^n}{[n]_{q_i}!}.$$

Theorem. $R_q[U_+^w] \cong \mathcal{U}_-^w$ under

$$d_\eta^{w,\lambda} \mapsto (d_\eta^{w,\lambda} \otimes \mathrm{id}) \mathcal{R}^w.$$

DKP algebras

Theorem. [Y.] Fix $w \in W$. For each $y \in W^{\leq w}$ define

$$I_w(y) = \{(d_\eta^{w,\lambda} \otimes \mathrm{id})(\mathcal{R}^w) \mid \lambda \in P_+, \eta \in (\mathcal{U}_+ T_w v_\lambda \cap \mathcal{U}_- T_y v_\lambda)^\perp\}.$$

Then:

(a) $I_w(y)$ is an H-invariant prime ideal of $\mathcal{U}_-^w$ and all H-invariant prime ideals of $\mathcal{U}_-^w$ are of this form.

(b) The correspondence $y \in W^{\leq w} \mapsto I_w(y)$ is an isomorphism from the poset $W^{\leq w}$ to the poset of H invariant prime ideals of $\mathcal{U}_-^w$ ordered under inclusion; that is $I_w(y) \subseteq I_w(y')$ for $y, y' \in W^{\leq w}$ if and only if $y \leq y'$.

(c) $I_w(y)$ is generated as a right ideal by

$$(d_\eta^{w,\omega_i} \otimes \mathrm{id})(\mathcal{R}^w) \quad \text{for} \quad \eta \in (\mathcal{U}_+ T_w v_{\omega_i} \cap \mathcal{U}^- T_y v_{\omega_i})^\perp, i = 1, \dots, r,$$

where $\omega_1, \dots, \omega_r$ are the fundamental weights of $\mathfrak{g}$.

Proof uses Theorems of Gorelik and Joseph (ring theoretic results along the lines of the results of Ramanathan and Kempf–Ramanathan).

Algebras of quantum matrices

$R_q[M_{m,n}]$ is the $\mathbb{C}$-algebra generated by x_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, with relations

$$x_{ij}x_{lj} = qx_{lj}x_{ij}, \quad \text{for } i < l,$$
$$x_{ij}x_{ik} = qx_{ik}x_{ij}, \quad \text{for } j < k,$$
$$x_{ij}x_{lk} = x_{lk}x_{ij}, \quad \text{for } i < l, j > k,$$
$$x_{ij}x_{lk} - x_{lk}x_{ij} = (q - q^{-1})x_{ik}x_{lj}, \quad \text{for } i < l, j < k,$$

$\mathbb{Z}^{m+n}$ acts on $R_q[M_{m,n}]$, by $(a_1, \ldots, a_m, b_1, \ldots, b_n) \cdot x_{ij} = q^{a_i - b_j} x_{ij}$.

Algebras of quantum matrices

$R_q[M_{m,n}]$ is the $\mathbb{C}$-algebra generated by x_{ij}, $1 \le i \le m$, $1 \le j \le n$, with relations

$$x_{ij}x_{lj} = q x_{lj}x_{ij}, \quad \text{for } i < l,$$
$$x_{ij}x_{ik} = q x_{ik}x_{ij}, \quad \text{for } j < k,$$
$$x_{ij}x_{lk} = x_{lk}x_{ij}, \quad \text{for } i < l, j > k,$$
$$x_{ij}x_{lk} - x_{lk}x_{ij} = (q - q^{-1})x_{ik}x_{lj}, \quad \text{for } i < l, j < k,$$

$\mathbb{Z}^{m+n}$ acts on $R_q[M_{m,n}]$, by $(a_1, \dots, a_m, b_1, \dots, b_n) \cdot x_{ij} = q^{a_i - b_j} x_{ij}$.

Corollary. [Y.] The $\mathbb{Z}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ are parametrized by $y \in S_{m+n}^{\le w_{m,n}}$. The ideal corresponding to y is generated by the sets of quantum minors

$$\Delta^q_{w_m^\circ(p_1(I)), (\overline{m+1, m+k} \setminus p_2(I)) - m}$$

for $k \in \overline{1, n}$, $I \subset \overline{1, m+n}$, $|I| = k$, $I \le c^m(\overline{1, k})$, $I \not\ge y(\overline{1, k})$ and

$$\Delta^q_{w_m^\circ(p_1(I) \setminus \overline{1, k-n}), (\overline{m+1, m+n} \setminus p_2(I)) - m}$$

for $k \in \overline{n+1, m+n-1}$, $I \subset \overline{1, m+n}$, $|I| = k$, $I \le c^m(\overline{1, k})$, $I \not\ge y(\overline{1, k})$.

DKP algebras -past results

1. Mériaux and Cauchon 2009 classified the H-primes of $\mathcal{U}_-^w$ without poset structure (milder assumptions on the ground filed), earlier Cauchon 2003 did the case of quantum matrices.

2. Launois 2007 described the poset of H-primes of quantum matrices, influential work of Goodearl and Lenagan 2001 on what it could look like.

3. Only explicit formulas for ideal generators of H-primes of 3×3 quantum matrices Goodearl–Lenagan 2001, simultaneously Goodearl–Launois–Lenagan and Casteels obtained generating sets for H-primes in the case of quantum matrices.

4. Garrett Johnson (UCSB) is working out a complete treatment of the H-spectra of the algebras of symmetric and antisymmetric matrices.

The Goodearl–Lenagan conjecture I

An ideal I of R has a polynormal generating sequence $y_1, \ldots, y_k$ if the set generates I and for all $i = 1, \ldots, k$ the image of y_i in $R/\langle y_1, \ldots, y_{i-1}\rangle$ is normal.

The Goodearl–Lenagan conjecture I

An ideal I of R has a polynormal generating sequence $y_1, \ldots, y_k$ if the set generates I and for all $i = 1, \ldots, k$ the image of y_i in $R/\langle y_1, \ldots, y_{i-1}\rangle$ is normal.

Goodearl–Lenagan Conjecture. All H-primes of $R_q[M_{m,n}]$ have polynormal generating sequences consisting of quantum minors.

The Goodearl–Lenagan conjecture I

An ideal I of R has a polynormal generating sequence $y_1, \ldots, y_k$ if the set generates I and for all $i = 1, \ldots, k$ the image of y_i in $R/\langle y_1, \ldots, y_{i-1}\rangle$ is normal.

Goodearl–Lenagan Conjecture. All H-primes of $R_q[M_{m,n}]$ have polynormal generating sequences consisting of quantum minors.

The standard R-matrix identities in $R_q[G]$ imply

$$d^{w,\lambda_1}_{\eta_1} d^{w,\lambda_2}_{\eta_2} = q^{\langle \eta_1 - w\lambda_1, \eta_2 + w\lambda_2 \rangle} d^{w,\lambda_2}_{\eta_2} d^{w,\lambda_1}_{\eta_1} + \sum_{\alpha \in Q_+, \alpha \neq 0} d^{w,\lambda_2}_{u_\alpha \eta_2} d^{w,\lambda_1}_{u_{-\alpha}\eta_1}, \quad \eta_i \in (V(\lambda_i)_w)^*$$

where $u_{\pm\alpha} \in (\mathcal{U}_\pm)_{\pm\alpha}$.

If $\eta \in (V_w(\lambda_i))^*_\mu$ set $ht(\eta) = \langle \mu, \omega_1^\vee + \ldots + \omega_r^\vee \rangle$.

The Goodearl–Lenagan conjecture I

An ideal I of R has a polynormal generating sequence $y_1, \dots, y_k$ if the set generates I and for all $i = 1, \dots, k$ the image of y_i in $R/\langle y_1, \dots, y_{i-1}\rangle$ is normal.

Goodearl–Lenagan Conjecture. All H-primes of $R_q[M_{m,n}]$ have polynormal generating sequences consisting of quantum minors.

The standard R-matrix identities in $R_q[G]$ imply

$$d^{w,\lambda_1}_{\eta_1} d^{w,\lambda_2}_{\eta_2} = q^{\langle \eta_1 - w\lambda_1, \eta_2 + w\lambda_2 \rangle} d^{w,\lambda_2}_{\eta_2} d^{w,\lambda_1}_{\eta_1} + \sum_{\alpha \in Q_+, \alpha \neq 0} d^{w,\lambda_2}_{u_\alpha \eta_2} d^{w,\lambda_1}_{u_{-\alpha}\eta_1}, \quad \eta_i \in (V(\lambda_i)_w)^*$$

where $u_{\pm\alpha} \in (\mathcal{U}_\pm)_{\pm\alpha}$.

If $\eta \in (V_w(\lambda_i))^*_\mu$ set $ht(\eta) = \langle \mu, \omega_1^\vee + \ldots + \omega_r^\vee \rangle$.

Theorem. [Y.] Fix an H-prime $I_y(w)$ of $\mathcal{U}_-^w$, $y \in W^{\leq w}$. Consider any linear ordering of the generating set from the previous theorem with the property that, if $\eta_1, \eta_2 \in (V(\omega_k)_w)^*$ and $ht(\eta_1) \leq ht(\eta_2)$, then $(d^{w,\omega_k}_{\eta_1} \otimes \mathrm{id})(\mathcal{R}^w)$ comes before $(d^{w,\omega_k}_{\eta_2} \otimes \mathrm{id})(\mathcal{R}^w)$. Any such sequence is a polynormal generating set of $I_y(w)$.

The Goodearl–Lenagan conjecture II

We obtain the following constructive proof of the Goodearl–Lenagan conjecture:

Corollary. Consider the $\mathbb{Z}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ corresponding to $y \in S_{m+n}^{\leq w_{m,n}}$ and a linear order on the generating set from the previous theorem with the property that, if $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$ satify $i_1 + \ldots + i_k \leq j_1 + \ldots + j_k$, then Δ_I comes before Δ_J. Any such sequence is a polynormal generating set of the prime ideal.

The Goodearl–Lenagan conjecture II

We obtain the following constructive proof of the Goodearl–Lenagan conjecture:

Corollary. Consider the $\mathbb{Z}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ corresponding to $y \in S_{m+n}^{\leq w_{m,n}}$ and a linear order on the generating set from the previous theorem with the property that, if $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$ satify $i_1 + \dots + i_k \leq j_1 + \dots + j_k$, then Δ_I comes before Δ_J. Any such sequence is a polynormal generating set of the prime ideal.

Corollary. The H-primes of $\mathcal{U}_-^w$ are graded normally separated.

Let $y_1 < y_2 \leq w$, i.e. $I_w(y_1) \subset I_w(y_2)$. Then for some k, $y_1\omega_k \neq y_2\omega_k$. Set $\xi = T_{y_1}\xi_\lambda$ where ξ_λ is the dual vector to the h.w.v. v_λ. Then

$$\xi \in (\mathcal{U}_- T_{y_2} v_{\omega_k})^\perp, \quad \xi \notin (\mathcal{U}_- T_{y_1} v_{\omega_k} \cap \mathcal{U}_+ T_w v_{\omega_k})^\perp$$

By iteratively changing $\xi \mapsto X_i^- \xi$ one can insure that in addition

$$X_i^- \xi \in (\mathcal{U}_- T_{y_1} v_{\omega_k} \cap \mathcal{U}_+ T_w v_{\omega_k})^\perp \quad \forall i = 1, \dots, r.$$

$\mathrm{Spec}\mathcal{U}_{-}^{w}$ is normally separated

Then

$$(d_{\xi}^{w,\omega_k} \otimes \mathrm{id})(\mathcal{R}^w) + I_w(y_1)$$

is a nonzero normal element in $\mathcal{U}_{-}^{w}/I_w(y_1)$ that belongs to $I_w(y_2)/I_w(y_1)$.

$\mathrm{Spec}\mathcal{U}_-^w$ is normally separated

Then

$$(d_\xi^{w,\omega_k} \otimes \mathrm{id})(\mathcal{R}^w) + I_w(y_1)$$

is a nonzero normal element in $\mathcal{U}_-^w/I_w(y_1)$ that belongs to $I_w(y_2)/I_w(y_1)$.

Theorem [Goodearl]. Assume that R is right noetherian. If $H - \mathrm{Spec}R$ is graded normally separated then $\mathrm{Spec}R$ is normally separated.

$\mathrm{Spec}\mathcal{U}_-^w$ is normally separated

Then

$$(d_\xi^{w,\omega_k} \otimes \mathrm{id})(\mathcal{R}^w) + I_w(y_1)$$

is a nonzero normal element in $\mathcal{U}_-^w/I_w(y_1)$ that belongs to $I_w(y_2)/I_w(y_1)$.

Theorem [Goodearl]. Assume that R is right noetherian. If $H - \mathrm{Spec}R$ is graded normally separated then $\mathrm{Spec}R$ is normally separated.

Theorem. $\mathrm{Spec}\mathcal{U}_-^w$ is normally separated.

Dimensions of strata of primes I

Recall that the stratum of $\mathrm{Spec}\mathcal{U}_-^w$ over each H-prime $I_y(w)$ in the Goodearl–Letzter stratification is homeomorphic to the spectrum of a Laurent polynomial ring. The latter is the center of the localization of $\mathcal{U}_-^w/I_y(w)$ by all nonzero homogeneous elements.

Dimensions of strata of primes I

Recall that the stratum of $\mathrm{Spec}\mathcal{U}_-^w$ over each H-prime $I_y(w)$ in the Goodearl–Letzter stratification is homeomorphic to the spectrum of a Laurent polynomial ring. The latter is the center of the localization of $\mathcal{U}_-^w/I_y(w)$ by all nonzero homogeneous elements.

Theorem [Bell–Casteels–Launois, Y.]. The dimension of the the Goodearl–Letzter stratum of $\mathrm{Spec}\mathcal{U}_-^w$ over the H-prime $I_y(w)$ is equal to

$$\dim \ker(1 + y^{-1}w) = \dim E_{-1}(y^{-1}w).$$

Dimensions of strata of primes I

Recall that the stratum of $\mathrm{Spec}\mathcal{U}^w_-$ over each H-prime $I_y(w)$ in the Goodearl–Letzter stratification is homeomorphic to the spectrum of a Laurent polynomial ring. The latter is the center of the localization of $\mathcal{U}^w_-/I_y(w)$ by all nonzero homogeneous elements.

Theorem [Bell–Casteels–Launois, Y.]. The dimension of the the Goodearl–Letzter stratum of $\mathrm{Spec}\mathcal{U}^w_-$ over the H-prime $I_y(w)$ is equal to

$$\dim \ker(1 + y^{-1}w) = \dim E_{-1}(y^{-1}w).$$

Denote the dual vector to the h.w.v. v_λ of $V_w(\lambda)$ by ξ_λ.

Fix $y \in W^{\leq w}$. For $\lambda \in P_+$ denote $a_\lambda = d^{w,\lambda}_{T_y \xi_\lambda}$. For $\lambda \in P$, $\lambda = \lambda_+ - \lambda_-$, $\lambda_\pm \in P_+$ (non-intersecting support) set

$$a_\lambda = (a_{\lambda_+})^{-1} a_{\lambda_-}.$$

Dimensions of strata of primes I

Recall that the stratum of $\mathrm{Spec}\mathcal{U}_-^w$ over each H-prime $I_y(w)$ in the Goodearl–Letzter stratification is homeomorphic to the spectrum of a Laurent polynomial ring. The latter is the center of the localization of $\mathcal{U}_-^w/I_y(w)$ by all nonzero homogeneous elements.

Theorem [Bell–Casteels–Launois, Y.]. The dimension of the the Goodearl–Letzter stratum of $\mathrm{Spec}\mathcal{U}_-^w$ over the H-prime $I_y(w)$ is equal to

$$\dim \ker(1 + y^{-1}w) = \dim E_{-1}(y^{-1}w).$$

Denote the dual vector to the h.w.v. v_λ of $V_w(\lambda)$ by ξ_λ.

Fix $y \in W^{\leq w}$. For $\lambda \in P_+$ denote $a_\lambda = d^{w,\lambda}_{T_y \xi_\lambda}$. For $\lambda \in P$, $\lambda = \lambda_+ - \lambda_-$, $\lambda_\pm \in P_+$ (non-intersecting support) set

$$a_\lambda = (a_{\lambda_+})^{-1} a_{\lambda_-}.$$

Then

$$a_\lambda d^{w,\mu}_\xi = q^{-\langle (y+w)\lambda, \nu - w\mu \rangle} d^{w,\mu}_\xi a_\lambda, \quad \forall \xi (\in V_w(\mu))^*_\nu$$

in $(\mathcal{U}_-^w/I_y(w))[a_\lambda^{-1}, \lambda \in P_+]$.

Dimensions of strata of primes II

Therefore the center of the localization of $\mathcal{U}_-^w / I_y(w)$ by all nonzero homogeneous elements contains the Laurent polynomial ring spanned by

$$a_\lambda, \quad \lambda \in P_+, (y + w)\lambda = 0.$$

Thus the stratum of $\mathrm{Spec}\mathcal{U}_-^w$ over $I_y(w)$ has dimension at least

$$\dim \ker(1 + y^{-1}w) = \dim E_{-1}(y^{-1}w).$$

Dimensions of strata of primes II

Therefore the center of the localization of $\mathcal{U}_-^w / I_y(w)$ by all nonzero homogeneous elements contains the Laurent polynomial ring spanned by

$$a_\lambda, \quad \lambda \in P_+, (y+w)\lambda = 0.$$

Thus the stratum of $\mathrm{Spec}\mathcal{U}_-^w$ over $I_y(w)$ has dimension at least

$$\dim \ker(1 + y^{-1}w) = \dim E_{-1}(y^{-1}w).$$

If its dimension is greater, then we pass to an integral form of the algebra over $\mathbb{Z}[q, q^{-1}]$ and specialize at $q = 1$. That would imply that the center of the Poisson field of rational functions on the open Richardson variety $R_{y,w}$ has trascendence degree strictly greater than

$$\dim \ker(1 + y^{-1}w) = \dim E_{-1}(y^{-1}w)$$

which is a contradiction.

Quantum partial flag varieties I

Choose a set of simple roots $I \subset \overline{1,r}$ and consider the standard parabolic subgroup $P_I \supset B_+$. Consider the multicone:

$$\mathrm{Spec}\Big(\bigoplus_{n_i \in \mathbb{Z}_{\geq 0}} H^0(G/P_I, \otimes_{i \notin I} \mathcal{L}_{\omega_i}^{n_i})\Big)$$

over G/P_I. Its coordinate ring is quantized to the subalgebra $R_q[G/P_I]$ of the restricted dual of $\mathcal{U}_q(\mathfrak{g})$ spanned by the matrix coefficients

$$c^{\lambda}_{\xi, v_\lambda}, \quad \lambda = \sum_{i \notin I} n_i \omega_i, n_i \in \mathbb{Z}_{\geq 0}, \xi \in V(\lambda)^*, v_\lambda - \text{h.w.v. of } V(\lambda).$$

The construction is due to Lakshmibai–Reshetikhin and Soibelman.

Quantum partial flag varieties I

Choose a set of simple roots $I \subset \overline{1,r}$ and consider the standard parabolic subgroup $P_I \supset B_+$. Consider the multicone:

$$\mathrm{Spec}\Big(\bigoplus_{n_i \in \mathbb{Z}_{\geq 0}} H^0(G/P_I, \otimes_{i \notin I} \mathcal{L}_{\omega_i}^{n_i}) \Big)$$

over G/P_I. Its coordinate ring is quantized to the subalgebra $R_q[G/P_I]$ of the restricted dual of $\mathcal{U}_q(\mathfrak{g})$ spanned by the matrix coefficients

$$c^{\lambda}_{\xi, v_\lambda}, \quad \lambda = \sum_{i \notin I} n_i \omega_i, n_i \in \mathbb{Z}_{\geq 0}, \xi \in V(\lambda)^*, v_\lambda - \text{h.w.v. of } V(\lambda).$$

The construction is due to Lakshmibai–Reshetikhin and Soibelman.

Problem. Classify the H-invariant prime ideals of $R_q[G/P_I]$ not containing the augmentation ideal.

Only two cases were previously known: full flag varieties Gorelik J. Algebra 2000, and Grassmannians Launois–Lenagan–Rigal Selecta Math. 2008.

Quantum partial flag varieties II

Denote by $H - \mathrm{Spec}_+(R_q[G/P_I])$ the set of H-invariant prime ideals of $R_q[G/P_I]$ not containing the augmentation ideal. Denote the quantum Schubert cell ideals:

$$Q(w)_I^+ = \mathrm{Span}\{c^{\lambda}_{\xi, v_\lambda} \mid \lambda = \sum_{i \notin I} n_i \omega_i, \xi \in V(\lambda)^*, \, \xi \perp \mathcal{U}_+ T_w v_\lambda\}, \quad w \in W^{W_I}.$$

They are $\mathcal{U}_q(\mathfrak{b}_+)$ invariant prime ideals of $R_q[G/P_I]$.

Quantum partial flag varieties II

Denote by $H-\mathrm{Spec}_+(R_q[G/P_I])$ the set of H-invariant prime ideals of $R_q[G/P_I]$ not containing the augmentation ideal. Denote the quantum Schubert cell ideals:

$$Q(w)_I^+ = \mathrm{Span}\{c^{\lambda}_{\xi, v_\lambda} \mid \lambda = \sum_{i \notin I} n_i \omega_i, \xi \in V(\lambda)^*, \, \xi \perp \mathcal{U}_+ T_w v_\lambda\}, \quad w \in W^{W_I}.$$

They are $\mathcal{U}_q(\mathfrak{b}_+)$ invariant prime ideals of $R_q[G/P_I]$.

We have the decomposition:

$$H-\mathrm{Spec}_+(R_q[G/P_I]) = \sqcup_{w \in W^{W_I}} X_I^w$$

according to the largest quantum Schubert cell ideal contained in $\mathcal{I} \in H-\mathrm{Spec}_+(R_q[G/P_I])$.

Quantum partial flag varieties II

Denote by $H - \mathrm{Spec}_+(R_q[G/P_I])$ the set of H-invariant prime ideals of $R_q[G/P_I]$ not containing the augmentation ideal. Denote the quantum Schubert cell ideals:

$$Q(w)_I^+ = \mathrm{Span}\{c^\lambda_{\xi, v_\lambda} \mid \lambda = \sum_{i \notin I} n_i \omega_i, \xi \in V(\lambda)^*, \, \xi \perp \mathcal{U}_+ T_w v_\lambda\}, \quad w \in W^{W_I}.$$

They are $\mathcal{U}_q(\mathfrak{b}_+)$ invariant prime ideals of $R_q[G/P_I]$.

We have the decomposition:

$$H - \mathrm{Spec}_+(R_q[G/P_I]) = \sqcup_{w \in W^{W_I}} X_I^w$$

according to the largest quantum Schubert cell ideal contained in $\mathcal{I} \in H - \mathrm{Spec}_+(R_q[G/P_I])$.

Denote $c^\lambda_w = c^\lambda_{T_w \xi_\lambda, v_\lambda}, c^I_w = \{c^\lambda_w \mid \lambda = \sum_{i \notin I} n_i \omega_i\}$.

Proposition. For all $w \in W^{W_I}$ the algebras

$$\big(\big(R_q[G/P_I]/Q(w)_I^+\big)[(c^I_w)^{-1}]\big)^H \quad \text{and} \quad \mathcal{U}_-^w$$

are isomorphic and for each $\mathcal{I} \in X_I^w$, $\mathcal{I} \cap c^I_w = \emptyset$. (Similar strategy to the one for the isomorphism between the 2 realizations of DKP algebras.)

Quantum partial flag varieties III

Theorem. [Y.] For an arbitrary partial flag variety G/P_I the H-invariant prime ideals of $R_q[G/P_I]$ (not containing the augmentation ideal) are parametrized by

$$\{(y_-, y_+) \in W \times W^{W_I} \mid y_- \leq y_+\}.$$

Denote by $\mathcal{I}^I_{y_-, y_+}$ the ideal corresponding to (y_-, y_+).

Quantum partial flag varieties III

Theorem. [Y.] For an arbitrary partial flag variety G/P_I the H-invariant prime ideals of $R_q[G/P_I]$ (not containing the augmentation ideal) are parametrized by

$$\{(y_-, y_+) \in W \times W^{W_I} \mid y_- \leq y_+\}.$$

Denote by $\mathcal{I}^I_{y_-, y_+}$ the ideal corresponding to (y_-, y_+).

Conjecture. Let $y_-, y'_- \in W$, $y_+, y'_+ \in W^{W_I}$, $y_- \leq y_+$, $y'_- \leq y'_+$. Then $\mathcal{I}^I_{y_-, y_+} \subseteq \mathcal{I}^I_{y'_-, y'_+}$ if and only if there exits $z \in W_I$ such that

$$y_- \geq y'_- z \text{ and } y_+ \leq y'_+ z.$$

Happy Birthday Ken!