Homotopy Theory in Algebraic Derived Categories

by

Mohammed Ahmed Musa Al Shumrani

A thesis submitted to the

Faculty of Information and Mathematical Sciences

at the University of Glasgow

for the degree of

Doctor of Philosophy

July 2006

Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow. I declare this thesis is entirely my own composition and no part of it has been submitted by me for any degree at any other university.

Acknowledgement

First I would like to thank Allah for giving me the health and the zeal and helping me to accomplish this thesis.

I would like to express my deepest thanks to my supervisor, Dr. Andrew Baker, for suggesting the topics studied in this thesis and for his assistance, guidance and encouragement during my study.

Many people have assisted in many ways during my study at Glasgow. In particular, I would like to thank my colleagues, Peter Thompson, Maurizio Martino, and Philipp Reinhard.

An acknowledgement must go to my great parents. Special thanks are due to my brothers and sisters. They have been a great help to me.

I would like to thank King Abdulaziz University for financing my research. Also, I would like to thank Mathematics Department in Glasgow University for its generous support. It has supported me to attend many conferences in my research area.

Finally, I would like to thank my wife and sons for their patience and encouragement during my stay in Glasgow.

Abstract

In this thesis, we introduce some new notions in the derived category $\mathcal{D}_{+(fg)}(R)$ of bounded below chain complexes of finite type over local commutative noetherian ring R with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{K}$ in chapter three and study their relations to each other. Also, we set up the Adams spectral sequence for chain complexes in $\mathcal{D}_{+(fg)}(R)$ in chapter four and study its convergence.

To accomplish this task, we give two background chapters. We give some good account of chain complexes in chapter one. We review some basic homological algebra and give definition and basic properties of chain complexes. Then we study the homotopy category of chain complexes and we end chapter one with section about spectral sequences.

Chapter two is about the derived category of a commutative ring. Section one is about localization of categories and left and right fractions. Then in section two, we give definition of triangulated categories and some of its basic properties and we end section two with definitions of homotopy limits and colimits. In section three, we show that the derived category is a triangulated category. In section four, we give definitions of the derived functors, the derived tensor product and the derived Hom.

In chapter three, we start section one by giving some facts about local rings and we end this section by showing that every bounded below chain complex of finite type has a minimal free resolution. In section two, we show a derived analog of the Whitehead Theorem. In section three, we construct Postnikov towers for chain complexes. In section four, we define the Steenrod algebra. In section five, six and seven, we define irreducible, atomic, minimal atomic, no mod $\mathfrak{m}$ detectable homology, $H^{\star}$-monogenic, nuclear chain complexes and the core of a chain complex. We show some various results relating these notions to each other and give some examples.

In chapter four, we set up the Adams spectral sequence in section one and study its properties. In section two, we study homology localization and local homology. In section three, we define $\mathbb{K}[0]$-nilpotent completion and we show that the

Adams spectral sequence for a chain complex Y converges strongly to the homology of the $\mathbb{K}[0]$-nilpotent completion of Y. In section four, we study the Adams spectral sequence's convergence where we show that the $\mathbb{K}[0]$-nilpotent completion for a bounded chain complex Y consisting of finitely generated free R-modules in each degree is isomorphic to the localization of Y with respect to the $H_\star(-, \mathbb{K})$-theory. In section five, we present some examples.

Contents

Chapter 1

Chain Complexes

In this chapter, we state and define some basic notions that are necessary for understanding what comes later. We give some preliminaries on chain complexes where the main references for these basic materials are [33], [30] and [19]. We start in section one by recalling some facts from commutative ring theory and basic notions of homological algebra. In section two, we give the definition of chain complexes and basic properties of them and then the definition of the category of chain complexes. In section three, we give the definition of the homotopy category of chain complexes and its some basic properties. In section four, we give the definition of spectral sequences and explain the convergence of spectral sequences.

Throughout this chapter and the following chapters, let R be an arbitrary commutative ring with identity.

1.1 Basic Homological Algebra

In this section, we review some basic definitions and facts from homological algebra and commutative ring theory. The main references for this section are [21], [30], [23] and [19].

Definition 1.1.1. An R-module M is *free* if it is a sum of copies of R.

Definition 1.1.2. An R-module P is called *projective* if in each diagram of R-

modules of the following form

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "f"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

with g an epimorphism, then there exists $h\colon P \longrightarrow M$ such that $gh = f$.

The following result is proved in [19, Lemma 5.4].

Lemma 1.1.3. *Every free R-module is projective.*

Definition 1.1.4. An R-module E is *injective* if for every R-module N and every submodule M of N, every $f\colon M \longrightarrow E$ can be extended to a map $g\colon N \longrightarrow E$. The diagram is

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "f"] \arrow[r] & N \arrow[ul, dotted, "g"'] \end{tikzcd}$$

Definition 1.1.5. An R-module F is *flat* if the functor $F \otimes_R -\colon$ R-**mod** $\longrightarrow$ R-**mod** where R-**mod** is the category of R-modules is exact.

The proof of the following result is in [30, Corollary 3.46].

Lemma 1.1.6. *Every projective R-module is flat.*

Definition 1.1.7. A *free (projective) resolution* of an R-module M is an exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

in which each P_n is a free (projective) module.

A proof of the following theorem is in [30, Theorem 3.8]

Theorem 1.1.8. *Every an R-module M has a free (projective) resolution.*

Definition 1.1.9. An *injective resolution* of an R-module M is an exact sequence

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow E^{n+1} \longrightarrow \cdots$$

in which each E^n is an injective R-module.

The following result is proved in [30, Theorem 3.28].

Theorem 1.1.10. *Every an R-module M has an injective resolution.*

If we suppress M from a projective resolution for M, then we get a *deleted projrctive* resolution for M. Similarly, if we suppress M from an injective resolution for M, then we get a *deleted injective* resolution for M.

The following theorem is proved in [19, Theorem 6.3].

Theorem 1.1.11. *The following properties of an R-module P are equivalent.*

(i) *P is projective.*

(ii) *For each epimorphism $f\colon M \longrightarrow N$,*

$$f_\star\colon \operatorname{Hom}_R(P, M) \longrightarrow \operatorname{Hom}_R(P, N)$$

is an epimorphism.

(iii) *If*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence, so is

$$0 \longrightarrow \operatorname{Hom}_R(P, L) \longrightarrow \operatorname{Hom}_R(P, M) \longrightarrow \operatorname{Hom}_R(P, N) \longrightarrow 0.$$

(iv) *Every short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$$

splits.

The following result is proved in [30, Theorem 3.16].

Theorem 1.1.12. *An R-module E is injective if and only if the functor* $\operatorname{Hom}_R(-, E)$ *is exact.*

The following important result is proved in [30, Theorem 3.6].

Theorem 1.1.13. *Let M be a finitely generated R-module. If $f\colon M \longrightarrow M$ is surjective, then f is also injective, and is thus an automorphism of M.*

Definition 1.1.14. R is *noetherian* if every ideal is finitely generated.

The following theorem is proved in [30, Theorem 4.1].

Theorem 1.1.15. *R is noetherian if and only if every submodule of a finitely generated R-module M is also finitely generated.*

The proof of the following lemma is straightforward.

Lemma 1.1.16. *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X' \arrow[d, "h'"] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

be a commutative diagram of R-modules. If f is an epimorphism and $h = 0$, then $h' = 0$. If g is a monomorphism and $h' = 0$, then $h = 0$.

Before we end this section, we give the definition of Hopf algebra and some results which we will need later.

Definition 1.1.17. An *algebra* over R is a graded R-module A together with homomorphisms of graded R-modules $\phi\colon A \otimes_R A \longrightarrow A$ and $\eta\colon R \longrightarrow A$ such that the following diagrams

$$\begin{tikzcd} A \otimes_R A \otimes_R A \arrow[r, "\mathrm{id} \otimes \phi"] \arrow[d, "\phi \otimes \mathrm{id}"'] & A \otimes_R A \arrow[d, "\phi"] \\ A \otimes_R A \arrow[r, "\phi"] & A \end{tikzcd}$$

and

$$\begin{tikzcd} A \otimes_R R \arrow[r, "\mathrm{id} \otimes \eta"] \arrow[d, "\cong"'] & A \otimes_R A \arrow[d, "\phi"] & R \otimes_R A \arrow[l, "\eta \otimes \mathrm{id}"'] \arrow[d, "\cong"'] \\ A \arrow[r, "\mathrm{id}"] & A & A \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$

are commutative. The homomorphism ϕ is called the *product* of the algebra A and η is called the *unit* of A. The algebra A is commutative if the following diagram

$$\begin{tikzcd} A \otimes_R A \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & A \otimes_R A \arrow[dl, "\phi"] \\ & A & \end{tikzcd}$$

is commutative where τ is the *twist* homomorphism such that for $a \in A_p$ and $b \in A_q$,

$$\tau(a \otimes b) = (-1)^{pq} b \otimes a.$$

Definition 1.1.18. If A is an algebra over R, a *left A-module* is a graded R-module N together with a A-action, that is, a homomorphism $\phi_N\colon A\otimes_R N \longrightarrow N$ of graded R-modules such that the following diagrams

$$\begin{tikzcd}
A \otimes_R A \otimes_R N \arrow[r, "\mathrm{id} \otimes \phi_N"] \arrow[d, "\phi \otimes \mathrm{id}"'] & A \otimes_R N \arrow[d, "\phi_N"'] \\
A \otimes_R N \arrow[r, "\phi_N"] & N
\end{tikzcd}$$

$$\begin{tikzcd}
R \otimes_R N \arrow[rr, "\eta \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & & A \otimes_R N \arrow[dl, "\phi_N"] \\
& N &
\end{tikzcd}$$

are commutative.

Definition 1.1.19. A *coalgebra* over R is a graded R-module A together with homomorphisms of graded R-modules $\psi\colon A \longrightarrow A\otimes_R A$ and $\epsilon\colon A \longrightarrow R$ such that the following diagrams

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "\psi"'] & A \otimes_R A \arrow[d, "\psi \otimes \mathrm{id}"'] \\
A \otimes_R A \arrow[r, "\mathrm{id} \otimes \psi"] & A \otimes_R A \otimes_R A
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes_R R & A \otimes_R A \arrow[l, "\mathrm{id} \otimes \epsilon"'] \arrow[r, "\epsilon \otimes \mathrm{id}"] & R \otimes_R A \\
A \arrow[u, "\cong"'] & A \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] \arrow[u, "\psi"'] & A \arrow[u, "\cong"']
\end{tikzcd}$$

are commutative. The homomorphism ψ is called the *coproduct* of the coalgebra A and ϵ is called the *counit* of A. The coalgebra A is cocommutative if the following diagram

$$\begin{tikzcd}
A \otimes_R A & & A \otimes_R A \arrow[ll, "\tau"'] \\
& A \arrow[ul, "\psi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

is commutative.

Definition 1.1.20. If A is a coalgebra over R, a *left A-comodule* is a graded R-module N with a A-coaction, that is, a homomorphism $\psi_N\colon N \longrightarrow A\otimes_R N$ of graded

R-modules such that the following diagrams

$$\begin{tikzcd}
N \arrow[r, "\psi_N"] \arrow[d, "\psi_N"'] & A \otimes_R N \arrow[d, "\psi \otimes \mathrm{id}"] \\
A \otimes_R N \arrow[r, "\mathrm{id} \otimes \psi_N"] & A \otimes_R A \otimes_R N
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes_R N \arrow[rr, "\epsilon \otimes \mathrm{id}"] & & R \otimes_R N \\
& N \arrow[ul, "\psi_N"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

are commutative.

Definition 1.1.21. A *bialgebra* A over R is an algebra and a coalgebra over R, such that the coproduct and the counit are both algebra homomorphisms. Equivalently, one may require that the product and the unit of the algebra both be coalgebra homomorphisms. The compatibility conditions can also be expressed by the following commutative diagrams:

$$\begin{tikzcd}
R \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta"'] & & R \\
& A \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes_R A \arrow[r, "\phi"] \arrow[d, "\psi \otimes \psi"'] & A \arrow[r, "\psi"] & A \otimes_R A \\
A \otimes_R A \otimes_R A \otimes_R A \arrow[rr, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] & & A \otimes_R A \otimes_R A \otimes_R A \arrow[u, "\phi \otimes \phi"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes_R A \arrow[rr, "\phi"] \arrow[dr, "\epsilon \otimes \epsilon"'] & & A \arrow[dl, "\epsilon"] \\
& R \otimes_R R \cong R &
\end{tikzcd}$$

$$\begin{tikzcd}
& R \cong R \otimes_R R \arrow[dl, "\eta"'] \arrow[dr, "\eta \otimes \eta"] & \\
A \arrow[rr, "\psi"] & & A \otimes_R A
\end{tikzcd}$$

Definition 1.1.22. A *Hopf algebra* is a bialgebra A over R together with a R-module homomorphism $c\colon A \longrightarrow A$, called the *antipode*, such that the following diagram

$$\begin{tikzcd}
A \otimes_R A \arrow[rr, "c\otimes\mathrm{id}"] & & A \otimes_R A \arrow[d, "\phi"] \\
A \arrow[u, "\psi"] \arrow[d, "\psi"'] \arrow[r, "\epsilon"] & R \arrow[r, "\eta"] & A \\
A \otimes_R A \arrow[rr, "\mathrm{id}\otimes c"] & & A \otimes_R A \arrow[u, "\phi"']
\end{tikzcd}$$

is commutative.

If A is a graded R-module, we denote by $A^\star$ the graded R-module such that $A_n^\star = \mathrm{Hom}_R(A_n, R)$. If $f\colon A \longrightarrow B$ is a homomorphism of graded R-modules, then $f^\star\colon B^\star \longrightarrow A^\star$ is the homomorphism of graded R-modules such that $f_n^\star = \mathrm{Hom}(f_n, \mathrm{id})$.

A graded R-module A is of *finite type* if each A_n is a finitely generated R-module. It is projective if each A_n is projective.

Theorem 1.1.23. *Suppose that A is a graded R-module which is projective of finite type, then*

(i) *$\phi\colon A \otimes_R A \longrightarrow A$ is a product in A if and only if $\phi^\star\colon A^\star \longrightarrow A^\star \otimes_R A^\star$ is a coproduct in $A^\star$,*

(ii) *$\eta\colon R \longrightarrow A$ is a unit for the product ϕ if and only if $\eta^\star\colon A^\star \longrightarrow R^\star = R$ is a counit for the coproduct $\phi^\star$,*

(iii) *(A, ϕ, η) is an algebra if and only if $(A^\star, \phi^\star, \eta^\star)$ is a coalgebra,*

(iv) *the algebra (A, ϕ, η) is commutative if and only if the coalgebra $(A^\star, \phi^\star, \eta^\star)$ is cocommutative.*

For the proof of the above theorem, see [23, Proposition 3.1].

Theorem 1.1.24. *Suppose (A, ϕ, η) is an algebra over R such that the graded R-module A is projective of finite type. If N is a graded R-module which is projective of finite type, then $\phi_N\colon A \otimes_R N \longrightarrow N$ defines the structure of a left A-module on N if and only if $\phi_{N^\star}\colon N^\star \longrightarrow A^\star \otimes_R N^\star$ defines the structure of a left $A^\star$-comodule on $N^\star$.*

For the proof of the above theorem, see [23, Proposition 3.2].

Theorem 1.1.25. *If A is a graded projective R-module of finite type, then $(A, \phi, \eta, \psi, \epsilon, c)$ is a Hopf algebra with product ϕ, coproduct ψ, unit η, counit ϵ and antipode c if and only if $(A^\star, \psi^\star, \epsilon^\star, \phi^\star, \eta^\star, c^\star)$ is a Hopf algebra with product $\psi^\star$, coproduct $\phi^\star$, unit $\epsilon^\star$, counit $\eta^\star$ and antipode $c^\star$.*

For the proof of the above theorem, see [23, Proposition 4.8].

1.2 Definition and Elementary Properties of Chain Complexes

In this section, we present several basic definitions and basic properties of chain complexes. The main references for this section are [30] and [33].

Definition 1.2.1. A *chain complex* $Y_\star$ of R-modules is a sequence of R-modules and R-module maps

$$Y_\star = \cdots \longrightarrow Y_{n+1} \xrightarrow{d_{n+1}} Y_n \xrightarrow{d_n} Y_{n-1} \longrightarrow \cdots$$

where $n \in \mathbb{Z}$ and $d_n d_{n+1} = 0$ for all n. The maps d_n are called the *differentials*. The elements of $\operatorname{Ker} d_n$ are called *n-cycles*, denoted $Z_n(Y_\star) = Z_n$ and the elements of $\operatorname{Im} d_{n+1}$ are called *n-boundaries*, denoted $B_n(Y_\star) = B_n$. Note that the condition $d_n d_{n+1} = 0$ means $\operatorname{Im} d_{n+1} \subset \operatorname{Ker} d_n$. So

$$0 \subseteq B_n \subseteq Z_n \subseteq Y_n$$

for all n. The *nth homology module* of $Y_\star$ is

$$H_n(Y_\star) = Z_n / B_n.$$

Definition 1.2.2. Dually, a *cochain complex* $Y^\star$ of R-modules is a sequence of R-modules and R-module maps

$$Y^\star = \cdots \longrightarrow Y^{n-1} \xrightarrow{d_{n-1}} Y^n \xrightarrow{d_n} Y^{n+1} \longrightarrow \cdots$$

where $n \in \mathbb{Z}$ and $d_n d_{n-1} = 0$ for all n. The elements of $\operatorname{Ker} d_n$ are called *n-cocycles*, denoted $Z^n(Y^\star) = Z^n$ and the elements of $\operatorname{Im} d_{n-1}$ are called *n-coboundaries*, denoted $B^n(Y^\star) = B^n$. Note that the condition $d_n d_{n-1} = 0$ means $\operatorname{Im} d_{n-1} \subset \operatorname{Ker} d_n$. So

$$0 \subseteq B^n \subseteq Z^n \subseteq Y^n$$

for all n. The *nth cohomology module* of $Y^\star$ is

$$H^n(Y^\star) = Z^n/B^n.$$

Throughout this chapter and the following chapters, we will omit the subscript and superscript and write Y for $Y_\star$ and $Y^\star$.

The chain complex Y is called *exact* at Y_n if $Z_n = B_n$, that is, $H_n(Y) = 0$. If $H_n(Y) = 0$ for each n, we say that Y is *acyclic*.

Definition 1.2.3. We say that a chain complex Y is *connective* if $H_i(Y) = 0$ for all $i < 0$.

Definition 1.2.4. We say that a chain complex Y is *of finite type* if it has finitely generated homology in each degree.

Definition 1.2.5. Let M and N be R-modules. Then

$$\operatorname{Tor}_n^R(M, N) = H_n(P_M \otimes_R N) = H_n(M \otimes_R Q_N),$$

where P_M is a deleted projective resolution of M and Q_N is a deleted projective resolution of N.

The following result is proved in [30, Theorem 8.7].

Theorem 1.2.6. *If F is flat, then $\operatorname{Tor}_n^R(F, N) = 0$ for all $n \geq 1$ and all R-modules N and similarly in the other variable.*

Definition 1.2.7. Let M and N be R-modules. Then

$$\operatorname{Ext}_R^n(M, N) = H^n(\operatorname{Hom}_R(M, E_N)) = H^n(\operatorname{Hom}_R(P_M, N)),$$

where E_N is a deleted injective resolution of N and P_M is a deleted projective resolution of M.

The following result is proved in [30, Theorem 7.6]

Theorem 1.2.8. *If N is an injective R-module, then $\operatorname{Ext}^n_R(M, N) = 0$ for all R-modules M and all $n \geq 1$.*

The following result is proved in [30, Theorem 7.7]

Theorem 1.2.9. *If M is projective R-module, then $\operatorname{Ext}^n_R(M, N) = 0$ for all R-modules N and all $n \geq 1$.*

Now consider a field k and two commutative k-algebras A and B. Let $A \otimes_k B = C$. Then we have the following important result which is proved in [8, Theorem XI 3.1].

Theorem 1.2.10. *Assume that A and B are noetherian. If M is finitely A-generated and M' is finitely B-generated, then there is an isomorphism*

$$\operatorname{Ext}^p_A(M, N) \otimes_k \operatorname{Ext}^q_B(M', N') \longrightarrow \operatorname{Ext}^{p+q}_C(M \otimes_k M', N \otimes_k N').$$

Definition 1.2.11. Let X and Y be chain complexes. Then a *chain map* $f\colon X \longrightarrow Y$ is a sequence of maps $f_n\colon X_n \longrightarrow Y_n$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "d'_{n+1}"'] & Y_n \arrow[r, "d'_n"'] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 1.2.12. A sequence of chain complexes

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is said to be a *short exact sequence* if the sequences of modules

$$0 \longrightarrow X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n \longrightarrow 0$$

are exact for every $n \in \mathbb{Z}$.

Definition 1.2.13. If Y is a chain complex and n is an integer, we define the *n-skeleton*, $Y^{[n]}$, to be the subcomplex of Y such that

$$(Y^{[n]})_i = \begin{cases} Y_i & \text{if } i \leq n, \\ 0 & \text{if } i > n. \end{cases}$$

It is clear that $H_i(Y^{[n]}) = H_i(Y)$ for $i < n$ and $H_i(Y^{[n]}) = 0$ if $i > n$. Also we can form the chain complex $Y[n]$ as follows:

$$(Y[n])_i = Y_{n+i}$$

with differential $(-1)^n d$. We call $Y[n]$ the *nth translation* of Y. We see that $H_i(Y[n]) = H_{n+i}(Y)$.

Also, we define nth translation on any chain map $f\colon X \longrightarrow Y$ by

$$(f[n])_i = f_{n+i}.$$

Definition 1.2.14. Let M be an R-module and $n \in \mathbb{Z}$ be a fixed integer. If we regard M as the nth term and all other terms 0, then this is a chain complex *concentrated in degree* n, written $M[-n]$.

Definition 1.2.15. A chain map $f\colon X \longrightarrow Y$ is called a *q-isomorphism* if the maps $f_\star\colon H_n(X) \longrightarrow H_n(Y)$ are all isomorphisms.

Definition 1.2.16. A *projective resolution* of a chain complex Y is a q-isomorphism $P \longrightarrow Y$ such that each P_i is a projective R-module.

Definition 1.2.17. An *injective resolution* of a chain complex Y is a q-isomorphism $Y \longrightarrow I$ such that each I_i is an injective R-module.

Definition 1.2.18. A *double complex* (or *bicomplex*) is a bigraded R-module $Y = \{Y_{p,q}\}$ with maps $d^h\colon Y_{p,q} \longrightarrow Y_{p-1,q}$ and $d^v\colon Y_{p,q} \longrightarrow Y_{p,q-1}$ such that

$$d^h d^h = d^v d^v = d^v d^h + d^h d^v = 0.$$

It is pictured as a lattice

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & Y_{p-1,q+1} \arrow[l] \arrow[d, "d^v"'] & Y_{p,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & Y_{p+1,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots & Y_{p-1,q} \arrow[l] \arrow[d, "d^v"'] & Y_{p,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & Y_{p+1,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots & Y_{p-1,q-1} \arrow[l] \arrow[d] & Y_{p,q-1} \arrow[l, "d^h"'] \arrow[d] & Y_{p+1,q-1} \arrow[l, "d^h"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The first two conditions $d^h d^h = d^v d^v = 0$ say that each row and each column is a chain complex.

Definition 1.2.19. If Y is a double complex, its *total complex* $\mathrm{Tot}^{\bigoplus}(Y)$ is the chain complex defined by

$$\mathrm{Tot}^{\bigoplus}(Y)_n = \bigoplus_{p+q=n} Y_{p,q}$$

with differential

$$d_n \colon \mathrm{Tot}^{\bigoplus}(Y)_n \longrightarrow \mathrm{Tot}^{\bigoplus}(Y)_{n-1}$$

given by $d = d^h + d^v$. Also, we have the *total complex* $\mathrm{Tot}^{\prod}(Y)$ which is defined by

$$\mathrm{Tot}^{\prod}(Y)_n = \prod_{p+q=n} Y_{p,q}$$

with differential

$$d_n \colon \mathrm{Tot}^{\prod}(Y)_n \longrightarrow \mathrm{Tot}^{\prod}(Y)_{n-1}$$

given by $d = d^h + d^v$.

The following lemma is proved in [30, Lemma 11.14].

Lemma 1.2.20. *If Y is a double complex, then both $\mathrm{Tot}^{\bigoplus}(Y)$ and $\mathrm{Tot}^{\prod}(Y)$ are chain complexes.*

Remark 1.2.21. A big commutative diagram whose rows and columns are chain complexes can be modified by a simple sign change to be a double complex. Let Y be a bigraded module with maps d^h and d^v. Assume that $d^h d^h = d^v d^v = 0$ and the diagram commutes. If $d^v_{p,q} \colon Y_{p,q} \longrightarrow Y_{p,q-1}$ is replaced by $d'^v_{p,q} = (-1)^p d^v_{p,q}$, then (Y, d^h, d'^v) is a double complex.

Definition 1.2.22. Let X and Y be chain complexes of R-modules. We form the double complex $X \otimes Y = \{X_p \otimes_R Y_q\}$ using the Remark 1.2.21, that is, with horizontal differentials $d \otimes 1$ and vertical differentials $(-1)^p \otimes d$. $X \otimes Y$ is called the *tensor product double complex*, and $\mathrm{Tot}^{\bigoplus}(X \otimes Y)$ is called the *(total) tensor product chain complex* of X and Y.

The following result is proved in [33, Theorem 3.6.3].

Theorem 1.2.23 (Künneth formula for complexes). *Let X and Y be chain complexes of R-modules. If X_n and $d(X_n)$ are flat for each n, then there is an exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X) \otimes_R H_q(Y) \longrightarrow H_n(\mathrm{Tot}^{\oplus}(X \otimes Y)) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X), H_q(Y)) \longrightarrow 0.$$

for each n.

Definition 1.2.24. Let X and Y be chain complexes. First we convert Y into a cochain complex Y with $Y^s = Y_{-s}$. We form the double cochain complex

$$\mathrm{Hom}(X, Y) = \{\mathrm{Hom}_R(X_p, Y^q)\}$$

using Remark 1.2.21. That is, if $f\colon X_p \longrightarrow Y^q$, then we define the horizontal differential $d^h f\colon X_{p+1} \longrightarrow Y^q$ by $(d^h f)(x) = f(dx)$, and we define the vertical differential $d^v f\colon X_p \longrightarrow Y^{q+1}$ by $(d^v f)(x) = (-1)^{p+q+1} d(fx)$ for $x \in X_p$. $\mathrm{Hom}(X, Y)$ is called the *Hom double complex*, and $\mathrm{Tot}^{\Pi}(\mathrm{Hom}(X, Y))$ is called the *(total) Hom cochain complex*. Note that we can reindex $\mathrm{Tot}^{\Pi}(\mathrm{Hom}(X, Y))$ to obtain *(total) Hom chain complex*.

The following theorem is proved in [33, Theorem 3.6.5].

Theorem 1.2.25 (Universal Coefficient Theorem for Cohomology). *Let X be a chain complex of projective R-modules such that each $d(X_n)$ is also projective. Then for every n and every R-module M, there is an exact sequence*

$$0 \longrightarrow \mathrm{Ext}_R^1(H_{n-1}(X), M) \longrightarrow H^n(\mathrm{Hom}_R(X, M)) \longrightarrow \mathrm{Hom}_R(H_n(X), M) \longrightarrow 0.$$

Next we define the mapping cone and mapping cylinder of a chain map.

Definition 1.2.26. Let $f\colon X \longrightarrow Y$ be a chain map. The *mapping cone* of f is the chain complex $\mathrm{cone}(f)$ whose degree n part is $X_{n-1} \bigoplus Y_n$. The differential in $\mathrm{cone}(f)$ is given by the formula

$$d(x, y) = (-d_X(x), d_Y(y) - f(x))$$

where $x \in X_{n-1}$ and $y \in Y_n$. That is, the differential is given by the matrix

$$\begin{bmatrix} -d_X & 0 \\ -f & d_Y \end{bmatrix}$$

The following result is proved in [33, Corollary 1.5.4].

Lemma 1.2.27. *A map $f\colon X \longrightarrow Y$ is a q-isomorphism if and only if the mapping cone chain complex* $\operatorname{cone}(f)$ *is exact.*

For every chain map $f\colon X \longrightarrow Y$, there is an exact sequence

$$0 \longrightarrow Y \longrightarrow \operatorname{cone}(f) \xrightarrow{\partial} X[-1] \longrightarrow 0$$

where the left map sends y to $(0, y)$ and the right map sends (x, y) to $-x$.

Definition 1.2.28. Let $f\colon X \longrightarrow Y$ be a chain map. The *mapping cylinder* of f is the chain complex $\operatorname{cyl}(f)$ whose degree n part is $X_n \bigoplus X_{n-1} \bigoplus Y_n$. The differential in $\operatorname{cyl}(f)$ is given by the formula

$$d(x_1, x_2, y) = (d_X(x_1) + x_2, -d_X(x_2), d_Y(y) - f(x_2)).$$

That is the differential is given by the matrix

$$\begin{bmatrix} d_X & \mathrm{id}_X & 0 \\ 0 & -d_X & 0 \\ 0 & -f & d_Y \end{bmatrix}$$

The following result is proved in [33, Lemma 1.5.6].

Lemma 1.2.29. *The subcomplex of elements $(0, 0, y)$ is isomorphic to Y and the corresponding inclusion $\alpha\colon Y \longrightarrow$* $\operatorname{cyl}(f)$ *is a q-isomorphism.*

Notice that the subcomplex of elements $(x, 0, 0)$ in $\operatorname{cyl}(f)$ is isomorphic to X, and the quotient $\operatorname{cyl}(f)/X$ is the mapping cone of f. Therefore we have the following exact sequence of chain complexes

$$0 \longrightarrow X \longrightarrow \operatorname{cyl}(f) \longrightarrow \operatorname{cone}(f) \longrightarrow 0.$$

There is a category of chain complexes of R-modules, denoted $\mathbf{Ch}(R)$, where the objects are chain complexes and morphisms are chain maps.

A proof of the following theorem can be found in [33, Theorem 1.2.3].

Theorem 1.2.30. *The category* $\mathbf{Ch}(R)$ *of chain complexes of* R*-modules is an abelian category.*

A chain complex Y is called *bounded* if $Y_n = 0$ unless $a \leq n \leq b$, *bounded above* if there is a bound b such that $Y_n = 0$ for all $n > b$ and *bounded below* if there is a bound a such that $Y_n = 0$ for all $n < a$. The bounded, bounded above and bounded below chain complexes form full subcategories of $\mathbf{Ch}(R)$ and are denoted $\mathbf{Ch}_b(R)$, $\mathbf{Ch}_-(R)$ and $\mathbf{Ch}_+(R)$, respectively. Denote the subcategory of non-negative complexes Y, $Y_n = 0$ for all $n < 0$, by $\mathbf{Ch}_{\geq 0}(R)$.

We define the *translation functor* $T\colon \mathbf{Ch}(R) \longrightarrow \mathbf{Ch}(R)$ on any object X by $T(X) = X[-1]$ and on any morphism $f\colon X \longrightarrow Y$ by $T(f) = f[-1]$. It is clear that T is an automorphism and its inverse T^{-1} is defined by $T^{-1}(X) = X[1]$.

The following theorem is one of the fundamental results on chain complexes and it is proved in [30, Theorem 6.3].

Theorem 1.2.31. *If*

$$0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{p} Z \longrightarrow 0$$

is short exact sequence of chain complexes, then there is a long exact sequence of modules

$$\cdots \longrightarrow H_n(Y) \xrightarrow{p_\star} H_n(Z) \xrightarrow{\partial} H_{n-1}(X) \xrightarrow{i_\star} H_{n-1}(Y) \longrightarrow \cdots$$

where the map $\partial\colon H_n(Z) \longrightarrow H_{n-1}(X)$ *is called the connecting homomorphism.*

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. We say that $\mathcal{A}$ *has enough projectives* if for every object A of $\mathcal{A}$ there is a surjection $P \longrightarrow A$ with P projective. We say that $\mathcal{A}$ *has enough injectives* if for every object A in $\mathcal{A}$ there is an injection $A \longrightarrow I$ with I injective.

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor. If $\mathcal{A}$ has enough projectives, we can construct the *left derived functors* L_iF $(i \geq 0)$ of F as follows. If A is an object of $\mathcal{A}$, choose a projective resolution $P \longrightarrow A$ and define

$$L_iF(A) = H_i(F(P)).$$

Note that since

$$F(P_1) \longrightarrow F(P_0) \longrightarrow F(A) \longrightarrow 0$$

is exact, we always have $L_0F(A) \cong F(A)$. A is said to be *left F-acyclic* if $L_nF(A) = 0$ for all $n \geq 1$.

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor. If $\mathcal{A}$ has enough injectives, we can construct the *right derived functors* R^iF $(i \geq 0)$ of F as follows. If A is an object of $\mathcal{A}$, choose an injective resolution $A \longrightarrow I$ and define

$$R^iF(A) = H^i(F(I)).$$

Note that since

$$0 \longrightarrow F(A) \longrightarrow F(I^0) \longrightarrow F(I^1)$$

is exact, we always have $R^0F(A) \cong F(A)$. A is said to be *right F-acyclic* if $R^iF(A) = 0$ for all $n \geq 1$.

1.3 The Homotopy Category of Chain Complexes

In this section, we give the definition of the homotopy category of chain complexes and present some of its properties. The main reference for this section is [33].

Definition 1.3.1. If $f\colon X \longrightarrow Y$ is a chain map, then f is *null homotopic* if there are maps $s_n\colon X_n \longrightarrow Y_{n+1}$

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "d_{n+1}"] & X_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "f_n"] & X_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "d_{n+1}"'] & Y_n \arrow[r, "d_n"'] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that

$$f = d_Y s + s d_X.$$

Let G be the set of all maps in $\mathrm{Hom}_{\mathbf{Ch}(R)}(X, Y)$ which are null homotopic.

Lemma 1.3.2. *The set G is a subgroup of* $\mathrm{Hom}_{\mathbf{Ch}(R)}(X, Y)$.

Proof. It is clear that the zero map is in G. Let f, g be null homotopic maps. By Definition 1.3.1, there are maps $s\colon X \longrightarrow Y[1]$ and $t\colon X \longrightarrow Y[1]$ such that $f = d_Y s + s d_X$ and $g = d_Y t + t d_X$. Then

$$f + g = d_Y(s+t) + (s+t)d_X.$$

That is, $f+g$ is null homotopic. Also, $-f = d_Y(-s) + (-s)d_X$, that is, $-f$ is null homotopic. Hence, G is a subgroup. □

Definition 1.3.3. If f and g are chain maps $X \longrightarrow Y$, then we say that f and g are *chain homotopic*, written $f \simeq g$, if $f - g \in G$, that is, if

$$f - g = d_Y s + s d_X.$$

Next we show that $\simeq$ is an equivalence relation on $\mathrm{Hom}_{\mathbf{Ch}(R)}(X, Y)$. It is clear that it is reflexive and symmetric. Assume that $f \simeq g$ and $g \simeq h$. Then there are maps $s\colon X \longrightarrow Y[1]$ and $t\colon X \longrightarrow Y[1]$ such that $f - g = d_Y s + s d_X$ and $g - h = d_Y t + t d_X$. So

$$f - h = d_Y(s+t) + (s+t)d_X.$$

That is, $f \simeq h$ and it follows that $\simeq$ is transitive. Hence, $\simeq$ is an equivalence relation.

Lemma 1.3.4. *Let X, Y and Z be chain complexes. Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be two chain maps. If either f or g is null homotopic, then gf is null homotopic.*

Proof. Assume that f is null homotopic. Then there exists $s\colon X \longrightarrow Y[1]$ such that $f = d_Y s + s d_X$. Therefore,

$$gf = g d_Y s + g s d_X = d_Z g s + g s d_X$$

where $gs\colon X \longrightarrow Z[1]$. This implies that gf is null homotopic. Now assume that g is null homotopic. Then there exists $t\colon Y \longrightarrow Z[1]$ such that $g = d_Z t + t d_Y$. Therefore,

$$gf = d_Z t f + t d_Y f = d_Z t f + t f d_X$$

where $tf\colon X \longrightarrow Z[1]$. This implies that gf is null homotopic. □

Let

$$\mathrm{Hom}_{\mathbf{K}(R)}(X,Y) = \mathrm{Hom}_{\mathbf{Ch}(R)}(X,Y)/G.$$

This is an abelian group of classes of homotopic maps between X and Y. Let X, Y and Z be chain complexes. By Lemma 1.3.4, the composition map

$$\mathrm{Hom}_{\mathbf{Ch}(R)}(Y,Z) \times \mathrm{Hom}_{\mathbf{Ch}(R)}(X,Y) \longrightarrow \mathrm{Hom}_{\mathbf{Ch}(R)}(X,Z)$$

induces a biadditive map

$$\mathrm{Hom}_{\mathbf{K}(R)}(Y,Z) \times \mathrm{Hom}_{\mathbf{K}(R)}(X,Y) \longrightarrow \mathrm{Hom}_{\mathbf{K}(R)}(X,Z).$$

Let $\mathbf{K}(R)$ be the quotient category of $\mathbf{Ch}(R)$ whose objects are chain complexes and morphisms are classes of homotopic maps, that is, $\mathrm{Hom}_{\mathbf{K}(R)}(X,Y)$ for every pair of objects X and Y. $\mathbf{K}(R)$ is called the *homotopy category of chain complexes.*

We define $\mathbf{K}_b(R)$, $\mathbf{K}_-(R)$ and $\mathbf{K}_+(R)$ to be the full subcategories of $\mathbf{K}(R)$ corresponding to the full subcategories $\mathbf{Ch}_b(R)$, $\mathbf{Ch}_-(R)$ and $\mathbf{Ch}_+(R)$ of bounded, bounded above and bounded below chain complexes.

The zero object in $\mathbf{K}(R)$ is the zero object in $\mathbf{Ch}(R)$. For every pair of objects in $\mathbf{K}(R)$, we define their direct sum as the direct sum in $\mathbf{Ch}(R)$. Therefore, we have the following result.

Theorem 1.3.5. *The category $\mathbf{K}(R)$ is an additive category.*

The following result is proved in [33, Lemma 1.4.5].

Lemma 1.3.6. *If f and g are homotopic chain maps $X \longrightarrow Y$, then they induce the same maps $H_n(X) \longrightarrow H_n(Y)$.*

It follows by the above lemma that if $f\colon X \longrightarrow Y$ is a q-isomorphism and $g\colon X \longrightarrow Y$ is homotopic to f, then g is also a q-isomorphism. Therefore, we say that a morphism in $\mathbf{K}(R)$ is a q-isomorphism if all of its representatives are q-isomorphisms.

Remark 1.3.7. Let X and Y be chain complexes. If we reindex Y as a cochain complex and form the total Hom cochain complex $\mathrm{Tot}^{\prod}(\mathrm{Hom}(X,Y))$, then an n-cocycle f is a sequence of maps $f_p\colon X_p \longrightarrow Y^{n-p}$ such that $f_p d = (-1)^n d f_{p+1}$, that is, a morphism of chain complexes from X to the translate $Y[-n]$ of Y. An n-coboundary is a morphism f that is null homotopic. Thus,

$$H^n \operatorname{Tot}^{\prod}(\operatorname{Hom}(X,Y)) = \operatorname{Hom}_{\mathbf{K}(R)}(X, Y[-n]).$$

Lemma 1.3.8. *If Y is a chain complex, then there is a natural isomorphism*

$$H_n(Y) \cong \operatorname{Hom}_{\mathbf{K}(R)}(R[-n], Y).$$

Proof. Let $[f] \in \operatorname{Hom}_{\mathbf{K}(R)}(R[-n], Y)$. Then $f(1) \in Y_n$ and $d(f(1)) = 0$. So $f(1) \in Z_n(Y)$. Therefore, there is a map

$$\phi\colon \operatorname{Hom}_{\mathbf{K}(R)}(R[-n], Y) \longrightarrow H_n(Y)$$

defined by $\phi([f]) = f(1) + B_n(Y)$. We claim that ϕ is well defined. Assume that $f \simeq g$. Then there exists $s\colon R[-n] \longrightarrow Y[1]$ such that $f - g = ds + sd$ by Definition 1.3.3. So $f(1) - g(1) = d(y)$ where $y = s(1)$. That is, $f(1) - g(1) \in B_n(Y)$. Thus, $f(1) + B_n(Y) = g(1) + B_n(Y)$. Hence, ϕ is well defined. We show that ϕ is one to one. Suppose that $[f], [g]\colon R[-n] \to Y$ such that $\phi([f]) = \phi([g])$. We claim that f is homotopic to g. We have $\phi([f]) = \phi([g])$. Then $f(1) + B_n(Y) = g(1) + B_n(Y)$, that is, $f(1) - g(1) + B_n(Y) = B_n(Y)$. Therefore, there exists $y \in Y_{n+1}$ such that $d(y) = f(1) - g(1)$. Let $h_n\colon R \longrightarrow Y_{n+1}$ defined by $1 \longmapsto y$. We extend h_n trivially to have a homotopy h. Hence, $f \simeq g$. Now let $\bar{y} \in H_n(Y)$. Choose $f\colon R \to Y_n$ such that $1 \longmapsto y$. This induces the map $f\colon R[-n] \longrightarrow Y$. Thus, for each $\bar{y} \in H_n(Y)$ there exists a morphism $[f] \in \operatorname{Hom}_{\mathbf{K}(R)}(R[-n], Y)$ such that $\phi([f]) = \bar{y}$. It is clear that ϕ is a homomorphism of R-modules. Therefore, $H_n(Y) \cong \operatorname{Hom}_{\mathbf{K}(R)}(R[-n], Y)$. Note that if $f\colon X \longrightarrow Y$, then it is clear that the following diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbf{K}(R)}(R[-n], X) \arrow[r] \arrow[d] & H_n(X) \arrow[d] \\
\operatorname{Hom}_{\mathbf{K}(R)}(R[-n], Y) \arrow[r] & H_n(Y)
\end{tikzcd}$$

is commutative. Hence, there is a natural isomorphism

$$H_n(Y) \cong \operatorname{Hom}_{\mathbf{K}(R)}(R[-n], Y). \qquad \square$$

Lemma 1.3.9. *Let $f\colon X \longrightarrow Y$ be a map of chain complexes. Then the following statements are equivalent.*

(i) *f is null homotopic.*

(ii) *$T(f)$ is null homotopic.*

Proof. Assume that f is null homotopic. Then there are maps $s_n\colon X_n \longrightarrow Y_{n+1}$ such that $f = d_Y s + s d_X$. Now note that

$$T(f)_{n+1} = f_n = d_Y^{n+1} s_n + s_{n-1} d_X^n = -d_{T(Y)}^{n+2} s_{n+1} - s_n d_{T(X)}^{n+1}.$$

for all n. Thus, $T(f)$ is null homotopic. Similarly, we can prove the converse. □

Therefore, Lemma 1.3.9 implies that the translation functor $T\colon \mathbf{Ch}(R) \longrightarrow \mathbf{Ch}(R)$ induces an automorphism of $\mathbf{K}(R)$. We call T again the translation functor of $\mathbf{K}(R)$.

1.4 Spectral Sequences

In this section, we give the definition of spectral sequences and explain the convergence of spectral sequences. The main references for this section are [6], [22], [33] and [30].

Definition 1.4.1. A *homology spectral sequence* in the category R-**mod** of R-modules consists of the following data:

(i) A family $\{E_{p,q}^r\}$ of R-modules for all integers p, q and $r \geq 1$.

(ii) R-maps

$$d_{p,q}^r\colon E_{p,q}^r \longrightarrow E_{p-r,q+r-1}^r$$

that are differentials in the sense that $d^r d^r = 0$.

(iii) Isomorphisms between $E_{p,q}^{r+1}$ and the homology of $E_{\star,\star}^r$ at the spot $E_{p,q}^r$:

$$E_{p,q}^{r+1} \cong \operatorname{Ker}(d_{p,q}^r)/\operatorname{Im}(d_{p+r,q-r+1}^r).$$

There is a category of homology spectral sequences. A morphism $f\colon E \longrightarrow \bar{E}$ is a family of R-maps $f^r_{p,q}\colon E^r_{p,q} \longrightarrow \bar{E}^r_{p,q}$ for all r of bidegree $(0,0)$ such that f^r commutes with the differentials, that is, $f^r d^r = \bar{d}^r f^r$ and each $f^{r+1}_{p,q}$ is induced by $f^r_{p,q}$ on homology.

Definition 1.4.2. A *cohomology spectral sequence* in the category R-**mod** of R-modules consists of the following data:

(i) A family $\{E_r^{p,q}\}$ of R-modules for all integers p, q and $r \geq 1$.

(ii) R-maps

$$d_r^{p,q}\colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

that are differentials in the sense that $d_r d_r = 0$.

(iii) Isomorphisms between $E_{r+1}^{p,q}$ and the homology of $E_r^{\star,\star}$ at the spot $E_r^{p,q}$:

$$E_{r+1}^{p,q} \cong \operatorname{Ker}(d_r^{p,q})/\operatorname{Im}(d_r^{p-r,q+r-1}).$$

There is a category of cohomology spectral sequences. A morphism $f\colon E \longrightarrow \bar{E}$ is a family of R-maps $f_r^{p,q}\colon E_r^{p,q} \longrightarrow \bar{E}_r^{p,q}$ for all r of bidegree $(0,0)$ such that f_r commutes with the differentials, that is, $f_r d_r = \bar{d}_r f_r$ and each $f_{r+1}^{p,q}$ is induced by $f_r^{p,q}$ on homology.

A *filtration* of a graded R-module G is

$$\cdots \subseteq F_{s-1}G_n \subseteq F_s G_n \subseteq F_{s+1}G_n \subseteq \cdots \subseteq G_n$$

for each n. The filtration is *exhaustive* if $G_n = \cup_s F_s G_n$ for each n and it is *Hausdorff* if $\cap_s F_s G_n = 0$ for each n. It is *complete* if $G_n = \lim_s G_n/F_s G_n$ for each n.

Definition 1.4.3. Given a homology spectral sequence $\{E^r_{s,t}, d^r : r \geq 1\}$ and a filtered graded R-module G, we say that the spectral sequence

(i) *converges weakly* to G if the filtration is exhaustive and we have isomorphisms $E^\infty_{s,t} \cong F_s G_{s+t}/F_{s-1}G_{s+t}$ for all s and t;

(ii) *converges* to G if (i) holds and the filtration of G is Hausdorff;

(iii) *converges strongly* to G if (i) holds and the filtration of G is complete and Hausdorff.

The following theorem is proved in [33, Comparison Theorem 5.2.12]

Theorem 1.4.4 (Comparison Theorem). *Let $E^r_{p,q}$ and $\bar{E}^r_{p,q}$ be two spectral sequences converge strongly to $H_\star$ and $\bar{H}_\star$, respectively. Suppose given a map $h\colon H_\star \longrightarrow \bar{H}_\star$ compatible with a morphism $f\colon E \longrightarrow \bar{E}$ of spectral sequences, that is, h maps F_pH_n to $F_p\bar{H}_n$ and the associated maps $F_pH_n/F_{p-1}H_n \longrightarrow F_p\bar{H}_n/F_{p-1}\bar{H}_n$ correspond to $E^\infty_{p,q} \longrightarrow \bar{E}^\infty_{p,q}$. If $f^r\colon E^r_{p,q} \longrightarrow \bar{E}^r_{p,q}$ is an isomorphism for all p and q and some r, then $f^s\colon E^s_{p,q} \longrightarrow \bar{E}^s_{p,q}$ is an isomorphism for all $r \leq s \leq \infty$ and $h_\star\colon H_\star \longrightarrow \bar{H}_\star$ is an isomorphism.*

Definition 1.4.5. Let D and E denote R-modules (which are bigraded in the relevant cases) and let $i\colon D \longrightarrow D$, $j\colon D \longrightarrow E$ and $k\colon E \longrightarrow D$ be module homomorphisms. We present these data as in the diagram:

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

and call $\{D, E, i, j, k\}$ an *exact couple* if this diagram is exact, that is, $\operatorname{Im} i = \operatorname{Ker} j$, $\operatorname{Im} j = \operatorname{Ker} k$ and $\operatorname{Im} k = \operatorname{Ker} i$.

Now we have the following important result which is proved in [33, Proposition 5.9.2].

Theorem 1.4.6. *An exact couple in which i, j and k have bidegrees $(1, -1)$, $(0, 0)$ and $(-1, 0)$ determines a homology spectral sequence $\{E^r_{s,t}, d^r : r \geq 1\}$.*

The following dual result is proved in [22, Theorem 2.8].

Theorem 1.4.7. *An exact couple in which i, j and k have bidegrees $(-1, 1)$, $(0, 0)$ and $(1, 0)$ determines a cohomology spectral sequence $\{E^{s,t}_r, d_r : r \geq 1\}$.*

A useful presentation of exact couples is the following *unrolled exact couple*

$$\begin{tikzcd} \cdots \arrow[r, "i"] & D^{s+1,\star} \arrow[rr, "i"] & & D^{s,\star} \arrow[rr, "i"] \arrow[dl, "j"] & & D^{s-1,\star} \arrow[r, "i"] \arrow[dl, "j"] & \cdots \\ & & E^{s,\star} \arrow[ul, "k"] & & E^{s-1,\star} \arrow[ul, "k"] & & \end{tikzcd}$$

The following important result is proved in [22, Corollary 2.10]

Lemma 1.4.8. *For $r \geq 1$, there is an exact sequence*

$$0 \longrightarrow D^{s,\star}/\operatorname{Ker}(i^r\colon D^{s,\star} \longrightarrow D^{s-r,\star}) + iD^{s+1,\star} \xrightarrow{\bar{j}} E_{r+1}^{s,\star} \xrightarrow{\bar{k}}$$
$$\operatorname{Im}(i^r\colon D^{s+r+1,\star} \longrightarrow D^{s+1,\star}) \cap \operatorname{Ker}(i\colon D^{s+1,\star} \longrightarrow D^{s,\star}) \longrightarrow 0.$$

Let $D^{\infty,\star} = \lim_s\{D^{s,\star}, i\}$ and $D^{-\infty,\star} = \operatorname{colim}_s\{D^{s,\star}, i\}$. Both $D^{\infty,\star}$ and $D^{-\infty,\star}$ have a decreasing filtration given by

$$F^s D^{\infty,\star} = \operatorname{Ker}(D^{\infty,\star} \longrightarrow D^{s,\star})$$

and

$$\bar{F}^s D^{-\infty,\star} = \operatorname{Im}(D^{s,\star} \longrightarrow D^{-\infty,\star}).$$

These two filtrations have the following properties which are proved in [22, Proposition 3.16].

Lemma 1.4.9. *For an exact couple, the filtration F on the limit $D^{\infty,\star}$ is Hausdorff and complete. The filtration $\bar{F}$ on the colimit $D^{-\infty,\star}$ is exhaustive.*

Definition 1.4.10. The spectral sequence associated to an exact couple

$$\{D^{s,\star}, E^{s,\star}, i, j, k\}$$

is said to be *conditionally convergent to the colimit $D^{-\infty,\star}$* if

$$D^{\infty,\star} = \lim_s\{D^{s,\star}, i\} = \{0\} = \lim_s{}^1\{D^{s,\star}, i\}.$$

We say the spectral sequence *conditionally convergent to the limit $D^{\infty,\star}$* if $D^{-\infty,\star} = \{0\}$.

The following important theorem and its corollary are proved in [22, Theorem 3.19].

Theorem 1.4.11. *Suppose $\{D^{s,\star}, E^{s,\star}, i, j, k\}$ is an exact couple satisfying $E^{s,\star} = \{0\}$ for all $s < 0$. Suppose further that the associated spectral sequence converges conditionally to $D^{-\infty,\star}$. Then the spectral sequence converges strongly to $D^{-\infty,\star}$ if and only if $\lim^1{}_r E_r^{s,\star} = \{0\}$ for all s.*

Corollary 1.4.12. *Suppose* $\{D^{s,\star}, E^{s,\star}, i, j, k\}$ *is an exact couple satisfying* $E^{s,\star} = \{0\}$ *for all* $s < 0$. *Suppose further that* $D^{\infty,\star} = 0$. *Then the spectral sequence converges to* $D^{-\infty,\star}$ *if and only if* $\lim^1{}_r E_r^{s,\star} = \{0\}$ *for all* s.

The following result is in [6, Theorem 7.4].

Theorem 1.4.13. *Suppose* $\{D^{s,\star}, E^{s,\star}, i, j, k\}$ *is an exact couple satisfying* $E^{s,\star} = \{0\}$ *for all* $s < 0$. *Suppose further that the associated spectral sequence converges conditionally to* $D^{\infty,\star}$. *Then the spectral sequence converges strongly to* $D^{\infty,\star}$ *if and only if* $\lim^1{}_r E_r^{s,\star} = \{0\}$ *for all* s.

Definition 1.4.14. A *Cartan-Eilenberg* resolution of a chain complex Y is an upper half-plane double complex P consisting of projective R-modules together with a chain map $\epsilon\colon P_{\star,0} \longrightarrow Y$ such that for each n,

(i)
$$0 \longleftarrow Y_n \longleftarrow P_{n,0} \longleftarrow P_{n,1} \longleftarrow \cdots,$$

(ii)
$$0 \longleftarrow Z_n(Y) \longleftarrow Z_n(P_0) \longleftarrow Z_n(P_1) \longleftarrow \cdots,$$

(iii)
$$0 \longleftarrow B_n(Y) \longleftarrow B_n(P_0) \longleftarrow B_n(P_1) \longleftarrow \cdots,$$

and

(iv)
$$0 \longleftarrow H_n(Y) \longleftarrow H_n(P_0) \longleftarrow H_n(P_1) \longleftarrow \cdots$$

are projective resolutions.

Note that $\mathrm{Tot}^{\oplus}(P) \longrightarrow Y$ is a q-isomorphism [33, Exercise 5.7.1].

The following result is proved in [33, Lemma 5.7.2].

Lemma 1.4.15. *Every chain complex* Y *of* R*-modules has a Cartan-Eilenberg resolution.*

The following result is proved in [30, Theorem 11.34].

Theorem 1.4.16 (Künneth Spectral Sequence). *Let X and Y be non-negative chain complexes with X flat. Then there is a strongly convergent first quadrant spectral sequence*

$$E^2_{p,q} = \bigoplus_{s+t=q} \mathrm{Tor}^R_p(H_s(X), H_t(Y)) \Longrightarrow H_{p+q}(\mathrm{Tot}^{\oplus}(X \otimes Y)).$$

The dual result for cohomology, see [30, Theorem 11.34], is the following.

Theorem 1.4.17 (Künneth Spectral Sequence). *Let X and Y be non-negative chain complexes. If either X is projective or Y is injective, there is a strongly convergent first quadrant spectral sequence*

$$E_2^{p,q} = \bigoplus_{s+t=q} \mathrm{Ext}^p_R(H_s(X), H_t(Y)) \Longrightarrow H^{p+q}(\mathrm{Tot}^{\prod}(\mathrm{Hom}(X, Y))).$$

Theorem 1.4.18 (Universal Coefficient Spectral Sequence). *Let X be non-negative chain complex of projective R-modules and M an R-module. Then there is a strongly convergent first quadrant spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^p_R(H_q(X), M) \Longrightarrow H^{p+q}(\mathrm{Hom}_R(X, M)).$$

Proof. Let $M \longrightarrow I$ be an injective resolution. Consider the first quadrant double cochain complex $\mathrm{Hom}(X, I)$. Since X_p is projective,

$$H^q(\mathrm{Tot}^{\prod}(\mathrm{Hom}(X, I))) = \mathrm{Hom}_R(X_p, H_q(I)).$$

Therefore, the first spectral sequence has

$${}^{I}E_2^{p,q} = \begin{cases} 0 & \text{if } q > 0, \\ H^p(\mathrm{Hom}_R(X, M)) & \text{if } q = 0. \end{cases}$$

It follows that this spectral sequence collapses to yield $H^p(\mathrm{Tot}^{\prod}(\mathrm{Hom}(X, I))) = H^p(\mathrm{Hom}_R(X, M))$. Since I^q is injective,

$$H^q(\mathrm{Hom}(X, I^n)) = \mathrm{Hom}_R(H_q(X), I^n).$$

So the second spectral sequence has

$${}^{II}E_2^{p,q} = \mathrm{Ext}^p_R(H_q(X), M).$$

Hence, there is a strongly convergent first quadrant spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_R(H_q(X), M) \Longrightarrow H^{p+q}(\mathrm{Hom}_R(X, M)).$$ □

A *filtration* F of a chain complex Y is an ordered family of chain subcomplexes

$$\cdots \subseteq F_{p-1}Y \subseteq F_pY \subseteq \cdots$$

of Y. The filtration F is called *bounded* if for each n, there are integers $s < t$ such that $F_sY_n = 0$ and $F_tY_n = Y_n$. The filtration F is called *stupid* if

$$(F_pY)_n = \begin{cases} 0 & \text{for } n > p, \\ Y_n & \text{for } n \leq p. \end{cases}$$

The following result is important and is proved in [33, Theorem 5.5.1].

Theorem 1.4.19. *Let Y be a chain complex. Suppose that the filtration on Y is bounded. Then there is an associated spectral sequence with*

$$E^1_{p,q} = H_{p+q}(F_pY/F_{p-1}Y)$$

converging strongly to $H_\star(Y)$.

Moreover, the following result is important and is proved in [22, Theorem 3.5].

Theorem 1.4.20. *Let $\phi\colon X \longrightarrow Y$ be a chain map respecting the filtration, that is, $\phi(F_nX) \subset F_nY$ for each n. Then ϕ induces a morphism of the associated spectral sequences. If for some n, $\phi_n\colon E_n(X) \longrightarrow E_n(Y)$ is an isomorphism, then $\phi_r\colon E_r(X) \longrightarrow E_r(Y)$ is an isomorphism for all r, $n \leq r \leq \infty$. If the filtrations are bounded, then ϕ induces an isomorphism $\phi_\star\colon H_\star(X) \longrightarrow H_\star(Y)$.*

Chapter 2

The Derived Category of a Commutative Ring

In this chapter, we give some preliminaries on triangulated categories and the derived category of a commutative ring where the main references of this chapter are [13], [33] and [25]. In section one, we give the definition of localizations and the left and right fractions. In section two, we define triangulated categories and present some of its elementary properties. In section three, we show that the derived category is a triangulated category. In section four, we give definitions of the derived functors, the derived tensor product and the derived Hom.

2.1 Localization and the Fractions

In this section, we define the derived category and give the definition of the left and right fractions. The main references for this section are [13] and [33].

Definition 2.1.1. If S is a collection of morphisms in a category $\mathcal{C}$, then a *localization* of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$ and a functor $q\colon \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$ with the following properties:

(i) $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$ for every $s \in S$.

(ii) Any functor $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ such that $F(s)$ is an isomorphism for all $s \in S$ can be factorized uniquely through q. That is, we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "q"] \arrow[d, "F"'] & S^{-1}\mathcal{C} \arrow[dl, dotted] \\ \mathcal{C}' & \end{tikzcd}$$

It follows that the category $S^{-1}\mathcal{C}$ is unique up to equivalence.

The following theorem is proved in [13, Theorem III.2.1].

Theorem 2.1.2. *Let $\mathcal{A}$ be an abelian category, $\mathbf{Ch}(\mathcal{A})$ the category of chain complexes over $\mathcal{A}$. There exists a category $\mathcal{D}(\mathcal{A})$ and a functor $q\colon \mathbf{Ch}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ with the following properties.*

(a) $q(f)$ is an isomorphism for any q-isomorphism f.

(b) Any functor $F\colon \mathbf{Ch}(\mathcal{A}) \longrightarrow \mathcal{D}$ transforming q-isomorphisms into isomorphisms can be uniquely factorized through $\mathcal{D}(\mathcal{A})$, that is, there exists a unique functor $G\colon \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}$ with $F = Gq$.

The category $\mathcal{D}(\mathcal{A})$ is called the derived category of the chain complexes of $\mathcal{A}$. In particular, if $\mathcal{A}$ is the category of R-modules, then we get $\mathcal{D}(R)$ the derived category of the commutative ring R.

The problem is that morphisms in $\mathcal{D}(\mathcal{A})$ are just formal expressions of the form

$$s_n^{-1} f_n \dots s_2^{-1} f_2 s_1^{-1} f_1$$

where f_i are morphisms in $\mathbf{Ch}(\mathcal{A})$ and s_i are q-isomorphisms. To work with such expression we need to simplify it and so we need the following definition.

Definition 2.1.3. A collection S of morphisms in a category $\mathcal{C}$ is called a *multiplicative system* in $\mathcal{C}$ if the following conditions are satisfied:

(i) S is closed under composition, that is $st \in S$ for any $s, t \in S$ whenever the composition is defined and $\mathrm{id}_X \in S$ for any object $X \in \mathcal{C}$

(ii) (Ore condition) for any f in $\mathcal{C}$, $s \in S$, there exist g in $\mathcal{C}$, $t \in S$ such that the following diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative. Moreover, the symmetric statement is also valid, that is the following diagram

$$\begin{tikzcd} W & Z \arrow[l, "g"'] \\ X \arrow[u, "t"] & Y \arrow[l, "f"] \arrow[u, "s"'] \end{tikzcd}$$

is commutative.

(iii) (Cancellation) Let f, g be two morphisms from X to Y, then the following two conditions are equivalent:

(a) $sf = sg$ for some $s \in S$ with source Y.

(b) $ft = gt$ for some $t \in S$ with target X.

2.1.1 The Left and Right Fractions

In this subsection, we give the definitions of the left and right fractions. Let $\mathcal{C}$ be a category and S a collection of morphisms in $\mathcal{C}$.

We call a chain in $\mathcal{C}$ of the form

$$fs^{-1} : X \xleftarrow{s} X_1 \xrightarrow{f} Y$$

a *left fraction* if s is in S. We say that fs^{-1} is equivalent to

$$X \xleftarrow{t} X_2 \xrightarrow{g} Y$$

if there exists a fraction

$$X \longleftarrow X_3 \longrightarrow Y$$

fitting into a commutative diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\ & X_2 \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

Next we show that the above relation is an equivalence relation. It is obvious that it is reflexive and symmetric. Now we show that it is transitive. Assume that

$$X \xleftarrow{s} X' \xrightarrow{f} Y$$

is equivalent to

$$X \xleftarrow{t} X'' \xrightarrow{g} Y$$

and

$$X \xleftarrow{t} X'' \xrightarrow{g} Y$$

equivalent to

$$X \xleftarrow{u} X''' \xrightarrow{e} Y.$$

That is we have the following commutative diagrams

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & Z' \arrow[l] \arrow[r] \arrow[u, "r"] \arrow[d, "h"'] & Y \\
& X'' \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& X'' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & Z'' \arrow[l] \arrow[r] \arrow[u, "p"] \arrow[d, "i"'] & Y \\
& X'' \arrow[ul, "u"] \arrow[ur, "e"'] &
\end{tikzcd}$$

with s, t, u, sr, tp all belonging to S. We claim that there is a commutative diagram

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & Z''' \arrow[l] \arrow[r] \arrow[u, "q"] \arrow[d, "j"'] & Y \\
& X''' \arrow[ul, "u"] \arrow[ur, "e"'] &
\end{tikzcd}$$

with $sq \in S$. Using Ore condition, we get the following commutative diagram

$$\begin{tikzcd}
W \arrow[r, "v"] \arrow[d, "k"'] & Z' \arrow[d, "sr"] \\
Z'' \arrow[r, "tp"] & X
\end{tikzcd}$$

where $v \in S$. Notice that the two morphisms $f_1 = hv$ and $f_2 = pk$ from W to X'' satisfy $tf_1 = tf_2$. Therefore the cancellation condition says that there exists a morphism $w\colon Z''' \longrightarrow W$ where $w \in S$ such that $f_1 w = f_2 w$. Now putting $q = rvw\colon Z''' \longrightarrow X'$ and $j = ikw\colon Z''' \longrightarrow X'''$, we see that

$$sq = srvw = tpkw = uikw = uj$$

and

$$fq = frvw = ghvw = gpkw = eikw = ej.$$

Therefore we get the following commutative diagram

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & Z''' \arrow[l] \arrow[r] \arrow[u, "q"] \arrow[d, "j"] & Y \\ & X''' \arrow[ul, "u"] \arrow[ur, "e"'] & \end{tikzcd}$$

Now we define the composition of equivalence classes of left fractions. Let

$$X \xleftarrow{s} X' \xrightarrow{f} Y$$

be a left fraction between X and Y and

$$Y \xleftarrow{t} Y' \xrightarrow{g} Z$$

a left fraction between Y and Z. Then using Ore condition, there exist an object U and morphisms $u\colon U \to X'$ in S and $h\colon U \to Y'$ such that

$$\begin{tikzcd} & U \arrow[r, "h"] \arrow[d, "u"'] & Y' \arrow[r, "g"] \arrow[d, "t"] & Z \\ X & X' \arrow[l, "s"'] \arrow[r, "f"] & Y & \end{tikzcd}$$

is a commutative diagram. It follows that

$$X \xleftarrow{su} U \xrightarrow{gh} Z$$

is a left fraction between X and Z. We show that the equivalence class of the composite is independent of the choice of X' and Y'. Let

$$X \xleftarrow{s'} X'' \xrightarrow{f'} Y$$

be a left fraction equivalent to

$$X \xleftarrow{s} X' \xrightarrow{f} Y.$$

That is, there exist an object V and morphisms $v\colon V \to X'$ and $v'\colon V \to X''$ such that the following diagram

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & V \arrow[l] \arrow[r] \arrow[u, "v"] \arrow[d, "v'"'] & Y \\ & X'' \arrow[ul, "s'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

is commutative and $sv = s'v'$ is in S. Using Ore condition, there exists an object U' and morphisms $u'\colon U' \to X''$ in S and $h'\colon U' \to Y'$ such that the following diagram

$$\begin{tikzcd} & U' \arrow[r, "h'"] \arrow[d, "u'"'] & Y' \arrow[r, "g"] \arrow[d, "t"] & Z \\ X & X'' \arrow[l, "s'"'] \arrow[r, "f'"] & Y & \end{tikzcd}$$

is commutative. It follows that

$$X \xleftarrow{s'u'} U' \xrightarrow{gh'} Z$$

is a left fraction between X and Z. Using Ore condition, we see that there exists an object W, a morphism $w\colon W \to V$ in S and a morphism $a\colon W \to U$ such that the following diagram

$$\begin{tikzcd} W \arrow[r, "a"] \arrow[d, "w"'] & U \arrow[d, "u"] \\ V \arrow[r, "v"] & X' \end{tikzcd}$$

is commutative. Using Ore condition again, we see that there exists an object W', a morphism $w'\colon W' \to V$ in S and a morphism $a'\colon W' \to U'$ such that the following diagram

$$\begin{tikzcd} W' \arrow[r, "a'"] \arrow[d, "w'"'] & U' \arrow[d, "u'"] \\ V \arrow[r, "v'"] & X'' \end{tikzcd}$$

is commutative. Using Ore condition for the third time, we see that there exists an object C and morphisms $c\colon C \to W$ and $c'\colon C \to W'$ in S such that the following diagram

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "c'"'] & W \arrow[d, "w"] \\ W' \arrow[r, "w'"] & V \end{tikzcd}$$

is commutative. Note that

$$suac = svwc = s'v'w'c' = s'u'a'c'$$

is in S since $s'v'$, w' and c' are in S. But

$$thac = fuac = fvwc = f'v'w'c' = f'u'a'c' = th'a'c'.$$

Therefore, using the cancellation condition, we see that there exists an object M and a morphism $m\colon M \to C$ in S such that

$$hacm = h'a'c'm.$$

Let $b = acm\colon M \to U$ and $b' = a'c'm\colon M \to U'$. Then we have that

$$sub = suacm = s'u'a'c'm = s'u'b'$$

is in S and $ghb = gh'b'$. That is the following diagram

$$\begin{tikzcd} & U \arrow[dl, "su"'] \arrow[dr, "gh"] & \\ X & M \arrow[l] \arrow[r] \arrow[u, "b"] \arrow[d, "b'"'] & Z \\ & U' \arrow[ul, "s'u'"] \arrow[ur, "gh'"'] & \end{tikzcd}$$

is commutative. Hence, the equivalence class of the composite is independent of the choice of X' and similarly we can verify that the composite is independent of the choice of Y'. Therefore, we have defined a product of the sets of equivalence classes of left fractions between X and Y and equivalence classes of left fractions between Y and Z into the set of equivalence classes of left fractions between X and Z. Now we show that the composition of equivalence classes of left fractions is associative. Consider the following left fractions

$$X \xleftarrow{s} X' \xrightarrow{f} Y$$

$$Y \xleftarrow{t} Y' \xrightarrow{g} Z$$

$$Z \xleftarrow{u} Z' \xrightarrow{h} W.$$

Using Ore condition three times, we see that we have the following commutative diagram

$$\begin{tikzcd}
 & M \arrow[r, "c'"] \arrow[d, "c"'] & V \arrow[r, "b'"] \arrow[d, "b"'] & Z' \arrow[r, "h"] \arrow[d, "u"'] & W \\
 & U \arrow[r, "a'"] \arrow[d, "a"'] & Y' \arrow[r, "g"] \arrow[d, "t"'] & Z & \\
X & X' \arrow[l, "s"'] \arrow[r, "f"] & Y. & &
\end{tikzcd}$$

in which a, b and c are in S. Note that the composition of the first two left fractions is represented by

$$X \xleftarrow{sa} U \xrightarrow{ga'} Z$$

and its composition with the third left fraction is represented by

$$X \xleftarrow{sac} M \xrightarrow{hb'c'} W.$$

While the composition of the last two left fractions is represented by

$$Y \xleftarrow{tb} V \xrightarrow{hb'} W$$

and its composition with the first left fraction is represented by

$$X \xleftarrow{sac} M \xrightarrow{hb'c'} W.$$

Hence, the composite is associative. Next we prove that

$$X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$$

is the identity morphism. Denote the above left fraction by id_X. Consider the following left fraction between X and Y.

$$X \xleftarrow{s} X' \xrightarrow{f} Y.$$

Thus, the following commutative diagram

$$\begin{tikzcd}
 & X' \arrow[r, "\mathrm{id}"] \arrow[d, "s"'] & X' \arrow[r, "f"] \arrow[d, "s"'] & Y \\
X & X \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & X &
\end{tikzcd}$$

implies that $fs^{-1}\,\mathrm{id}_X = fs^{-1}$. Similarly, the following commutative diagram

$$\begin{tikzcd} & W' \arrow[r, "g"] \arrow[d, "\mathrm{id}"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \\ W & W' \arrow[l, "t"'] \arrow[r, "g"] & X & \end{tikzcd}$$

implies that $\mathrm{id}_X\, gt^{-1} = gt^{-1}$ where

$$W \xleftarrow{t} W' \xrightarrow{g} X$$

is a left fraction between W and X. Hence, id_X is the identity morphism.

We define a *right fraction* between X and Y to be a chain of the form

$$s^{-1}f : X \xrightarrow{f} Y_1 \xleftarrow{s} Y.$$

Similarly, we define a relation on right fractions as follows. We say that $s^{-1}f$ is equivalent to

$$X \xrightarrow{g} Y_2 \xleftarrow{t} Y$$

if there exists a fraction

$$X \longrightarrow Y_3 \longleftarrow Y$$

fitting into a commutative diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} & Y_1 \arrow[d] & \\ X \arrow[ur, "f"] \arrow[r] \arrow[dr, "g"'] & Y_3 & Y \arrow[ul, "s"'] \arrow[l] \arrow[dl, "t"] \\ & Y_2 \arrow[u] & \end{tikzcd}$$

Similarly, we can show that the above relation is an equivalence relation.

The following important result is proved in [13, Lemma III.2.8].

Theorem 2.1.4. *Let S be a multiplicative system in a category $\mathcal{C}$. Then the category $S^{-1}\mathcal{C}$ can be described as follows. $S^{-1}\mathcal{C}$ has the same objects as $\mathcal{C}$ and $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Y)$ is the family of equivalence classes of left fractions between X and Y. The universal functor $q\colon \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$ sends $f\colon X \longrightarrow Y$ to $X = X \xrightarrow{f} Y$.*

Remark 2.1.5. $S^{-1}\mathcal{C}$ can be constructed using equivalence classes of right fractions.

The following result is proved in [33, Corollary 10.3.11].

Lemma 2.1.6. *If $\mathcal{C}$ is an additive category, then so is $S^{-1}\mathcal{C}$ and q is an additive functor.*

2.2 Triangulated Categories

In this section, we present the axioms and basic properties of triangulated categories. The main references for this section are [25] and [33].

Definition 2.2.1. A *triangle* in some category of complexes ($\mathbf{K}(\mathcal{A})$, $\mathcal{D}(\mathcal{A}), \cdots$) where $\mathcal{A}$ is an abelian category is a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[-1].$$

Definition 2.2.2. A triangle of the form

$$X \xrightarrow{u} Y \xrightarrow{v} \mathrm{cone}(u) \xrightarrow{\partial} X[-1]$$

is called a *strict triangle* on u.

Definition 2.2.3. We say that a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[-1]$$

is *distinguished* if it is isomorphic to a strict triangle on $u'\colon X' \longrightarrow Y'$ in the sense that there is a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & X[-1] \arrow[d, "f[-1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & \mathrm{cone}(u') \arrow[r, "\partial"] & X'[-1] \end{tikzcd}$$

where f, g and h are isomorphisms in the corresponding category.

Definition 2.2.4. Let $\mathcal{C}$ be an additive category. We say that $\mathcal{C}$ is a *triangulated category* if it is equipped with an automorphism $T\colon \mathcal{C} \longrightarrow \mathcal{C}$ called the translation functor and with a class of triangles called distinguished triangles which are subject to the following four axioms:

(TR1) (a) Every morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX.$$

(b) Any triangle isomorphic to a distinguished one is itself distinguished.

(c) The triangle

$$X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow TX$$

is a distinguished triangle.

(TR2) A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

is distinguished if and only if the triangle

$$Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$$

is distinguished.

(TR3) For any diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$

where the rows are distinguished triangles and the first square is commutative, there is a morphism $h\colon Z \longrightarrow Z'$, not necessarily unique, which makes the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, dotted, "h"'] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$

commutative.

(TR4) (The octahedral axiom). Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Y'$ be two composable morphisms. Let us be given distinguished triangles

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$$

$$X \xrightarrow{gf} Y' \longrightarrow Z' \longrightarrow TX$$

$$Y \xrightarrow{g} Y' \longrightarrow Y'' \longrightarrow TY$$

Then we can complete this to a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d] & TX \arrow[d, "\mathrm{id}"] \\
X \arrow[r, "gf"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & TX \arrow[d] \\
0 \arrow[r] \arrow[d] & Y'' \arrow[r, "\mathrm{id}"] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \arrow[d] \\
TX \arrow[r, "Tf"] & TY \arrow[r] & TZ \arrow[r] & T^2X
\end{tikzcd}$$

where the first and second row and second column are given three distinguished triangles, and every row and column in the diagram is a distinguished triangle.

The following important theorem is proved in [33, Proposition 10.2.4].

Theorem 2.2.5. $\mathbf{K}(R)$ *is a triangulated category.*

The following result is in [33, Corollary 10.2.5].

Corollary 2.2.6. *Let $\mathcal{C}$ be a full subcategory of $\mathbf{Ch}(R)$ and $\mathcal{K}$ its corresponding quotient category. Suppose that $\mathcal{C}$ is an additive category and is closed under translation and the formation of mapping cones. Then $\mathcal{K}$ is a triangulated category.*

Therefore, we deduce from Corollary 2.2.6 that $\mathbf{K}_b(R)$, $\mathbf{K}_-(R)$ and $\mathbf{K}_+(R)$ are triangulated categories.

2.2.1 Basic Properties of Triangulated Categories

We present some elementary properties of triangulated categories. We start by giving the following result.

Lemma 2.2.7. *Let $\mathcal{C}$ be a triangulated category with a translation functor T. If*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

is a distinguished triangle in $\mathcal{C}$, then the composites vu, wv and Tuw are zero.

Proof. Consider the following diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] & TX \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & TX \end{tikzcd}$$

Then by (TR3), we can complete the diagram to get the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d] & TX \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & TX \end{tikzcd}$$

and now we see that $vu = 0$. Also from the above and axiom (TR2) we deduce that $wv = 0$ and $Tuw = 0$. □

Definition 2.2.8. Let $\mathcal{C}$ be a triangulated category with a translation functor T. Let $\mathcal{A}$ be an abelian category. An additive functor $H\colon \mathcal{C} \to \mathcal{A}$ is called *homological* if for every distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX,$$

the sequence

$$\cdots \longrightarrow H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(TX) \longrightarrow \cdots$$

is exact in the abelian category $\mathcal{A}$.

Definition 2.2.9. Let $\mathcal{C}$ be a triangulated category with a translation functor T. Let $\mathcal{A}$ be an abelian category. An additive functor $H\colon \mathcal{C} \to \mathcal{A}$ is called *cohomological* if for every distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX,$$

the sequence

$$\cdots \longrightarrow H(Z) \xrightarrow{H(v)} H(Y) \xrightarrow{H(u)} H(X) \xrightarrow{H(w)} H(T^{-1}X) \longrightarrow \cdots$$

is exact in the abelian category $\mathcal{A}$.

The following lemma is proved in [25, Lemma 1.1.10].

Lemma 2.2.10. *Let $\mathcal{C}$ be a triangulated category. Let W be an object of $\mathcal{C}$. Then the functor $\operatorname{Hom}_{\mathcal{C}}(W,-)$ is homological.*

The following lemma is proved in [25, Remark 1.1.11].

Lemma 2.2.11. *Let $\mathcal{C}$ be a triangulated category. Let W be an object of $\mathcal{C}$. Then the functor $\operatorname{Hom}_{\mathcal{C}}(-,W)$ is cohomological.*

Now consider the triangulated category $\mathbf{K}(R)$ and the chain complex $R[-n]$. Then for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[-1],$$

the sequence

$$\cdots \to \operatorname{Hom}_{\mathbf{K}(R)}(R[-n],X) \xrightarrow{u_\star} \operatorname{Hom}_{\mathbf{K}(R)}(R[-n],Y) \xrightarrow{v_\star} \\ \operatorname{Hom}_{\mathbf{K}(R)}(R[-n],Z) \xrightarrow{w_\star} \operatorname{Hom}_{\mathbf{K}(R)}(R[-n],X[-1]) \to \cdots$$

is exact. Using Lemma 1.3.8, we have the following result.

Corollary 2.2.12. *If*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[-1]$$

is a distinguished triangle, then the following homology sequence

$$\cdots \longrightarrow H_n(X) \xrightarrow{u_\star} H_n(Y) \xrightarrow{v_\star} H_n(Z) \xrightarrow{w_\star} H_{n-1}(X) \longrightarrow \cdots$$

is exact.

Lemma 2.2.13. *Let $f\colon X \longrightarrow Y$ be a morphism in $\mathbf{K}(R)$. Then the following conditions are equivalent.*

(i) The morphism f is a q-isomorphism.

(ii) The cone of f is acyclic.

Proof. Consider the following distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow \operatorname{cone}(f) \longrightarrow X[-1].$$

By Corollary 2.2.12, we have the following long exact sequence of homology

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_\star} H_n(Y) \longrightarrow H_n(\mathrm{cone}(f)) \longrightarrow H_{n-1}(X) \xrightarrow{f_\star} \cdots$$

Since f is a q-isomorphism, $f_\star$ is an isomorphism. Thus, $H_n(\mathrm{cone}(f)) = 0$ for each n. Hence, $\mathrm{cone}(f)$ is acyclic. Now assume that $\mathrm{cone}(f)$ is acyclic. From the above long exact sequence of homology

$$\cdots \longrightarrow H_{n+1}(\mathrm{cone}(f)) \longrightarrow H_n(X) \xrightarrow{f_\star} H_n(Y) \longrightarrow H_n(\mathrm{cone}(f)) \longrightarrow \cdots$$

we deduce that $f_\star$ is an isomorphism, that is, f is a q-isomorphism. □

Remark 2.2.14. Lemma 2.2.13 also holds if we replace $\mathbf{K}(R)$ by any of its full subcategories $\mathbf{K}_b(R)$, $\mathbf{K}_-(R)$ and $\mathbf{K}_+(R)$ of bounded, bounded above and bounded below chain complexes of R-modules.

Lemma 2.2.15. *(Five Lemma) Let $\mathcal{C}$ be a triangulated category with a translation functor T. Consider the following diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & TX \arrow[d, "Tf"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd}$$

where the rows are distinguished triangles. If f and g are isomorphisms in $\mathcal{C}$, then so is h.

Proof. Assume that f and g are isomorphisms. Now consider the following diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Z', Y) \arrow[r] \arrow[d, "g_\star"'] & \mathrm{Hom}_{\mathcal{C}}(Z', Z) \arrow[r] \arrow[d, "h_\star"'] & \mathrm{Hom}_{\mathcal{C}}(Z', TX) \arrow[r] \arrow[d, "Tf_\star"'] & \cdots \\ \cdots \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Z', Y') \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Z', Z') \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Z', TX') \arrow[r] & \cdots \end{tikzcd}$$

We see that the diagram is commutative and the rows are exact by Lemma 2.2.10. Also, $f_\star$, $g_\star$, $Tf_\star$ and $Tg_\star$ are isomorphisms. Then the Five Lemma implies that $h_\star$ is an isomorphism. Therefore, there exists $a\colon Z' \longrightarrow Z$ such that $h_\star(a) = ha = \mathrm{id}_{Z'}$. Also, consider the following diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(TX', Z) \arrow[r] \arrow[d, "Tf^\star"'] & \mathrm{Hom}_{\mathcal{C}}(Z', Z) \arrow[r] \arrow[d, "h^\star"'] & \mathrm{Hom}_{\mathcal{C}}(Y', Z) \arrow[r] \arrow[d, "g^\star"'] & \cdots \\ \cdots \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(TX, Z) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Z, Z) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Y, Z) \arrow[r] & \cdots \end{tikzcd}$$

We see that the diagram is commutative and the rows are exact by Lemma 2.2.11. Also, $f^\star$, $g^\star$, $Tf^\star$, and $Tg^\star$ are isomorphisms. By the Five Lemma, $h^\star$ is an isomorphism. Therefore, there exists $b\colon Z' \longrightarrow Z$ such that $h^\star(b) = bh = \mathrm{id}_Z$. Thus,

$$b = b(ha) = (bh)a = a.$$

Hence, h is an isomorphism. $\square$

Lemma 2.2.16. *Every distinguished triangle, in a triangulated category $\mathcal{C}$ with a translation functor T, is determined up to isomorphism by any of its morphisms.*

Proof. Let $u\colon X \longrightarrow Y$ be given. By (TR1), u can be completed to a triangle. Now let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

and

$$X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} TX$$

be two distinguished triangles completing u. Therefore, we have the following diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\mathrm{id}"'] & Y \arrow[r, "v"] \arrow[d, "\mathrm{id}"'] & Z \arrow[r, "w"] & TX \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "u"] & Y \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX \end{tikzcd}$$

By (TR3), this diagram can be completed to have

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\mathrm{id}"'] & Y \arrow[r, "v"] \arrow[d, "\mathrm{id}"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & TX \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "u"] & Y \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX \end{tikzcd}$$

But $\mathrm{id}\colon X \longrightarrow X$ and $\mathrm{id}\colon Y \longrightarrow Y$ are isomorphisms. Now Lemma 2.2.15 implies that h is an isomorphism. Thus, Z is well defined up to isomorphism. $\square$

Remark 2.2.17. Let $\mathcal{C}$ be a triangulated category with translation functor T. Suppose we have a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX,$$

and a factoring of the identity on X as

$$X \xrightarrow{u} Y \xrightarrow{u'} X$$

Then we have a canonical isomorphism $Y \cong X \oplus Z$. This is easily seen by the following diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\mathrm{id}"'] & Y \arrow[r, "v"] \arrow[d, "h"'] & Z \arrow[r, "w"] \arrow[d, "\mathrm{id}"'] & TX \arrow[d, "\mathrm{id}"'] \\ X \arrow[r] & X \oplus Z \arrow[r] & Z \arrow[r] & TX \end{tikzcd}$$

where h is clearly an isomorphism by the Five Lemma.

Remark 2.2.18. Let $\mathcal{C}$ be a triangulated category with a translation functor T. Suppose that we have the following distinguished triangle

$$X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z \longrightarrow TX$$

and a morphism $\alpha\colon Y \longrightarrow W$ such that the composite $\alpha\beta$ is zero. Then there exists a morphism $\bar{\alpha}\colon Z \longrightarrow W$ such that $\alpha = \bar{\alpha}\gamma$. For we have the following exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(TX, W) \to \mathrm{Hom}_{\mathcal{C}}(Z, W) \xrightarrow{\gamma^\star} \mathrm{Hom}_{\mathcal{C}}(Y, W) \xrightarrow{\beta^\star} \mathrm{Hom}_{\mathcal{C}}(X, W)$$

and if $\beta^\star(\alpha) = 0$, then there exists $\bar{\alpha} \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$ such that $\gamma^\star(\bar{\alpha}) = \alpha$. Dually, given the same distinguished triangle and a morphism $\alpha\colon W \longrightarrow Y$ such that the composite $\gamma\alpha = 0$, there exists a morphism $\bar{\alpha}\colon W \longrightarrow X$ such that $\beta\bar{\alpha} = \alpha$.

2.2.2 Homotopy Limits and Colimits

In this subsection, we define the notions of homotopy limit and homotopy colimit.

Definition 2.2.19. Suppose that $\mathcal{C}$ is a triangulated category with a translation functor T and assume that countable products exist in $\mathcal{C}$. Let

$$X_0 \xleftarrow{j_0} X_1 \xleftarrow{j_1} X_2 \xleftarrow{j_2} X_3 \longleftarrow \cdots$$

be a sequence of objects and morphisms in $\mathcal{C}$. The *homotopy limit* of the sequence, denoted $\mathrm{holim}_i\, X_i$, is by definition given up to non-canonical isomorphism by the following distinguished triangle

$$\mathrm{holim}_i\, X_i \longrightarrow \textstyle\prod_{i\geq 0} X_i \xrightarrow{1-g} \prod_{i\geq 0} X_i \longrightarrow T\,\mathrm{holim}_i\, X_i$$

where g is induced by the maps j_1, j_2, ...

Definition 2.2.20. Suppose that $\mathcal{C}$ is a triangulated category with a translation functor T and assume that countable coproducts exist in $\mathcal{C}$. Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} \cdots$$

be a sequence of objects and morphisms in $\mathcal{C}$. The *homotopy colimit* of the sequence, denoted $\operatorname{hocolim}_i X_i$, is by definition given up to non-canonical isomorphism by the following distinguished triangle

$$\coprod_{i\geq 0} X_i \xrightarrow{1-shift} \coprod_{i\geq 0} X_i \longrightarrow \operatorname{hocolim}_i X_i \longrightarrow T\coprod_{i\geq 0} X_i$$

where the shift map is the direct sum of $j_{i+1}\colon X_i \longrightarrow X_{i+1}$.

The following result is proved in [25, Lemma 1.6.5].

Lemma 2.2.21. *If we have two sequences*

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

and

$$Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots$$

then non-canonically

$$\operatorname*{hocolim}_i \{X_i \oplus Y_i\} \cong \{\operatorname*{hocolim}_i X_i\} \oplus \{\operatorname*{hocolim}_i Y_i\}.$$

The following result is proved in [25, Lemma 1.6.7].

Lemma 2.2.22. *Let*

$$X_0 \xrightarrow{0} X_1 \xrightarrow{0} X_2 \xrightarrow{0} \cdots$$

be a sequence. Then the homotopy colimit of this sequence is 0.

The following result is proved in [25, Lemma 1.7.1].

Proposition 2.2.23. *Let*

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

be a sequence. Suppose we take any increasing sequence of integers

$$0 \leq i_0 < i_1 < i_2 < i_3 < \cdots .$$

Then we can form the subsequence

$$X_{i_0} \longrightarrow X_{i_1} \longrightarrow X_{i_2} \longrightarrow \cdots$$

Then the two sequences have isomorphic homotopy colimits.

Definition 2.2.24. Let

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

be a sequence. We say that this sequence is *pro-zero* if for each r, there exists $s > r$ such that $X_s \longrightarrow X_r$ is zero.

It follows that if a sequence is pro-zero, then its homotopy limit is 0.

The following result is proved in [25, Proposition 1.6.8].

Proposition 2.2.25. *Let* $e\colon X \longrightarrow X$ *be idempotent, that is,* $e^2 = e$*. Then there are morphisms* ϕ *and* ψ

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} X$$

with $\phi\psi = \mathrm{id}_Y$ *and* $\psi\phi = e$*. Moreover,*

$$Y \cong \mathrm{hocolim}(X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots).$$

2.3 The Derived Category

In this section, we show that the derived category $\mathcal{D}(R)$ is a triangulated category. The main references for this section are [13] and [33].

Proposition 2.3.1. *If* S *is the collection of* q*-isomorphisms in* $\mathbf{K}(R)$*, then* S *is multiplicative system.*

Proof. Assume that S is the collection of q-isomorphisms in $\mathbf{K}(R)$. We show that S is multiplicative system. Axiom (i) is obvious. Now we show Ore condition. Let $f\colon X \longrightarrow Y$ and $s\colon Z \longrightarrow Y$ be given. Using (TR1), form the following distinguished triangle

$$Z \xrightarrow{s} Y \xrightarrow{u} C \xrightarrow{\partial} Z[-1].$$

Also, embed $uf\colon X \longrightarrow C$ into the distinguished triangle

$$W \xrightarrow{t} X \xrightarrow{uf} C \xrightarrow{v} W[-1].$$

By (TR3), there is a morphism g such that the diagram

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "g"'] & X \arrow[r, "uf"] \arrow[d, "f"] & C \arrow[r, "v"] \arrow[d, "\mathrm{id}"] & W[-1] \arrow[d, "Tg"] \\ Z \arrow[r, "s"] & Y \arrow[r, "u"] & C \arrow[r, "\partial"] & Z[-1] \end{tikzcd}$$

is commutative. We show that t is a q-isomorphism. Lemma 2.2.13 implies that $H_\star(C) = 0$ since s is q-isomorphism. Now the long exact homology sequence of the top distinguished triangle implies that t is q-isomorphism, that is, $t \in S$. Similarly, we can prove the symmetric assertion. Next we show the cancellation condition holds. Let $f, g\colon X \longrightarrow Y$. Let $s\colon Y \longrightarrow Y'$ be in S with $sf = sg$. We show that there exists $t\colon X' \longrightarrow X$ such that $ft = gt$. Using (TR1), we have the following distinguished triangle

$$Z \xrightarrow{u} Y \xrightarrow{s} Y' \longrightarrow Z[-1].$$

Since $s \in S$, we have that $H_\star(Z) = 0$. Therefore, we have the following exact sequence

$$\mathrm{Hom}_{\mathbf{K}(R)}(X, Z) \xrightarrow{u_\star} \mathrm{Hom}_{\mathbf{K}(R)}(X, Y) \xrightarrow{s_\star} \mathrm{Hom}_{\mathbf{K}(R)}(X, Y').$$

But $s(f-g) = 0$. Thus, there exists $h\colon X \longrightarrow Z$ such that $f - g = uh$. Using (TR1), we have the following commutative diagram

$$X' \xrightarrow{t} X \xrightarrow{h} Z \longrightarrow X'[-1].$$

But $H_\star(Z) = 0$, which implies that t is q-isomorphism, that is, $t \in S$. Since $ht = 0$, we have $(f-g)t = 0$, that is, $ft = gt$. Hence, S is multiplicative system. □

Definition 2.3.2. The localization $S^{-1}\mathbf{K}(R)$ of the homotopy category of chain complexes $\mathbf{K}(R)$ is the *derived category* $\mathcal{D}(R)$ where S is the collection of q-isomorphisms in $\mathbf{K}(R)$.

The following result is proved in [13, Proposition III.4.2]

Theorem 2.3.3. *The localization of* $\mathbf{K}(R)$ *by q-isomorphisms is equivalent to the localization of* $\mathbf{Ch}(R)$ *by q-isomorphisms. The same is true for* $\mathbf{K}_\star(R)$ *and* $\mathbf{Ch}_\star(R)$ *where* $\star = +, -$ *or* b*.*

2.3.1 $\mathcal{D}(R)$ is Triangulated

In this subsection, we show the following theorem which says that the derived category is a triangulated category.

Theorem 2.3.4. $\mathcal{D}(R)$ *is a triangulated category.*

Proof. First note that Lemma 2.1.6 and Theorem 2.2.5 combine together to give additivity of $\mathcal{D}(R)$ and the formula $T(fs^{-1}) = T(f)T(s)^{-1}$ defines a translation functor T on $\mathcal{D}(R)$. To prove (TR1), it is enough to check that every morphism can be completed to a distinguished triangle. Let $X \xrightarrow{u} Y$ be in $\mathcal{D}(R)$ represented by the fraction

$$X \xleftarrow{s} Z \xrightarrow{u'} Y.$$

Since $\mathbf{K}(R)$ is triangulated by Theorem 2.2.5, we can complete u' to a distinguished triangle

$$Z \xrightarrow{u'} Y \xrightarrow{v} W \xrightarrow{w} TZ$$

Now consider the following triangle

$$X \xrightarrow{u} Y \xrightarrow{v} W \xrightarrow{Tsw} TX$$

in $\mathcal{D}(R)$. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "u'"] \arrow[d, "s"'] & Y \arrow[r, "v"] \arrow[d, "\mathrm{id}"'] & W \arrow[r, "w"] \arrow[d, "\mathrm{id}"'] & TZ \arrow[d, "Ts"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & W \arrow[r, "Tsw"] & TX
\end{tikzcd}$$

Since s is invertible in $\mathcal{D}(R)$,

$$X \xrightarrow{u} Y \xrightarrow{v} W \xrightarrow{Tsw} TX$$

is distinguished. (TR2) obviously follows from the definitions and from the properties of T. Next we show (TR3). Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

and

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} TX'$$

be two distinguished triangles in $\mathcal{D}(R)$ with morphisms $f\colon X \longrightarrow X'$ and $g\colon Y \longrightarrow Y'$ such that $gu = u'f$. We claim that there exists a morphism $h\colon Z \longrightarrow Z'$ such that the following diagram is commutative

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & TX \arrow[d, "Tf"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX'
\end{tikzcd}$$

We can assume that the given distinguished triangles in $\mathcal{D}(R)$ are represented by distinguished triangles in $\mathbf{K}(R)$ and the morphisms f, g in $\mathcal{D}(R)$ are represented by left fractions

$$X \xleftarrow{s} X'' \xrightarrow{\bar{f}} X'$$

and

$$Y \xleftarrow{t} Y'' \xrightarrow{\bar{g}} Y',$$

respectively. We must construct the arrows r and $\bar{h}$ in the following diagram

$$\begin{tikzcd}
& X'' \arrow[rr, "u''"] \arrow[dl, "s"'] \arrow[dd, "\bar{f}" near end] & & Y'' \arrow[rr, "v''"] \arrow[dl, "t"'] \arrow[dd, "\bar{g}" near end] & & Z'' \arrow[rr, "w''"] \arrow[dl, "r"'] \arrow[dd, "\bar{h}" near end] & & TX'' \arrow[dl, "Ts"] \arrow[dd, "T\bar{f}"] \\
X \arrow[rr, "u" near start] & & Y \arrow[rr, "v" near start] & & Z \arrow[rr, "w" near start] & & TX & \\
& X' \arrow[rr, "u'"] & & Y' \arrow[rr, "v'"] & & Z' \arrow[rr, "w'"] & & TX'
\end{tikzcd}$$

We claim that by changing, if necessary, the fraction representing $f\colon X \longrightarrow X'$ we can guarantee the existence of a morphism $u''\colon X'' \longrightarrow Y''$ in $\mathbf{K}(R)$ such that both squares containing this morphism are commutative. Using the Ore condition, we complete the following diagram to a commutative square in $\mathbf{K}(R)$

$$\begin{tikzcd}
\bar{X} \arrow[r, "\bar{u}"] \arrow[d, "\bar{t}"'] & Y'' \arrow[d, "t"] \\
X'' \arrow[r, "us"'] & Y
\end{tikzcd}$$

where $\bar{t} \in S$. Replace X'' by $\bar{X}$, s by $s\bar{t}$, $\bar{f}$ by $\bar{f}\bar{t}$. It is clear that

$$X \xleftarrow{s\bar{t}} \bar{X} \xrightarrow{\bar{f}\bar{t}} X'$$

represents the same morphism $f\colon X \longrightarrow X'$ in $\mathcal{D}(R)$. Next, $\bar{u}\colon \bar{X} \longrightarrow Y''$ makes one of the two squares commutative, the square $t\bar{u} = us\bar{t}$ while the second square commutes in $\mathcal{D}(R)$ but not necessarily in $\mathbf{K}(R)$ where we have

$$u'\bar{f}s^{-1} = \bar{g}t^{-1}u = \bar{g}\bar{u}(\bar{t})^{-1}s^{-1}$$

since $u'f = gu$. So $u'\bar{f}\bar{t} = \bar{g}\bar{u}$ in $\mathcal{D}(R)$. To make the second square commutative in $\mathbf{K}(R)$, we must change the representative of f once more. Let us consider two morphisms $u'\bar{f}\bar{t}, \bar{g}\bar{u}\colon \bar{X} \longrightarrow Y'$ in $\mathbf{K}(R)$. As they are equal in $\mathcal{D}(R)$, there exists $q\colon \bar{\bar{X}} \longrightarrow \bar{X}$ where $q \in S$. Then we take $\bar{\bar{X}}$ as the new X'' and the rest is clear. Now we complete $u''\colon X'' \longrightarrow Y''$ to the following distinguished triangle in $\mathbf{K}(R)$

$$X'' \xrightarrow{u''} Y'' \xrightarrow{v''} Z'' \xrightarrow{w''} TX''.$$

Using (TR3) for $\mathbf{K}(R)$, we choose $\bar{h}$ making the diagram commutative. Similarly, we construct r and since s, t are in S we see that $r \in S$. Denote by h the morphism $Z \longrightarrow Z'$ in $\mathcal{D}(R)$ represented by the left fraction

$$Z \xleftarrow{r} Z'' \xrightarrow{\bar{h}} Z'.$$

Hence, (TR3) holds for $\mathcal{D}(R)$. Next we show (TR4). Suppose that we have the following three distinguished triangles in $\mathcal{D}(R)$

$$X \xrightarrow{\alpha} Y \longrightarrow Z \longrightarrow TX$$

$$X \xrightarrow{\beta\alpha} Y' \longrightarrow Z' \longrightarrow TX$$

$$Y \xrightarrow{\beta} Y' \longrightarrow Y'' \longrightarrow TY$$

We claim that we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & Y \arrow[r] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d] & TX \arrow[d, "\mathrm{id}"] \\
X \arrow[r, "\beta\alpha"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & TX \arrow[d] \\
0 \arrow[r] \arrow[d] & Y'' \arrow[r, "\mathrm{id}"] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \arrow[d] \\
TX \arrow[r, "T\alpha"] & TY \arrow[r] & TZ \arrow[r] & T^2X
\end{tikzcd}$$

Let α and β be represented by some left fractions

$$X \xleftarrow{s} U \xrightarrow{f} Y$$

and

$$Y \xleftarrow{t} V \xrightarrow{g} Y'$$

with $s, t \in S$. Composition is represented by

$$\begin{tikzcd}
 & W \arrow[r, "h"] \arrow[d, "t'"'] & V \arrow[r, "g"] \arrow[d, "t"] & Y' \\
X & U \arrow[l, "s"'] \arrow[r, "f"] & Y &
\end{tikzcd}$$

where $t' \in S$. Therefore, $\beta\alpha$ is represented by the left fraction

$$X \xleftarrow{st'} W \xrightarrow{gh} Y'.$$

We see that the left fraction

$$X \xleftarrow{st'} W \xrightarrow{ft'} Y$$

represents in $\mathcal{D}(R)$ the same morphism α. Now consider the following three distinguished triangles in $\mathbf{K}(R)$

$$W \xrightarrow{h} V \longrightarrow \operatorname{cone}(h) \longrightarrow TW,$$

$$V \xrightarrow{g} Y' \longrightarrow \operatorname{cone}(g) \longrightarrow TV,$$

$$W \xrightarrow{gh} Y' \longrightarrow \operatorname{cone}(gh) \longrightarrow TW.$$

We have the following diagram in $\mathcal{D}(R)$

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "st'"'] & V \arrow[r] \arrow[d, "t"'] & \operatorname{cone}(h) \arrow[r] \arrow[d, dotted, "\exists\ r"] & TW \arrow[d, "Ts"] \\
X \arrow[r, "\alpha"] & Y \arrow[r] & Z \arrow[r] & TX
\end{tikzcd}$$

in which the left square is commutative and s, t are isomorphisms in $\mathcal{D}(R)$. By the axiom (TR3) for $\mathcal{D}(R)$, which we just proved, there exists a morphism $r\colon \operatorname{cone}(h) \longrightarrow Z$ in $\mathcal{D}(R)$ that makes the diagram commutative. By Lemma 2.2.15, we have that r

is an isomorphism since st', t are isomorphisms in $\mathcal{D}(R)$. Similarly, there exists an isomorphism $r'\colon \operatorname{cone}(g) \longrightarrow Y''$ in $\mathcal{D}(R)$ such that

$$\begin{tikzcd}
V \arrow[r, "g"] \arrow[d, "t"'] & Y' \arrow[r] \arrow[d, "\mathrm{id}"'] & \operatorname{cone}(g) \arrow[r] \arrow[d, "r'"'] & TV \arrow[d, "Tt"] \\
Y \arrow[r, "\beta"] & Y' \arrow[r] & Y'' \arrow[r] & TY
\end{tikzcd}$$

is an isomorphism of distinguished triangles. Also, there exists an isomorphism $r''\colon \operatorname{cone}(gh) \longrightarrow Z'$ in $\mathcal{D}(R)$ such that

$$\begin{tikzcd}
W \arrow[r, "gh"] \arrow[d, "st'"'] & Y' \arrow[r] \arrow[d, "\mathrm{id}"'] & \operatorname{cone}(gh) \arrow[r] \arrow[d, "r''"'] & TW \arrow[d, "Tst'"] \\
X \arrow[r, "\beta\alpha"] & Y' \arrow[r] & Z' \arrow[r] & TX
\end{tikzcd}$$

is an isomorphism of distinguished triangles. Now since $\mathbf{K}(R)$ is triangulated, Theorem 2.2.5, we can complete the above three distinguished triangles to the following commutative diagram

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "\mathrm{id}"'] & V \arrow[r] \arrow[d, "g"'] & \operatorname{cone}(h) \arrow[r] \arrow[d] & TW \arrow[d, "\mathrm{id}"] \\
W \arrow[r, "gh"] \arrow[d] & Y' \arrow[r] \arrow[d] & \operatorname{cone}(gh) \arrow[r] \arrow[d] & TW \arrow[d] \\
0 \arrow[r] \arrow[d] & \operatorname{cone}(g) \arrow[r, "\mathrm{id}"] \arrow[d] & \operatorname{cone}(g) \arrow[r] \arrow[d] & 0 \arrow[d] \\
TW \arrow[r, "Th"] & TV \arrow[r] & T\operatorname{cone}(h) \arrow[r] & T^2W
\end{tikzcd}$$

Hence, (TR4) holds for $\mathcal{D}(R)$. $\square$

The proof of the following result is in [13, Proposition IV.2.8].

Proposition 2.3.5. *Every exact sequence of chain complexes*

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$$

in $\mathbf{Ch}(R)$ *can be completed to a distinguished triangle in* $\mathcal{D}(R)$ *by an appropriate morphism* $Z \longrightarrow X[-1]$.

Remark 2.3.6. Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[-1]$$

be a distinguished triangle. Consider $\mathrm{Hom}_{\mathcal{D}(R)}(W_s, -)_\star$ where $s \geq 0$. We know that the following sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(W_s, X)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(W_s, Y)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(W_s, Z)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(W_s, X)_{n-1} \longrightarrow \cdots$$

is a long exact sequence of abelian groups for each $s \geq 0$. But the category **Ab** of abelian groups satisfies $(AB5)$, that is, **Ab** is cocomplete and filtered colimits of exact sequences are exact. So the following sequence

$$\cdots \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}(R)}(W_s, X)_n \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}(R)}(W_s, Y)_n \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}(R)}(W_s, Z)_n \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}(R)}(W_s, X)_{n-1} \longrightarrow \cdots$$

is exact.

Similarly, let

$$X_s \longrightarrow Y_s \longrightarrow Z_s \longrightarrow X_s[-1]$$

be a distinguished triangle for each $s \geq 0$. Consider $\mathrm{Hom}_{\mathcal{D}(R)}(-, N)_\star$. Then the following sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(Z_s, N)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(Y_s, N)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(X_s, N)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(Z_s, N)_{n+1} \longrightarrow \cdots$$

is a long exact sequence of abelian groups for each $s \geq 0$. Therefore, the following sequence

$$\cdots \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}(R)}(Z_s, N)_n \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}(R)}(Y_s, N)_n \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}(R)}(X_s, N)_n \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}(R)}(Z_s, N)_{n+1} \longrightarrow \cdots$$

is exact.

2.3.2 Localizing Subcategories

In this subsection, we compare the localization of a category $\mathcal{C}$ with the localizations of its subcategories.

The following result is proved in [13, Proposition III.2.10].

Proposition 2.3.7. *Let $\mathcal{C}$ be a category, S be a multiplicative system in $\mathcal{C}$ and $\mathcal{B}$ be a full subcategory of $\mathcal{C}$. Suppose also $S \cap \mathcal{B}$ be a multiplicative system in $\mathcal{B}$. $\mathcal{B}$ is called a* localizing subcategory *if any of the following equivalent conditions holds.*

(i) *The natural functor $S^{-1}\mathcal{B} \longrightarrow S^{-1}\mathcal{C}$ is fully faithful.*

(ii) *Whenever $C \longrightarrow B$ is a morphism in S with B in $\mathcal{B}$, there is a morphism $B' \longrightarrow C$ in $\mathcal{C}$ with B' in $\mathcal{B}$ such that the composite $B' \longrightarrow B$ is in S.*

(iii) *Whenever $B \longrightarrow C$ is a morphism in S with B in $\mathcal{B}$, there is a morphism $C \longrightarrow B'$ in $\mathcal{C}$ with B' in $\mathcal{B}$ such that the composite $B \longrightarrow B'$ is in S.*

The following result is proved in [33, Corollary 10.3.14].

Lemma 2.3.8. *If $\mathcal{B}$ is a localizing subcategory of $\mathcal{C}$, and for every object C in $\mathcal{C}$ there is a morphism $C \longrightarrow B$ in S with B in $\mathcal{B}$, then $S^{-1}\mathcal{B} \cong S^{-1}\mathcal{C}$. Suppose in addition that $S \cap \mathcal{B}$ consists of isomorphisms. Then*

$$\mathcal{B} \cong S^{-1}\mathcal{B} \cong S^{-1}\mathcal{C}.$$

The subcategories $\mathbf{K}_b(R)$, $\mathbf{K}_+(R)$ and $\mathbf{K}_-(R)$ of $\mathbf{K}(R)$ are localizing for the collection S of q-isomorphisms. Thus, their localizations are the full subcategories $\mathcal{D}_b(R)$, $\mathcal{D}_+(R)$ and $\mathcal{D}_-(R)$ whose objects are the chain complexes which are bounded, bounded below and bounded above, respectively.

The following result is proved in [13, Corollary IV.2.7].

Corollary 2.3.9. *$\mathcal{D}_b(R)$, $\mathcal{D}_-(R)$ and $\mathcal{D}_+(R)$ are triangulated categories.*

Lemma 2.3.10. *Let P be a bounded below chain complex of projectives. Let $s\colon X \longrightarrow P$ be a q-isomorphism where X is a bounded below chain complex. Then there exists a morphism of chain complexes $t\colon P \longrightarrow X$ such that st is homotopic to id_P.*

Proof. Since s is q-isomorphism, $\mathrm{cone}(s)$ is exact by Lemma 2.2.13. For brevity, let $\mathrm{cone}(s) = C$. Also, let d_C and d_P denote the differentials of C and P, respectively. There is a map $f\colon P \longrightarrow C$. Next we show that f is null homotopic. We construct the homotopy by induction. We may assume that we begin with $n = 0$, consider the following diagram.

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "f_0"] & 0 \\ C_1 \arrow[r, "d_C^1"] & C_0 \arrow[r] & 0 \end{tikzcd}$$

We have a map $\bar{k}_0 \colon P_0 \longrightarrow C_1$ since d_C^1 is surjective and P_0 is projective. Assume we constructed $\bar{k}_{n-1} \colon P_{n-1} \longrightarrow C_n$ such that

$$f_{n-1} = d_C^n \bar{k}_{n-1} + \bar{k}_{n-2} d_p^{n-1}.$$

If we show that $\operatorname{Im}(f_n - \bar{k}_{n-1} d_P^n) \subset \operatorname{Im} d_C^{n+1}$, then we will have the diagram

$$\begin{tikzcd} & P_n \arrow[d, "f_n - \bar{k}_{n-1} d_P^n"] & \\ C_{n+1} \arrow[r, "d_C^{n+1}"] & \operatorname{Im} d_C^{n+1} \arrow[r] & 0 \end{tikzcd}$$

and projectivity of P_n will give a map $\bar{k}_n \colon P_n \longrightarrow C_{n+1}$ such that $f_n = d_C^{n+1} \bar{k}_n + \bar{k}_{n-1} d_P^n$. Now we show that $\operatorname{Im}(f_n - \bar{k}_{n-1} d_P^n) \subset \operatorname{Im} d_C^{n+1}$. Since C is exact, it suffices to show that $d_C^n(f_n - \bar{k}_{n-1} d_p^n) = 0$. Since $d_C^n \bar{k}_{n-1} = f_{n-1} - \bar{k}_{n-2} d_P^{n-1}$, we have

$$d_C^n(f_n - \bar{k}_{n-1} d_P^n) = d_C^n f_n - f_{n-1} d_P^n = 0.$$

since f is a morphism of chain complexes. Thus, f is null homotopic, say, by a homotopy $\bar{k} = (t, k)$. We have $f(p) = (d_C \bar{k} + \bar{k} d_P)(p)$. But

$$\begin{aligned} (d_C \bar{k} + \bar{k} d_P)(p) &= d_C(t(p), k(p)) + (t d_P(p), k d_P(p)) \\ &= (-d_X t(p), d_P k(p) + s t(p)) + (t d_P(p), k d_P(p)). \end{aligned}$$

Since $f(p) = (0, p)$, we get

$$t d_P(p) - d_X t(p) = 0$$

and

$$p = d_P k(p) + s t(p) + k d_P(p).$$

That is, t is a morphism of chain complexes and st is homotopic to id_P. □

The following result is proved in [33, Corollary 10.3.9].

Lemma 2.3.11. *If two parallel maps $f, g\colon X \longrightarrow Y$ in $\mathbf{K}(R)$ become identified in $\mathcal{D}(R)$, then $fs = gs$ for some $s\colon Z \longrightarrow X$ in S.*

Theorem 2.3.12. *Let P be a bounded below chain complex of projectives. Then for any X, the map $\phi\colon \mathrm{Hom}_{\mathbf{K}(R)}(P, X) \to \mathrm{Hom}_{\mathcal{D}(R)}(P, X)$ is an isomorphism of R-modules.*

Proof. First we show that ϕ is onto. Let $\alpha\colon P \longrightarrow X$ be a morphism in $\mathcal{D}(R)$. Let

$$P \xleftarrow{s} Z \xrightarrow{f} X$$

be a representative of α. By Lemma 2.3.10, there exists $t\colon P \longrightarrow Z$ such that $st = \mathrm{id}_P$. Thus, $\phi(ft) = ft$. But $ft = fs^{-1}\colon P \longrightarrow X$ is equivalent to fs^{-1}. Thus, ϕ is onto. Next we show that ϕ is one to one. Let $f, g\colon P \longrightarrow X$ in $\mathbf{K}(R)$. Assume that f, g become identified in $\mathcal{D}(R)$. Then Lemma 2.3.11 says that there exists $s\colon Z \longrightarrow P$ such that $fs = gs$. But there exists $t\colon P \longrightarrow Z$ such that $st = \mathrm{id}_P$. Therefore, $f = fst = gst = g$ in $\mathbf{K}(R)$. Thus, ϕ is one to one. Finally, note that $\phi\colon \mathrm{Hom}_{\mathbf{K}(R)}(P, X) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(P, X)$ is a group homomorphism by Lemma 2.1.6. Also,

$$\phi(rf_n(p)) = \phi(f_n(rp)) = f_n(rp) = rf_n(p) = r\phi(f_n(p))$$

for each $r \in R$, $p \in P$, n and chain map $f\colon P \longrightarrow X$. This implies that ϕ is an R-module homomorphism. Hence, ϕ is an isomorphism of R-modules. □

Remark 2.3.13. Dually, if I is a bounded above chain complex of injectives, then

$$\mathrm{Hom}_{\mathbf{K}(R)}(X, I) \cong \mathrm{Hom}_{\mathcal{D}(R)}(X, I).$$

Theorem 2.3.14. *The localization $\mathcal{D}_+(R)$ of $\mathbf{K}_+(R)$ is equivalent to the full subcategory $\mathbf{K}_+(\mathcal{P})$ of bounded below chain complexes of projectives in $\mathbf{K}_+(R)$:*

$$\mathcal{D}_+(R) \cong \mathbf{K}_+(\mathcal{P}).$$

Proof. Let X be in $\mathbf{K}_+(R)$ and let $X \longrightarrow Y$ be a q-isomorphism where $Y \in \mathbf{K}_+(\mathcal{P})$. Lemma 1.4.15 says that X has a Cartan-Eilenberg resolution $P \longrightarrow X$ with $\mathrm{Tot}^{\oplus}(P)$ in $\mathbf{K}_+(\mathcal{P})$. But $\mathrm{Tot}^{\oplus}(P) \longrightarrow Y$ is a q-isomorphism. Therefore, $\mathbf{K}_+(\mathcal{P})$ is a localizing subcategory of $\mathbf{K}_+(R)$ by Proposition 2.3.7. Thus, $\mathcal{D}_+(R) \cong S^{-1}\mathbf{K}_+(\mathcal{P})$. By Lemma

2.3.8, it suffices to show that every q-isomorphism in $\mathbf{K}_+(\mathcal{P})$ is an isomorphism. Let P and Q be bounded below chain complexes of projectives and $s\colon P \longrightarrow Q$ a q-isomorphism. Lemma 2.3.10 implies that there is a morphism $t\colon Q \longrightarrow P$ such that $st = \mathrm{id}_Q$. In fact t is a q-isomorphism. So applying Lemma 2.3.10 we have a morphism $u\colon P \longrightarrow Q$ with $tu = \mathrm{id}_P$. Thus, s is an isomorphism in $\mathbf{K}_+(\mathcal{P})$ with $s^{-1} = t$. Hence,

$$\mathbf{K}_+(\mathcal{P}) \cong S^{-1}\mathbf{K}_+(\mathcal{P}) \cong \mathcal{D}_+(R). \qquad \square$$

In the following theorem, let R be noetherian.

Theorem 2.3.15. *Let $\mathbf{M}(R)$ denote the category of all finitely generated R-modules. Let $\mathcal{D}_{(fg)}(R)$ denote the full subcategory of $\mathcal{D}(R)$ consisting of chain complexes Y whose homology modules $H_n(Y)$ are all finitely generated. Then,*

$$\mathcal{D}_+(\mathbf{M}(R)) \cong \mathcal{D}_{+(fg)}(R).$$

where $\mathcal{D}_+(\mathbf{M}(R))$ denotes the derived category whose objects are bounded below chain complexes of finitely generated R-modules and $\mathcal{D}_{+(fg)}(R)$ denotes the derived category whose objects are bounded below chain complexes whose homology modules $H_n(Y)$ are all finitely generated.

Proof. We show that $\mathbf{K}_{+(fg)}(R)$ is a localizing subcategory of $\mathbf{K}_+(\mathbf{M}(R))$. Now let $Y \longrightarrow X$ be a q-isomorphism where X is in $\mathbf{K}_{+(fg)}(R)$. So $H_n(Y)$ is finitely generated for each n. There exists a Cartan-Eilenberg resolution $P \longrightarrow Y$ by Lemma 1.4.15. It is clear that $\mathrm{Tot}^{\oplus}(P)$ is in $\mathbf{K}_{+(fg)}(R)$ and the composite $\mathrm{Tot}^{\oplus}(P) \longrightarrow X$ is a q-isomorphism. Thus, $\mathbf{K}_{+(fg)}(R)$ is a localizing subcategory of $\mathbf{K}_+(\mathbf{M}(R))$ by Proposition 2.3.7. Since each object $Z \in \mathbf{K}_+(\mathbf{M}(R))$ has a Cartan-Eilenberg resolution $P \longrightarrow Z$ with $H_n(\mathrm{Tot}^{\oplus}(P)$ finitely generated for each n, we have

$$S^{-1}\mathbf{K}_+(\mathbf{M}(R)) \cong S^{-1}\mathbf{K}_{+(fg)}(R)$$

by Lemma 2.3.8. Hence,

$$\mathcal{D}_+(\mathbf{M}(R)) \cong \mathcal{D}_{+(fg)}(R). \qquad \square$$

2.4 Derived Functors

In this section, we study derived functors. We define the derived tensor product and the derived Hom. The main references for this section are [33], [13] and [25].

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. We write $\mathbf{K}_-(\mathcal{A})$, $\mathcal{D}_-(\mathcal{A})$ for the homotopy category and derived category of bounded above chain complexes of $\mathcal{A}$, respectively. Also, we write $\mathbf{K}_+(\mathcal{A})$, $\mathcal{D}_+(\mathcal{A})$ for the homotopy category and the derived category of bounded below chain complexes of $\mathcal{A}$, respectively.

Note that $\mathbf{K}_-(\mathcal{A})$, $\mathcal{D}_-(\mathcal{A})$, $\mathbf{K}_+(\mathcal{A})$ and $\mathcal{D}_+(\mathcal{A})$ are triangulated categories by [33, Corollary 10.2.5], [33, Corollary 10.4.3], [33, Corollary 10.2.5] and [33, Corollary 10.4.3], respectively.

Definition 2.4.1. Let $\mathbf{K}_1$, $\mathbf{K}_2$ be triangulated categories. A *morphism* $F\colon \mathbf{K}_1 \longrightarrow \mathbf{K}_2$ of triangulated categories is an additive functor that commutes with the translation functor T and sends distinguished triangles to distinguished triangles. There is a category of triangulated categories and their morphisms. We say that $\mathbf{K}_1$ is a *triangulated subcategory* of $\mathbf{K}_2$ if $\mathbf{K}_1$ is a full subcategory of $\mathbf{K}_2$, the inclusion is a morphism of triangulated categories and if every distinguished triangle in $\mathbf{K}_1$ is distinguished in $\mathbf{K}_2$.

The proof of the following result is in [13, Proposition III.6.2].

Proposition 2.4.2. *Assume that $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is an exact functor.*

(a) *The functor $\mathbf{K}_\star(F)\colon \mathbf{K}_\star(\mathcal{A}) \longrightarrow \mathbf{K}_\star(\mathcal{B})$ transforms q-isomorphisms into q-isomorphisms so that it induces a functor $\mathcal{D}_\star(F)\colon \mathcal{D}_\star(\mathcal{A}) \longrightarrow \mathcal{D}_\star(\mathcal{B})$.*

(b) *$\mathcal{D}_\star(F)$ is an exact functor, that is, it transforms distinguished triangles into distinguished triangles.*

where $\star$ stands for b, $+$, $-$, or $\emptyset$.

Definition 2.4.3. A *right derived functor* of an additive left exact functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a pair consisting of an exact functor $\mathbf{R}_-F\colon \mathcal{D}_-(\mathcal{A}) \longrightarrow \mathcal{D}_-(\mathcal{B})$ and a natural transformation ζ from

$$q\mathbf{K}_-(F) : \mathbf{K}_-(\mathcal{A}) \longrightarrow \mathbf{K}_-(\mathcal{B}) \longrightarrow \mathcal{D}_-(\mathcal{B})$$

to

$$(\mathbf{R}_-F)q : \mathbf{K}_-(\mathcal{A}) \longrightarrow \mathcal{D}_-(\mathcal{A}) \longrightarrow \mathcal{D}_-(\mathcal{B})$$

which is universal in the sense that if $G\colon \mathcal{D}_-(\mathcal{A}) \longrightarrow \mathcal{D}_-(\mathcal{B})$ is another exact functor equipped with a natural transformation $\epsilon\colon q\mathbf{K}_-(F) \longrightarrow Gq$, then there exists a unique natural transformation $\eta\colon \mathbf{R}_-F \longrightarrow G$ making the diagram

$$\begin{tikzcd} & q\mathbf{K}_-(F) \arrow[dl, "\zeta"'] \arrow[dr, "\epsilon"] & \\ (\mathbf{R}_-F)q \arrow[rr, "\eta q"] & & Gq \end{tikzcd}$$

commutative. Similarly, a *left derived functor* of a right exact functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a pair consisting of an exact functor $\mathbf{L}_+F\colon \mathcal{D}_+(\mathcal{A}) \longrightarrow \mathcal{D}_+(\mathcal{B})$ together with a natural transformation $\zeta\colon (\mathbf{L}_+F)q \longrightarrow q\mathbf{K}_+(F)$ satisfying the dual universal property (G factors through $\eta\colon G \longrightarrow \mathbf{L}_+F$).

The universal property implies that if $\mathbf{R}_-F$ and $\mathbf{L}_+F$ exist, then they are unique up to natural isomorphism.

The following result is proved in [33, Existence Theorem 10.5.6].

Theorem 2.4.4. *Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. If $\mathcal{A}$ has enough injectives, then the right derived functor $\mathbf{R}_-F$ exists on $\mathcal{D}_-(\mathcal{A})$, and if I is a bounded above chain complex of injectives, then*

$$\mathbf{R}_-F(I) \cong q\mathbf{K}_-(F)(I).$$

Dually, if $\mathcal{A}$ has enough projectives, then the left derived functor $\mathbf{L}_+F$ exists on $\mathcal{D}_+(\mathcal{A})$ and if P is a bounded below chain complex of projectives, then

$$\mathbf{L}_+F(P) \cong q\mathbf{K}_+(F)(P).$$

2.4.1 The Derived Tensor Product

In this subsection, we will give the definition of the derived tensor product and present its properties.

Definition 2.4.5. The *derived tensor product* of two chain complexes X and Y is

$$X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y = \mathbf{L}_+ \operatorname{Tot}^{\oplus}(X \underset{R}{\otimes} -)Y.$$

Lemma 2.4.6. *If $X \longrightarrow X'$ is a q-isomorphism and X, X', and Y are bounded below chain complexes, then*

$$X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y \cong X' \overset{\mathrm{L}}{\underset{R}{\otimes}} Y.$$

Proof. Suppose Y is a chain complex of flat modules. Theorem 2.4.4 implies that

$$X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y = \mathrm{Tot}^{\oplus}(X \otimes Y)$$

and

$$X' \overset{\mathrm{L}}{\underset{R}{\otimes}} Y = \mathrm{Tot}^{\oplus}(X' \otimes Y).$$

But

$$E^1_{p,q}(X) = H_q(X) \underset{R}{\otimes} Y_p \Longrightarrow H_{p+q}(X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y)$$

and

$$E^1_{p,q}(X') = H_q(X') \underset{R}{\otimes} Y_p \Longrightarrow H_{p+q}(X' \overset{\mathrm{L}}{\underset{R}{\otimes}} Y)$$

by Theorem 1.4.16. It is clear that $E^1_{p,q}(X) \cong E^1_{p,q}(X')$. Thus,

$$H_{p+q}(X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y) \cong H_{p+q}(X' \overset{\mathrm{L}}{\underset{R}{\otimes}} Y)$$

by the Comparison Theorem 1.4.4. Hence, $X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y \cong X' \overset{\mathrm{L}}{\underset{R}{\otimes}} Y$. □

The following theorem is proved in [33, Theorem 10.6.3].

Theorem 2.4.7. *The derived tensor product is a bifunctor*

$$\overset{\mathrm{L}}{\underset{R}{\otimes}} \colon \mathcal{D}_+(R) \times \mathcal{D}_+(R) \longrightarrow \mathcal{D}_+(R).$$

Its homology is

$$\mathrm{Tor}^R_n(X, Y) \cong H_n(X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y).$$

Definition 2.4.8. A *symmetric monoidal product* on a category $\mathcal{C}$ is a bifunctor $\bigotimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, a unit $U \in \mathcal{C}$ and coherent natural isomorphisms $(X \bigotimes Y) \bigotimes Z \cong X \bigotimes (Y \bigotimes Z)$ (the associativity isomorphism), $X \bigotimes Y \cong Y \bigotimes X$ (the twist isomorphism) and $U \bigotimes X \cong X$ (the unit isomorphism). A *symmetric monoidal category* is a category $\mathcal{C}$ with a symmetric monoidal product.

If X, Y and Z are chain complexes in $\mathcal{D}_+(R)$, then by [33, Example 10.8.1], there is a natural isomorphism

$$X \overset{\mathrm{L}}{\underset{R}{\otimes}} (Y \overset{\mathrm{L}}{\underset{R}{\otimes}} Z) \cong (X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y) \overset{\mathrm{L}}{\underset{R}{\otimes}} Z.$$

Also, by [33, Exercise 10.6.2], there is a natural isomorphism

$$X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y \cong Y \overset{\mathrm{L}}{\underset{R}{\otimes}} X.$$

Moreover, it is clear that there is a natural isomorphism $R[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} X \cong X$. Therefore, We deduce that the derived category $\mathcal{D}_+(R)$ of bounded below chain complexes of R-modules is a symmetric monoidal category.

2.4.2 The Derived Hom

In this subsection, we will give the definition of the derived Hom and present its properties.

Definition 2.4.9. The *derived Hom* of two chain complexes X and Y is

$$\mathrm{RHom}_R(X, Y) = \mathbf{R}_- \operatorname{Tot}^{\prod} \operatorname{Hom}(X, -)Y.$$

Lemma 2.4.10. *If $Y \longrightarrow Y'$ is a q-isomorphism and X is a bounded below chain complex, then*

$$\mathrm{RHom}_R(X, Y) \cong \mathrm{RHom}_R(X, Y').$$

Proof. Suppose X is a chain complex of projectives. Then,

$$\mathrm{RHom}_R(X, Y) \cong \operatorname{Tot}^{\prod} \operatorname{Hom}(X, Y)$$

and

$$\mathrm{RHom}_R(X, Y') \cong \operatorname{Tot}^{\prod} \operatorname{Hom}(X, Y').$$

Therefore,

$$\begin{aligned} H_n(\operatorname{Tot}^{\prod} \operatorname{Hom}(X, Y)) &= \operatorname{Hom}_{\mathbf{K}(R)}(X, Y[-n]) \\ &\cong \operatorname{Hom}_{\mathcal{D}(R)}(X, Y[-n]) \\ &\cong \operatorname{Hom}_{\mathcal{D}(R)}(X, Y'[-n]) \\ &\cong \operatorname{Hom}_{\mathbf{K}(R)}(X, Y'[-n]) \\ &= H_n(\operatorname{Tot}^{\prod} \operatorname{Hom}(X, Y')). \end{aligned}$$

where the first isomorphism is induced by Theorem 2.3.12. Hence,

$$\operatorname{RHom}_R(X, Y) \cong \operatorname{RHom}_R(X, Y'). \qquad \square$$

The following lemma is proved in [33, Lemma 10.7.3].

Lemma 2.4.11. *If $X \longrightarrow X'$ is a q-isomorphism and Y is a bounded above chain complex, then*

$$\operatorname{RHom}_R(X', Y) \cong \operatorname{RHom}_R(X, Y).$$

Definition 2.4.12. If X and Y are chain complexes, then

$$\operatorname{Ext}^n_R(X, Y) = \operatorname{Hom}_{\mathcal{D}(R)}(X, Y[-n]).$$

The following result is proved in [33, Theorem 10.7.4].

Theorem 2.4.13. *The derived Hom is a bifunctor*

$$\operatorname{RHom}_R \colon \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}_-(R) \longrightarrow \mathcal{D}(R).$$

Dually,

$$\operatorname{RHom}_R \colon \mathcal{D}_+(R)^{\mathrm{op}} \times \mathcal{D}(R) \longrightarrow \mathcal{D}(R).$$

In both cases its cohomology is

$$\operatorname{Ext}^n_R(X, Y) \cong H^n(\operatorname{RHom}_R(X, Y)).$$

The following result is proved in [33, Theorem 10.8.7].

Theorem 2.4.14 (Adjoint Isomorphism). *If Y is a bounded below chain complex, then*

$$- \overset{\mathrm{L}}{\underset{R}{\otimes}} Y : \mathcal{D}_+(R) \longrightarrow \mathcal{D}_+(R)$$

is left adjoint to the functor

$$\operatorname{RHom}_R(Y, -) \colon \mathcal{D}_-(R) \longrightarrow \mathcal{D}_-(R).$$

That is, for X in $\mathcal{D}_+(R)$ and Z in $\mathcal{D}_-(R)$ there is a natural isomorphism

$$\operatorname{Hom}_{\mathcal{D}(R)}(X, \operatorname{RHom}_R(Y, Z)) \cong \operatorname{Hom}_{\mathcal{D}(R)}(X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y, Z).$$

This isomorphism arises by applying H^0 *to the isomorphism*

$$\mathrm{RHom}_R(X, \mathrm{RHom}_R(Y, Z)) \cong \mathrm{RHom}_R(X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y, Z)$$

in $\mathcal{D}_-(R)$. *The adjunction morphisms are*

$$\eta_X \colon X \longrightarrow \mathrm{RHom}_R(Y, X \overset{L}{\underset{R}{\otimes}} Y)$$

and

$$\epsilon_Z \colon \mathrm{RHom}_R(Y, Z) \overset{L}{\underset{R}{\otimes}} Y \longrightarrow Z.$$

Chapter 3

Minimal Atomic Chain Complexes

In this chapter, we define some new notions which are invariant in the derived category. These notions have been defined in a topological framework in [5]. After introducing these concepts we establish the connection between them.

Introduction

First we know what we mean by an irreducible (or simple) R-module M, namely $0 \neq M$ and M has no proper submodules. Also, an atomic module is an R-module for which every non-trivial self map is an isomorphism. If M is an irreducible R-module, then M is atomic by Schur's lemma. However, atomic does not imply irreducible.

Example 3.0.15. Let F be a field and $A = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) : a, b, c \in F\}$ be the ring of triangular matrices over F. Let $M = \{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) : a, b \in F\}$. Then it is clear that M is a module over A. Note that $\{\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right) : a \in F\}$ is a submodule of M. So M is not irreducible. If $0 \neq \phi \colon M \longrightarrow M$, then it can be proved that ϕ is invertible and hence M is an atomic module.

We generalize both concepts and define others. In section one, we recall some facts about local commutative rings, give the definition of a minimal chain complex and show that for any chain complex Y of finitely generated homology there is a minimal free chain complex X and a q-isomorphism $f \colon X \longrightarrow Y$. In section two, we state and prove a derived analog of the Whitehead Theorem. In section three, we construct Postnikov towers. In section four, we define an analog of the Steenrod algebra. In section five, we present some definitions and in the following section, we show a result that characterizes irreducible chain complexes and we prove that minimal atomic chain complexes and irreducible chain complexes are the same. In the last section, we define the notions of a nuclear chain complex and a core of a chain complex and we show that a nuclear chain complex is minimal atomic.

3.1 Local Rings

In this section, we review some basic facts about local commutative rings. The main references for these facts are [30] and [21]. We recall that R is an arbitrary commutative ring.

Definition 3.1.1. R is *local* if it has a unique maximal ideal.

We will give a number of examples of local rings.

Example 3.1.2. (a) Every field is local.

(b) If F is a field, then the ring of formal power series $F[[x]]$ over F is local.

(c) If P is a prime ideal in R, then the localization $S^{-1}R$ is a local ring where $S = R - P$. For example, the ring of localized integers $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : (b,p) = 1\}$ is a local ring.

(d) If P is a maximal ideal in R, then the completion $\hat{R}_P$ is a local ring. For example, the ring of p-adic integers $\hat{\mathbb{Z}}_p$ is a local ring.

(e) If I is a maximal ideal in R, then R/I^n is a local ring.

Definition 3.1.3. If R is a local ring with maximal ideal $\mathfrak{m}$, then the field $R/\mathfrak{m}$ is called the *residue field* of R.

The following result is proved in [30, Theorem 4.47].

Lemma 3.1.4 (Nakayama's Lemma). *If R is a local ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module with $\mathfrak{m}M = M$, then $M = 0$.*

If W is a set of generators of an R-module M, then we say that W is *minimal* if any proper subset of W does not generate M.

The following theorem is proved in [21, Theorem 2.3].

Theorem 3.1.5. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m}$ and let M be a finitely generated R-module. Set $\bar{M} = M/\mathfrak{m}M$. Now $\bar{M}$ is a finite-dimensional vector space over $R/\mathfrak{m}$, and we write n for its dimension. Then*

(i) *If we take a basis $\{u'_1, \ldots, u'_n\}$ for $\bar{M}$ over $R/\mathfrak{m}$, and choose an inverse image $u_i \in M$ of each u'_i, then $\{u_1, \ldots, u_n\}$ is a minimal generating set of M,*

(ii) *conversely every minimal generating set of M is obtained in this way, and so has n elements.*

The following result is proved in [30, Theorem 4.44].

Theorem 3.1.6. *If R is a local ring, every finitely generated projective module M is free.*

The following lemma is important and will be used later. Its proof is in [28, Proposition 1.5].

Lemma 3.1.7. *Let R be a local commutative noetherian ring with maximal ideal $\mathfrak{m}$ and M be an R-module. Let $F \longrightarrow M$ be a free resolution of M. Then the following conditions are equivalent.*

(i) *For each $i \geq 1$, the map $\phi_i \colon F_i \longrightarrow F_{i-1}$ is defined by a matrix with coefficients in $\mathfrak{m}$. (Note that this condition is independent of the choice of bases for F_i and F_{i-1}.)*

(ii) *For each $i \geq 0$, the map $\theta_i \colon F_i \longrightarrow \operatorname{Ker} \phi_{i-1}$ is defined by a minimal set of generators (for $i = 0$, θ_0 is the map from F_0 onto M).*

Definition 3.1.8. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m}$ and let M be a finitely generated R-module. An exact sequence

$$\cdots \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \longrightarrow 0$$

is called a *minimal free resolution* of M if it satisfies the following conditions:

(a) each L_n is a finitely generated free R-module,

(b) $d_n L_n \subset \mathfrak{m} L_{n-1}$ for each n,

(c) $R/\mathfrak{m} \otimes_R L_0 \longrightarrow R/\mathfrak{m} \otimes_R M$ is an isomorphism.

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{K} = R/\mathfrak{m}$. Assume that $\mathfrak{m}$ is generated by a finite regular sequence $m_1, \dots, m_n \in R$, that is, m_1 is not a zero divisor and for $i > 1$, each m_i is not a zero divisor on $R/(m_1, \dots, m_{i-1})$.

Since $\mathfrak{m}$ is generated by a finite regular sequence, we have the following result which is proved in [33, Corollary 4.5.5].

Proposition 3.1.9. *There is a* Koszul free resolution P *of* $\mathbb{K}$ *where* $P = E_R(e_i : 1 \leq i \leq n)$ *is a differential graded algebra with* e_i *in degree 1 and differential given by* $d(e_i) = m_i$. *In this case we have that*

$$\operatorname{Ext}^{\star}_R(\mathbb{K}, \mathbb{K}) \cong E_{\mathbb{K}}(e_i : 1 \leq i \leq n)$$

Also, we have the following result proved in [21, Theorem 16.2].

Lemma 3.1.10. *For* $s \geq 1$, $\mathfrak{m}^s/\mathfrak{m}^{s+1}$ *is* $R/\mathfrak{m}$*-module with a basis consisting of the residue classes of the distinct monomials of degree* s *in the* m_i.

3.1.1 Minimal Free Resolutions

In this subsection and the following sections, we assume that R is a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{K} = R/\mathfrak{m}$.

Definition 3.1.11. A chain complex (Y, d) is *minimal* if the induced differential $d \otimes \mathrm{id}_{\mathbb{K}}$ on $Y \otimes \mathbb{K}[0]$ satisfies $d \otimes \mathrm{id}_{\mathbb{K}} = 0$.

Now we give the following important theorem which proves the existence of a minimal free resolution of any chain complex of finite type in $\mathbf{Ch}_+(R)$. The usefulness of minimal free resolutions will become clear when we study the Adams spectral sequence in the next chapter.

This theorem is [28, Theorem 2.4]. We think it is not well known and thus we give its proof here.

Theorem 3.1.12. *Let Y be a chain complex of finite type in $\mathbf{Ch}_+(R)$. Then there is a minimal free chain complex G in $\mathbf{Ch}_+(R)$ and a q-isomorphism $f\colon G \longrightarrow Y$.*

Proof. We prove this theorem using induction. If i_0 is such that $H_i(Y) = 0$ for $i \leq i_0$, then we can let $F_i = 0$ for $i \leq i_0$ and the zero map in degrees $\leq i_0$ from F to Y is obviously a q-isomorphism.

Now assume that we have defined finitely generated free R-modules F_i and maps f_i for $i \leq n$ such that the following diagram

$$\begin{tikzcd} & F_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & F_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\ \cdots \arrow[r] Y_{n+1} \arrow[r] & Y_n \arrow[r] & Y_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

is commutative and

(a) $H_i(f)\colon H_i(F) \longrightarrow H_i(Y)$ is an isomorphism for $i < n$.

(b) $g\colon \operatorname{Ker}(d_n) \longrightarrow H_n(Y)$ is surjective.

Now we construct F_{n+1} and f_{n+1} such that (a) and (b) hold for $n+1$. First note that $\operatorname{Ker}(g)$ maps to $B_n(Y)$ since we have the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}(g) \arrow[r] \arrow[d, dotted] & \operatorname{Ker}(d_n) \arrow[r, "g"] \arrow[d] & H_n(Y) \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\ 0 \arrow[r] & B_n(Y) \arrow[r] & Z_n(Y) \arrow[r] & H_n(Y) \arrow[r] & 0 \end{tikzcd}$$

We have that the $\operatorname{Ker}(g)$ is finitely generated since R is noetherian. Let $\{y_1, \ldots, y_n\}$ be a generating set of $\operatorname{Ker}(g)$. Assume that E_1 is free on $\{x_1, \ldots, x_n\}$. Define $\psi\colon E_1 \longrightarrow \operatorname{Ker}(g)$ by $\psi(x_i) = y_i$. Therefore, we have the following commutative diagram where the map h_1 exists since E_1 is free.

$$\begin{tikzcd} & E_1 \arrow[dl, "h_1"'] \arrow[d] & \\ Y_{n+1} \arrow[r] & B_n(Y) \arrow[r] & 0 \end{tikzcd}$$

That is, the following diagram

$$\begin{tikzcd} E_1 \arrow[r, "\lambda"] \arrow[d, "h_1"'] & F_n \arrow[d, "f_n"'] \\ Y_{n+1} \arrow[r] & Y_n \end{tikzcd}$$

is commutative where the map λ is the composite of $\psi\colon E_1 \longrightarrow \mathrm{Ker}(g)$ with the inclusion map $\mathrm{Ker}(g) \longrightarrow F_n$. We have that $H_{n+1}(Y)$ is finitely generated. Let $\{z_1, \ldots, z_m\}$ be a generating set of $H_{n+1}(Y)$. Assume that E_2 is free on $\{t_1, \ldots, t_m\}$. Define $\phi\colon E_2 \longrightarrow H_{n+1}(Y)$ by $\phi(t_i) = z_i$. Then we have the following commutative diagram

$$\begin{tikzcd} & E_2 \arrow[dl, dotted] \arrow[d, "\phi"] & \\ Z_{n+1}(Y) \arrow[r] & H_{n+1}(Y) \arrow[r] & 0 \end{tikzcd}$$

where the dotted arrow exists since E_2 is free. Let the map $h_2\colon E_2 \longrightarrow Y_{n+1}$ be the composite of the map $E_2 \longrightarrow Z_{n+1}(Y)$ with the inclusion map $Z_{n+1}(Y) \longrightarrow Y_{n+1}$. Map E_2 to zero in F_n. Let $F_{n+1} = E_1 \oplus E_2$ and $f_{n+1} = h_1 + h_2$. The differential $d_{n+1} = \lambda + 0$. Then we can see that the following diagram

$$\begin{tikzcd} F_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & F_n \arrow[d, "f_n"'] \\ Y_{n+1} \arrow[r] & Y_n \end{tikzcd}$$

is commutative. By construction, it is clear that $H_n(f)\colon H_n(F) \longrightarrow H_n(Y)$ is an isomorphism and $\mathrm{Ker}(d_{n+1}) \longrightarrow H_{n+1}(Y)$ is surjective. Hence, we have constructed a free resolution of Y.

This gives some free resolution of Y and to get a minimal one, we can proceed as follows. We show that F is a sum of a minimal chain complex G and an exact chain complex H of free modules. Then G is a minimal free resolution of Y. If F is not minimal, then the matrix (a_{ij}) defining $d_n\colon F_n \longrightarrow F_{n-1}$ must have a unit element since R is local. We can transform (a_{ij}) by a finite number of elementary row and column operations to the following form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (a'_{ij}) & \\ 0 & & & \end{bmatrix}$$

This means that we have a diagram

$$\begin{tikzcd} F_n \arrow[r, "d_n"] \arrow[d, "\cong"'] & F_{n-1} \arrow[d, "\cong"'] \\ R \oplus F'_n \arrow[r, "(\mathrm{id}, d'_n)"] & R \oplus F'_{n-1} \end{tikzcd}$$

and the chain complex F is the direct sum of

$$\cdots \longrightarrow F_{n+1} \longrightarrow F'_n \longrightarrow F'_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots$$

and

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\mathrm{id}} R \longrightarrow 0 \longrightarrow \cdots .$$

This process can be continued until we are left with a minimal free resolution G of Y and the sum of pieces of the form

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\mathrm{id}} R \longrightarrow 0 \longrightarrow \cdots$$

in various degrees. Putting these latter pieces together gives H which is an exact chain complex of free modules. Hence, there is a minimal free chain complex G in $\mathbf{Ch}_+(R)$ and a q-isomorphism $f\colon G \longrightarrow Y$. $\square$

3.2 The Derived Whitehead Theorem

In this section, a chain complex means a chain complex Y in the derived category $\mathcal{D}_{+(fg)}(R)$ of bounded below chain complexes whose homology modules $H_i(Y)$ are of finite type.

In this section, we state and prove a derived analog of the Whitehead Theorem. We begin by introducing some definitions and proving some results.

First note that Lemma 1.3.8 and Theorem 2.3.12 combine together to give that for any chain complex Y,

$$H_n(Y) \cong \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[-n], Y).$$

Define

$$H_n(Y, \mathbb{K}) = H_n(Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]),$$

the nth homology of the derived tensor product $Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$. We have the reduction map

$$\rho\colon H_n(Y) \longrightarrow H_n(Y, \mathbb{K})$$

induced from the evident morphism

$$Y \cong Y \overset{\mathrm{L}}{\underset{R}{\otimes}} R[0] \longrightarrow Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0].$$

Let P be a minimal projective resolution of Y by Theorem 3.1.12. Then

$$\begin{aligned} H_n(Y, \mathbb{K}) &= H_n(Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \\ &\cong H_n(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \\ &= H_n(\cdots \xrightarrow{0} P_1 \otimes_R \mathbb{K} \xrightarrow{0} P_0 \otimes_R \mathbb{K} \to 0) \\ &= P_n \otimes_R \mathbb{K}. \end{aligned}$$

Hence, $H_n(Y, \mathbb{K})$ is a $\mathbb{K}$-module.

Similarly, we can define

$$\begin{aligned} H^n(Y, \mathbb{K}) &= H^n(\mathrm{RHom}_R(Y, \mathbb{K}[0])) \\ &= H^0(\mathrm{RHom}_R(Y, \mathbb{K}[-n])). \end{aligned}$$

If P is a minimal projective resolution of Y, then

$$\begin{aligned} H^n(Y, \mathbb{K}) &= H^n(\mathrm{RHom}_R(Y, \mathbb{K}[0])) \\ &\cong H^n(\mathrm{Hom}^{\Pi}(P, \mathbb{K}[0])) \\ &= H^n(0 \to \mathrm{Hom}_R(P_0, \mathbb{K}) \xrightarrow{0} \mathrm{Hom}_R(P_1, \mathbb{K}) \xrightarrow{0} \cdots) \\ &= \mathrm{Hom}_R(P_n, \mathbb{K}). \end{aligned}$$

Hence, $H^n(Y, \mathbb{K})$ is a $\mathbb{K}$-module.

One of the applications of the existence of a minimal projective resolution is the following result.

Lemma 3.2.1. *If Y is a chain complex, then*

$$H^n(Y, \mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(H_n(Y, \mathbb{K}), \mathbb{K}).$$

Proof. Let Y be a chain complex. By Theorem 3.1.12, there exists a minimal projective resolution $P \longrightarrow Y$. Therefore,

$$\begin{aligned}
H^n(Y, \mathbb{K}) &= H^n(\mathrm{RHom}_R(Y, \mathbb{K}[0])) \\
&\cong H^n(\mathrm{RHom}_R(P, \mathbb{K}[0])) \\
&\cong H^n(\mathrm{Hom}^{\prod}(P, \mathbb{K}[0])) \\
&= H^n(\mathrm{Hom}_R(P, \mathbb{K})) \\
&= \mathrm{Hom}_R(P_n, \mathbb{K}) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(P_n \otimes_R \mathbb{K}, \mathbb{K}) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(H_n(P, \mathbb{K}), \mathbb{K}) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(H_n(Y, \mathbb{K}), \mathbb{K}).
\end{aligned}$$

Hence,

$$H^n(Y, \mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(H_n(Y, \mathbb{K}), \mathbb{K}). \qquad \square$$

Definition 3.2.2. A chain complex Y is called *n-connected* if $H_i(Y) = 0$ for all $i \leq n$.

Definition 3.2.3. A morphism $\alpha\colon X \longrightarrow Y$ in $\mathcal{D}_{+(fg)}(R)$ is called an *n-isomorphism* if $\alpha_\star\colon H_i(X) \longrightarrow H_i(Y)$ is an isomorphism for each $i \leq n$.

Theorem 3.2.4. *Let Y be a chain complex. If Y is n-connected, then $H_i(Y, \mathbb{K}) = 0$ for all $i \leq n$ and $\rho\colon H_{n+1}(Y) \longrightarrow H_{n+1}(Y, \mathbb{K})$ is an epimorphism.*

Proof. Assume that Y is n-connected, that is, $H_i(Y) = 0$ for all $i \leq n$. We claim that $H_i(Y, \mathbb{K}) = 0$ for all $i \leq n$. There exists a minimal projective chain complex P and a q-isomorphism $P \longrightarrow Y$ by Theorem 3.1.12. But $Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ by Lemma 2.4.6. Therefore, by Theorem 1.4.16 there exists a Künneth spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^R_s(H_t(P), \mathbb{K}) \Longrightarrow H_{s+t}(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]).$$

Since $H_t(P) \cong H_t(Y) = 0$ for each $t \leq n$, we have $E^2_{s,t} = 0$ for each s and $t \leq n$. Thus, $E^\infty_{s,t} = 0$ for each s and $t \leq n$. So, $H_i(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) = 0$ for each $i \leq n$. Hence, $H_i(Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) = 0$ for all $i \leq n$. Now notice that

$$E^\infty_{0,n+1} \cong H_{n+1}(Y) \otimes_R \mathbb{K}.$$

Therefore, $H_{n+1}(Y, \mathbb{K}) \cong H_{n+1}(Y) \otimes_R \mathbb{K}$. Hence, $\rho\colon H_{n+1}(Y) \longrightarrow H_{n+1}(Y) \otimes_R \mathbb{K}$ is reduction mod $\mathfrak{m}$, that is, it is an epimorphism. $\square$

Theorem 3.2.5 (Derived Whitehead Theorem). *Let $\alpha\colon X \longrightarrow Y$ be a morphism in $\mathcal{D}_{+(fg)}(R)$.*

(i) *If α is an n-isomorphism, then $H_i(X, \mathbb{K}) \longrightarrow H_i(Y, \mathbb{K})$ is an isomorphism for all $i \leq n$.*

(ii) *If $H_i(X, \mathbb{K}) \longrightarrow H_i(Y, \mathbb{K})$ is an isomorphism for all $i \leq n$, then $\alpha_\star\colon H_i(X) \longrightarrow H_i(Y)$ is an isomorphism for all $i < n$ and an epimorphism for $i = n$.*

Proof. Without loss of generality, we assume X and Y are connective. First we prove (i). Assume that α is an n-isomorphism. Then $\alpha_\star\colon H_i(X) \longrightarrow H_i(Y)$ is an isomorphism for each $i \leq n$. We must show that $H_i(X, \mathbb{K}) \longrightarrow H_i(Y, \mathbb{K})$ is an isomorphism for all $i \leq n$. By Theorem 3.1.12, there exist minimal projective resolutions $P \longrightarrow X$ and $Q \longrightarrow Y$. Therefore, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] & Y \\ P \arrow[u, "g"] \arrow[r, "\beta"] & Q \arrow[u, "h"'] \end{tikzcd}$$

in which $\beta = h^{-1}\alpha g$. Let $f\colon P \longrightarrow Q$ be a chain map representing the morphism β. But α is n-isomorphism. So f is n-isomorphism. Consider

$$\phi = f \otimes \mathrm{id}\colon P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \longrightarrow Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0].$$

Now we claim that $\phi_\star\colon H_i(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \longrightarrow H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])$ is an isomorphism for each $i \leq n$.

We will consider two spectral sequences E_r and $\bar{E}_r$ associated to filtrations on the complexes $P \otimes \mathbb{K}[0]$ and $Q \otimes \mathbb{K}[0]$, respectively, induced by the stupid filtrations on P and Q, respectively. Note that both filtrations are bounded. By Theorem 1.4.16, there exist Künneth spectral sequences

$$E^2_{s,t} \cong \mathrm{Tor}^R_s(H_t(P), \mathbb{K}) \Longrightarrow H_{s+t}(P \otimes \mathbb{K}[0])$$

and

$$\bar{E}^2_{s,t} \cong \mathrm{Tor}^R_s(H_t(Q), \mathbb{K}) \Longrightarrow H_{s+t}(Q \otimes \mathbb{K}[0]).$$

Since $H_t(P) \cong H_t(Q)$ for each $t \leq n$, we have that $E^2_{s,t} \cong \bar{E}^2_{s,t}$ for each s and $t \leq n$. ϕ induces a map from the filtration of $P \otimes \mathbb{K}[0]$ to the filtration of $Q \otimes \mathbb{K}[0]$ and thus a homomorphism of spectral sequences $E_r \longrightarrow \bar{E}_r$. Since $d^3_{s,t}$ is of bidegree $(-3, 2)$, we can deduce that $E^3_{s,t} \cong \bar{E}^3_{s,t}$ in the following cases.

(i) $s < 3$ and $t \leq n$,

(ii) $s \geq 3$ and $t \leq n - 2$.

Also, since $d^4_{s,t}$ is of bidegree $(-4, 3)$, we can deduce that $E^4_{s,t} \cong \bar{E}^4_{s,t}$ in the following cases.

(i) $s < 3$ and $t \leq n$,

(ii) $s = 3$ and $t \leq n - 2$,

(iii) $4 \leq s \leq 6$ and $t \leq n - 3$,

(iv) $s > 6$ and $t \leq n - 5$.

Continuing this way, we can deduce that $E^\infty_{s,t} \cong \bar{E}^\infty_{s,t}$ for each $s + t \leq n$. Using Theorem 1.4.20, we have that $\phi_\star\colon H_i(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \longrightarrow H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])$ is an isomorphism for all $i \leq n$. Therefore, $\beta_\star\colon H_i(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \longrightarrow H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])$ is an isomorphism for each $i \leq n$. But $g_\star\colon H_i(P, \mathbb{K}) \longrightarrow H_i(X, \mathbb{K})$ is an isomorphism for each i and $h_\star\colon H_i(Q, \mathbb{K}) \longrightarrow H_i(Y, \mathbb{K})$ is an isomorphism for each i by Lemma 2.4.6. Hence, $H_i(X, \mathbb{K}) \longrightarrow H_i(Y, \mathbb{K})$ is an isomorphism for all $i \leq n$.

Next we show (ii). Assume that $H_i(X, \mathbb{K}) \longrightarrow H_i(Y, \mathbb{K})$ is an isomorphism for all $i \leq n$. We claim that $\alpha_\star\colon H_i(X) \longrightarrow H_i(Y)$ is an isomorphism for all $i < n$ and an epimorphism for $i = n$. We have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] & Y \\ P \arrow[u, "g"] \arrow[r, "\beta"] & Q \arrow[u, "h"'] \end{tikzcd}$$

in which $\beta = h^{-1}\alpha g$ and P and Q are minimal projective resolutions for X and Y, respectively. Let $f\colon P \longrightarrow Q$ be a chain map representing the morphism β. We show that $f_\star\colon H_i(P) \longrightarrow H_i(Q)$ is an isomorphism for each $i < n$ and an epimorphism for $i = n$. We have that $g_\star\colon H_i(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \longrightarrow H_i(X \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])$ is an isomorphism for all i as well as $h_\star\colon H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \longrightarrow H_i(Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])$ is an isomorphism for all i. Thus, $H_i(P, \mathbb{K}) \longrightarrow H_i(Q, \mathbb{K})$ is an isomorphism for all $i \leq n$. Choose $q_1, \ldots, q_n \in Q_i$ whose images form a basis of the $\mathbb{K}$-vector space $Q_i \otimes_R \mathbb{K}$. By Nakayama's Lemma, $q_1, \ldots, q_n$ generate Q_i. Similarly, choose $p_1, \ldots, p_m \in P_i$ whose images form a basis of the $\mathbb{K}$-vector space $P_i \otimes_R \mathbb{K}$. By Nakayama's Lemma, $p_1, \ldots, p_m$ generate P_i. Note that P_i and Q_i are finitely generated free since R is local by Theorem 3.1.6. But $P_i \otimes_R \mathbb{K} \cong Q_i \otimes_R \mathbb{K}$. Thus, $f_i\colon P_i \longrightarrow Q_i$ is onto and $n = m$. Hence, f_i is an isomorphism. Therefore, f_i is an isomorphism for all $i \leq n$. We deduce that $f_\star$ is an isomorphism for each $i < n$ and an epimorphism for $i = n$. Therefore, we have that $\beta_\star\colon H_i(P) \longrightarrow H_i(Q)$ is an isomorphism for each $i < n$ and an epimorphism for $i = n$. Hence, we have $\alpha_\star\colon H_i(X) \longrightarrow H_i(Y)$ is an isomorphism for all $i < n$ and an epimorphism for $i = n$. □

3.3 Postnikov towers

In this section, we show how to construct a Postnikov tower for a chain complex in $\mathcal{D}_+(R)$ and without loss of generality, we assume that chain complexes are connective. The main references for Postnikov towers in topology are [24] and [9].

Theorem 3.3.1. *For each chain complex Y in $\mathcal{D}_+(R)$, there exists a tower*

$$\cdots \xrightarrow{\beta_3} Y\{2\} \xrightarrow{\beta_2} Y\{1\} \xrightarrow{\beta_1} Y\{0\}\,,$$

as well as morphisms $\alpha_n\colon Y \longrightarrow Y\{n\}$ such that the diagram

$$\begin{tikzcd} & Y\{n\} \arrow[dd, "\beta_n"] \\ Y \arrow[ur, "\alpha_n"] \arrow[dr, "\alpha_{n-1}"'] & \\ & Y\{n-1\} \end{tikzcd}$$

commutes for each n. Moreover, $H_i(Y\{n\}) = 0$ for $i > n$ and $\alpha_{i_\star}\colon H_i(Y) \longrightarrow H_i(Y\{n\})$ is an isomorphism for all $i \le n$.

Proof. We may change Y up to q-isomorphism to assume that Y is a chain complex of projective modules. We prove this theorem by induction. We start at $n = 0$. Consider the chain complex $H_0(Y)[0]$. Then

$$H^0(Y, H_0(Y)) = \operatorname{Hom}_{\mathbf{K}(R)}(Y, H_0(Y)[0])$$

and the universal coefficient spectral sequence Theorem 1.4.18 implies that

$$H^0(Y, H_0(Y)) = \operatorname{Hom}_R(H_0(Y), H_0(Y)).$$

Therefore,

$$\operatorname{Hom}_{\mathbf{K}(R)}(Y, H_0(Y)[0]) = \operatorname{Hom}_R(H_0(Y), H_0(Y)).$$

Choose a chain map $g\colon Y \longrightarrow H_0(Y)[0]$ which corresponds to the identity on $H_0(Y)$. Let $f\colon Y\{0\} \longrightarrow H_0(Y)[0]$ be a projective resolution of $H_0(Y)[0]$. Let $\alpha_0 = f^{-1}g\colon Y \longrightarrow Y\{0\}$ be in $\mathcal{D}_+(R)$. Then α_0 induces an isomorphism on H_0. Now form the following distinguished triangle

$$F \xrightarrow{\pi} Y \xrightarrow{\alpha_0} Y\{0\} \longrightarrow F[-1].$$

Then the corresponding homology long exact sequence

$$\cdots \longrightarrow H_i(F) \xrightarrow{\pi_\star} H_i(Y) \xrightarrow{\alpha_{0\star}} H_i(Y\{0\}) \longrightarrow \cdots$$

implies that

$$H_i(F) = \begin{cases} H_i(Y) & i > 0, \\ 0 & i \le 0. \end{cases}$$

We have

$$H^1(F, H_1(Y)) = \operatorname{Hom}_{\mathbf{K}_+(R)}(F, H_1(Y)[-1]).$$

By the universal coefficient spectral sequence, we have

$$H^1(F, H_1(Y)) = \operatorname{Hom}_R(H_1(Y), H_1(Y)).$$

Let $i\colon F \longrightarrow H_1(Y)[-1]$ represent the identity on $H_1(Y)$, that is,

$$i_\star\colon H_1(F) \cong H_1(H_1(Y)[-1]) = H_1(Y).$$

Consider

$$\cdots \to \mathrm{Hom}_{\mathbf{K}_+(R)}(F, H_1(Y)[-1]) \xrightarrow{\partial^\star} \mathrm{Hom}_{\mathbf{K}_+(R)}(Y\{0\}[1], H_1(Y)[-1]) \\ \xrightarrow{\alpha_{0\star}} \mathrm{Hom}_{\mathbf{K}_+(R)}(Y[1], H_1(Y)[-1]) \to \cdots .$$

Then let

$$k^2 = \partial^\star(i) \in \mathrm{Hom}_{\mathbf{K}_+(R)}(Y\{0\}[1], H_1(Y)[-1]).$$

Form the following distinguished triangle

$$Y\{0\}[1] \xrightarrow{k^2} H_1(Y)[-1] \xrightarrow{\gamma} Y\{1\} \xrightarrow{\beta_1} Y\{0\}.$$

From the following homology long exact sequence

$$\cdots \longrightarrow H_i(Y\{1\}) \xrightarrow{\beta_{1\star}} H_i(Y\{0\}) \xrightarrow{k^2_\star} H_i(H_1(Y)[-2]) \xrightarrow{\gamma_\star} \cdots$$

we find that

$$H_i(Y\{1\}) = \begin{cases} H_i(Y) & i \leq 1, \\ 0 & i > 1. \end{cases}$$

Then the following commutative diagram in $\mathbf{K}_+(R)$

$$\begin{tikzcd} Y\{0\}[1] \arrow[r, "k^2"] & H_1(Y)[-1] \\ Y\{0\}[1] \arrow[u, "\mathrm{id}"] \arrow[r] & F \arrow[u, "i"'] \end{tikzcd}$$

implies that there exists a morphism $\alpha_1 \colon Y \longrightarrow Y\{1\}$ in $\mathbf{K}_+(R)$ such that the following diagram commutes in $\mathbf{K}_+(R)$.

$$\begin{tikzcd} Y\{0\}[1] \arrow[r, "k^2"] & H_1(Y)[-1] \arrow[r, "\gamma"] & Y\{1\} \arrow[r, "\beta_1"] & Y\{0\} \\ Y\{0\}[1] \arrow[u, "\mathrm{id}"] \arrow[r] & F \arrow[u, "i"] \arrow[r, "\pi"] & Y \arrow[u, "\alpha_1"] \arrow[r, "\alpha_0"] & Y\{0\} \arrow[u, "\mathrm{id}"] \end{tikzcd}$$

Therefore, we have the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & H_1(H_1(Y)[-1]) \arrow[r, "\gamma_\star"] & H_1(Y\{1\}) \arrow[r] & 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & H_1(F) \arrow[u, "i_\star"] \arrow[r, "\pi_\star"] & H_1(Y) \arrow[u, "\alpha_{1\star}"] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

By the Five Lemma, we have $\alpha_{1_\star}$ is an isomorphism since $i_\star$ is an isomorphism. Also, we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & H_0(Y\{1\}) \arrow[r] & H_0(Y) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & H_0(Y) \arrow[r, "\alpha_{0_\star}"] \arrow[u, "\alpha_{1_\star}"] & H_0(Y) \arrow[r] \arrow[u, "\mathrm{id}"] & 0 \arrow[u]
\end{tikzcd}$$

By the Five Lemma, we have $\alpha_{1_\star}$ is an isomorphism. Assume that we have constructed $Y\{n\}$ such that the diagram

$$\begin{tikzcd}
& Y\{n\} \arrow[dd, "\beta_n"] \\
Y \arrow[ur, "\alpha_n"] \arrow[dr, "\alpha_{n-1}"'] & \\
& Y\{n-1\}
\end{tikzcd}$$

commutes, $H_i(Y\{n\}) = 0$ for $i > n$ and $\alpha_{n_\star}\colon H_i(Y) \longrightarrow H_i(Y\{n\})$ is an isomorphism for all $i \leq n$. Next we construct $Y\{n+1\}$. We may change $Y\{n\}$ up to q-isomorphism to assume that $Y\{n\}$ is a chain complex of projective modules. Form the following distinguished triangle

$$Q \xrightarrow{\epsilon} Y \xrightarrow{\alpha_n} Y\{n\} \longrightarrow Q[-1].$$

Then it follows from the corresponding homology long exact sequence that

$$H_i(Q) = \begin{cases} H_i(Y) & i > n, \\ 0 & i \leq n. \end{cases}$$

The remainder of the proof continues as the case $n = 0$. Hence, the theorem is proved. □

Note that inductively we can define k^n to be the morphism $Y\{n-2\}[1] \longrightarrow H_{n-1}(Y)[-n+1]$ and k^n is called the nth *k-invariant* of the chain complex Y.

3.4 The Steenrod Algebra and its dual

In this section, we define an analogue of the mod p Steenrod algebra.

Let $P \longrightarrow \mathbb{K}[0]$ be a minimal projective resolution. It follows that there exists a morphism $P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \longrightarrow \mathbb{K}[0]$ and hence a morphism

$$\phi\colon \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \longrightarrow \mathbb{K}[0].$$

Also, note that in $\mathcal{D}(R)$, we have

$$\begin{aligned}\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] &\cong \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \\ &\cong (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]).\end{aligned}$$

Therefore,

$$\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong (P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} (P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]).$$

since $P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$.

Note that the degree n part of the chain complex $P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ is $P_n \otimes_R \mathbb{K}$ which is free over $\mathbb{K}$ and $d(P_n \otimes_R \mathbb{K}) = 0$ since P is minimal projective resolution. Using Künneth formula for complexes Theorem 1.2.23, we see that

$$\mathrm{Tor}_1^{\mathbb{K}}(H_i(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]), H_j(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])) = 0$$

since $H_i(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])$ is free over $\mathbb{K}$ for each i. It follows that

$$H_n((P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} (P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])) \cong \bigoplus_{i=0}^{n} H_i(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \otimes_{\mathbb{K}} H_{n-i}(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]).$$

Moreover, the natural map $R \longrightarrow \mathbb{K}$ induces the following morphism

$$\eta\colon R[0] \longrightarrow \mathbb{K}[0].$$

Next we show that $\mathbb{K}[0]$ with the morphisms ϕ and η is a commutative monoid in $\mathcal{D}_{+(fg)}(R)$. Since $X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y \cong Y \overset{\mathrm{L}}{\underset{R}{\otimes}} X$ and $X \overset{\mathrm{L}}{\underset{R}{\otimes}} (Y \overset{\mathrm{L}}{\underset{R}{\otimes}} Z) \cong (X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y) \overset{\mathrm{L}}{\underset{R}{\otimes}} Z$ for any chain complexes X, Y and Z in $\mathcal{D}_{+(fg)}(R)$ and because of Lemma 2.4.6, we see that

$$\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} P,$$

$$(\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong (P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{R}{\otimes}} P$$

and

$$\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \cong P \overset{\mathrm{L}}{\underset{R}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} P).$$

Therefore, we have the following commutative diagrams

$$\begin{tikzcd}
& (P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{R}{\otimes}} P \arrow[dl, "\cong"'] \arrow[dr] & \\
P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} P & (P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{R}{\otimes}} P \arrow[l, "\cong"'] \arrow[r] \arrow[u, "\mathrm{id}"] \arrow[d, "\cong"'] & \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} P \\
& P \overset{\mathrm{L}}{\underset{R}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} P) \arrow[ul, "\cong"] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} P \arrow[dl, "\cong"'] \arrow[dr] & \\
\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} P & \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} P \arrow[l, "\cong"'] \arrow[r] \arrow[u, "\mathrm{id}"] \arrow[d, "\cong"'] & \mathbb{K}[0] \\
& P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[ul, "\cong"] \arrow[ur] &
\end{tikzcd}$$

The above two commutative diagrams prove commutativity of the following diagram

$$\begin{tikzcd}
\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[r, "\mathrm{id} \otimes \phi"] \arrow[d, "\phi \otimes \mathrm{id}"'] & \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[d, "\phi"'] \\
\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[r, "\phi"] & \mathbb{K}[0]
\end{tikzcd}$$

Similarly, we can show that the following diagrams

$$\begin{tikzcd}
\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} R[0] \arrow[r, "\mathrm{id} \otimes \eta"] \arrow[d, "\cong"'] & \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[d, "\phi"'] \\
\mathbb{K}[0] \arrow[r, "\mathrm{id}"] & \mathbb{K}[0]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[d, "\phi"'] & R[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[l, "\eta \otimes \mathrm{id}"'] \arrow[d, "\cong"'] \\
\mathbb{K}[0] & \mathbb{K}[0] \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[dl, "\phi"] \\
& \mathbb{K}[0] &
\end{tikzcd}$$

are commutative where τ is the twist morphism. Hence, $\mathbb{K}[0]$ is a commutative monoid in $\mathcal{D}_{+(fg)}(R)$.

Note that we have the following morphism

$$(\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \cong \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \xrightarrow{\phi \otimes \mathrm{id}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0].$$

We shall denote this morphism by Δ. Also, we have the following morphism

$$\begin{aligned} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} R[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \xrightarrow{\mathrm{id} \otimes \eta \otimes \mathrm{id}} & \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \\ & \xrightarrow{\cong} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]). \end{aligned}$$

We shall denote this morphism by Ψ. Now if Y is a chain complex in $\mathcal{D}_{+(fg)}(R)$, then we have the following morphism

$$\begin{aligned} Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong Y \overset{\mathrm{L}}{\underset{R}{\otimes}} R[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \xrightarrow{\mathrm{id} \otimes \eta \otimes \mathrm{id}} & Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \\ & \xrightarrow{\cong} (Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]). \end{aligned}$$

We shall denote this morphism by Ω. Moreover, we have the following morphism

$$R[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \xrightarrow{\eta \otimes \mathrm{id}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0].$$

We shall denote this morphism also by η. We see that there are homomorphisms

$$\Gamma = \Delta_\star \colon H_\star(\mathbb{K}[0], \mathbb{K}) \otimes_{\mathbb{K}} H_\star(\mathbb{K}[0], \mathbb{K}) \longrightarrow H_\star(\mathbb{K}[0], \mathbb{K})$$

$$\lambda = \eta_\star \colon \mathbb{K} \longrightarrow H_\star(\mathbb{K}[0], \mathbb{K})$$

$$\Sigma = \Psi_\star \colon H_\star(\mathbb{K}[0], \mathbb{K}) \longrightarrow H_\star(\mathbb{K}[0], \mathbb{K}) \otimes_{\mathbb{K}} H_\star(\mathbb{K}[0], \mathbb{K})$$

$$\epsilon = \phi_\star \colon H_\star(\mathbb{K}[0], \mathbb{K}) \longrightarrow \mathbb{K}$$

$$c = \tau_\star \colon H_\star(\mathbb{K}[0], \mathbb{K}) \longrightarrow H_\star(\mathbb{K}[0], \mathbb{K})$$

$$\Theta = \Omega_\star \colon H_\star(Y, \mathbb{K}) \longrightarrow H_\star(Y, \mathbb{K}) \otimes_{\mathbb{K}} H_\star(\mathbb{K}[0], \mathbb{K}).$$

Since $H_\star(\mathbb{K}[0], \mathbb{K})$ is free over $H_\star(R[0], \mathbb{K}) = \mathbb{K}$, using [32, Theorem 17.8], we have that $H_\star(\mathbb{K}[0], \mathbb{K})$ is a Hopf algebra over $\mathbb{K}$ with commutative product Γ, unit λ, coproduct Σ, counit ϵ and antipode map c. Moreover, $H_\star(Y, \mathbb{K})$ is a comodule over $H_\star(\mathbb{K}[0], \mathbb{K})$ for any chain complex Y in $\mathcal{D}_{+(fg)}(R)$ where $\Omega_\star$ is the coaction map.

We have noted earlier that $H^\star(\mathbb{K}[0], \mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(H_\star(\mathbb{K}[0], \mathbb{K}), \mathbb{K})$ and since $H_\star(\mathbb{K}[0], \mathbb{K})$ is a graded projective $\mathbb{K}$-module of finite type, then its dual $H^\star(\mathbb{K}[0], \mathbb{K})$ is a Hopf algebra over $\mathbb{K}$ with product $\Sigma^\star$, unit $\epsilon^\star$, cocomutative coproduct $\Gamma^\star$, counit $\lambda^\star$ and antipode map $c^\star$ by Theorem 1.1.25. Moreover,

$$\Theta^\star \colon H^\star(Y, \mathbb{K}) \otimes_{\mathbb{K}} H^\star(\mathbb{K}[0], \mathbb{K}) \longrightarrow H^\star(Y, \mathbb{K})$$

defines the structure of a $H^\star(\mathbb{K}[0], \mathbb{K})$-module on $H^\star(Y, \mathbb{K})$ by Theorem 1.1.24. Therefore, $H^\star(Y, \mathbb{K})$ is a module over $H^\star(\mathbb{K}[0], \mathbb{K})$ for every chain complex Y in $\mathcal{D}_{+(fg)}(R)$.

Definition 3.4.1. The *mod* $\mathfrak{m}$ *Steenrod algebra* $\mathcal{A}^\star$ is the graded $\mathbb{K}$-module with

$$\mathcal{A}^n = H^n(\mathbb{K}[0], \mathbb{K})$$

for all n. Note that $\mathcal{A}^0 = \mathbb{K}$ and $\mathcal{A}^1 \cong \mathbb{K} \oplus \cdots \oplus \mathbb{K}$ n-times where n is the size of minimal generating set of $\mathfrak{m}$.

Remark 3.4.2. Observe that Theorem 2.4.13 implies that

$$\mathcal{A}^n = H^n(\mathbb{K}[0], \mathbb{K}) \cong \mathrm{Ext}^n_R(\mathbb{K}[0], \mathbb{K}[0]) = \mathrm{Hom}_{\mathcal{D}(R)}(\mathbb{K}[0], \mathbb{K}[-n]).$$

3.5 Definitions

In this section, we give some definitions of new notions.

From now on until the end of this chapter, we work in the derived category $\mathcal{D}_{+(fg)}(R)$ of bounded below chain complexes Y whose homology modules $H_i(Y)$ are of finite type and we will consider only chain complexes Y with $Y_i = 0$ for all $i < 0$ and $H_0(Y) \neq 0$. The condition $H_0(Y) \neq 0$ corresponds to the Hurewicz dimension 0 in [5].

We begin with definitions of concepts that are invariant in the derived category. In the following definitions, consider chain complexes X and Y as stated above.

Definition 3.5.1. A morphism $\alpha\colon X \longrightarrow Y$ in $\mathcal{D}_{+(fg)}(R)$ is a *d-monomorphism* if

$$\alpha_\star\colon H_0(X) \otimes_R \mathbb{K} \longrightarrow H_0(Y) \otimes_R \mathbb{K}$$

and

$$\alpha_\star\colon H_n(X) \longrightarrow H_n(Y)$$

are monomorphisms for all $n \geq 0$.

Definition 3.5.2. Y is *irreducible* if any d-monomorphism $\alpha\colon X \longrightarrow Y$ is a d-isomorphism.

Definition 3.5.3. Y is *atomic* if any self morphism $\alpha\colon Y \longrightarrow Y$ that induces an isomorphism on H_0 is a d-isomorphism.

Definition 3.5.4. Y is *minimal atomic* if it is atomic and any d-monomorphism $\alpha\colon X \longrightarrow Y$ from an atomic chain complex X to Y is d-isomorphism.

Definition 3.5.5. Y *has no mod* $\mathfrak{m}$ *detectable homology* if the reduction morphism $\rho\colon H_n(Y) \longrightarrow H_n(Y;\mathbb{K})$ is zero for all $n > 0$.

Definition 3.5.6. Y is *$H^\star$-monogenic* if $H^\star(Y;\mathbb{K})$ is a cyclic module over the mod $\mathfrak{m}$ Steenrod algebra $\mathcal{A}^\star$.

We will prove the following theorem later when we define the notion of a nuclear chain complex.

Theorem 3.5.7. *If Y is a chain complex and $u \in H_0(Y)$ with $0 \neq \bar{u} \in H_0(Y, \mathbb{K})$, then there is a d-monomorphism $\alpha\colon X \longrightarrow Y$ such that X is atomic with $H_0(X)$ a cyclic R-module.*

The above theorem implies the following important result.

Corollary 3.5.8. *Every irreducible chain complex is atomic.*

3.6 Minimal atomic and irreducible chain complexes

The first result in this section characterizes irreducible chain complexes Y which have $H_0(Y)$ a cyclic R-module.

Theorem 3.6.1. *If Y is a chain complex with $H_0(Y)$ a cyclic R-module, then Y is irreducible if and only if Y has no mod $\mathfrak{m}$ detectable homology.*

Proof. Suppose that Y is a chain complex with $H_0(Y)$ a cyclic R-module. Assume that Y is irreducible. Assume that Y has mod $\mathfrak{m}$ detectable homology, that is, $\rho\colon H_n(Y) \longrightarrow H_n(Y, \mathbb{K})$ is non-zero for $n > 0$. Then there is $f\colon R[-n] \longrightarrow Y$ such that $0 \neq \rho(f) \in H_n(Y, \mathbb{K})$. Thus, there exists $0 \neq \alpha\colon Y \longrightarrow \mathbb{K}[-n]$ where $\alpha \in H^n(Y, \mathbb{K})$. Form the following distinguished triangle

$$Y \xrightarrow{\alpha} \mathbb{K}[-n] \xrightarrow{\beta} X \xrightarrow{\gamma} Y[-1]$$

Then we have the following long exact sequence

$$\cdots \longrightarrow 0 \longrightarrow H_{n+1}(X) \xrightarrow{\gamma_\star} H_{n+1}(Y[-1]) \xrightarrow{\alpha_\star} \mathbb{K} \xrightarrow{\beta_\star} H_n(X) \longrightarrow \cdots$$

It is clear that Y is irreducible if and only if $Y[-1]$ is irreducible. Thus, γ is d-monomorphism which is not d-isomorphism. This contradicts the fact that Y is irreducible. Hence, Y has no mod $\mathfrak{m}$ detectable homology.

Conversely, assume that Y has no mod $\mathfrak{m}$ detectable homology. We show that Y is irreducible. Let $\alpha\colon X \longrightarrow Y$ be a d-monomorphism. We claim that α is d-isomorphism. Let

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[-1]$$

be a distinguished triangle. Thus, we have the following long exact sequence

$$\cdots \longrightarrow H_n(X) \xrightarrow{\alpha_\star} H_n(Y) \xrightarrow{\beta_\star} H_n(Z) \xrightarrow{\gamma_\star} H_{n-1}(X) \longrightarrow \cdots$$

Then it is clear that α is d-isomorphism if and only if $H_i(Z)$ is zero for all i. Suppose that $H_\star(Z) \neq 0$. Let n be minimal such that $H_n(Z) \neq 0$. Thus, $\rho_1\colon H_n(Z) \longrightarrow H_n(Z, \mathbb{K})$ is non-zero. Now consider the following commutative diagram

$$\begin{tikzcd}
H_n(Y) \arrow[r, "\beta_\star"] \arrow[d, "\rho_2"'] & H_n(Z) \arrow[d, "\rho_1"] \\
H_n(Y, \mathbb{K}) \arrow[r] & H_n(Z, \mathbb{K})
\end{tikzcd}$$

We have that $\beta_\star$ is an epimorphism since α is d-monomorphism. Thus, ρ_2 is not zero since ρ_1 is not by Lemma 1.1.16. This contradicts that Y has no mod $\mathfrak{m}$ detectable homology. Therefore, $H_i(Z)$ is zero for all i. Hence, Y is irreducible. □

Remark 3.6.2. If $H_0(Y)$ is not a cyclic R-module, then Theorem 3.6.1 does not hold. For example, the chain complex $R[0] \oplus R[0]$ has no mod $\mathfrak{m}$ detectable homology but is not irreducible since

$$R[0] \xrightarrow{(0,\mathrm{id})} R[0] \oplus R[0]$$

is a d-monomorphism which is not a d-isomorphism.

We will now present some examples of irreducible chain complexes.

Example 3.6.3. Let M be a cyclic R-module and consider the chain complex $X = M[0]$. Then clearly X has no mod $\mathfrak{m}$ detectable homology with $H_0(X)$ a cyclic R-module. Therefore, X is irreducible by Theorem 3.6.1.

Example 3.6.4. A projective resolution P of a cyclic R-module M obviously has no mod $\mathfrak{m}$ detectable homology. Hence, it is irreducible.

Example 3.6.5. Consider the following chain complex Y

$$0 \longrightarrow R \xrightarrow{i} R \oplus R \longrightarrow 0$$

where R is in degree 1 and i is the map $(0, \mathrm{id})$. Then it is clear that $H_0(Y) = R$ is a cyclic R-module. Notice that $H_i(Y) = 0$ for all $i > 0$. Thus, the reduction map is zero for $i > 0$. Therefore, Y has no mod $\mathfrak{m}$ detectable homology. Hence, Y is irreducible.

We now give the following interesting example.

Example 3.6.6. Let $R = E_{\mathbb{C}}(x)$ be the exterior algebra over $\mathbb{C}$ with generator x of degree one. Note that R is a noetherian local ring with maximal ideal $\mathfrak{m} = x\mathbb{C}$. The residue field $\mathbb{K} = R/\mathfrak{m} \cong \mathbb{C}$. Let Y be the following chain complex

$$0 \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow 0.$$

Then it is clear that $H_0(Y) = \mathbb{C}$, $H_1(Y) = 0$ and $H_2(Y) = \mathfrak{m}$.

On the other hand, $Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{C}[0]$ is the following chain complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \longrightarrow 0.$$

Therefore, we deduce that $\rho \colon H_2(Y) \longrightarrow H_2(Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{C}[0])$ is zero. Thus, Y has no mod $\mathfrak{m}$ detectable homology. Hence, Y is irreducible.

Theorem 3.6.7. *Let Y be $H^\star$-monogenic. Then Y has no mod $\mathfrak{m}$ detectable homology.*

Proof. We have that Y is $H^\star$-monogenic. Thus, $H^\star(\mathbb{K}[0],\mathbb{K}) \longrightarrow H^\star(Y,\mathbb{K})$ is an epimorphism. Thus, $H_\star(Y,\mathbb{K}) \longrightarrow H_\star(\mathbb{K}[0],\mathbb{K})$ is monomorphism. Now consider the following commutative diagram.

$$\begin{tikzcd} H_n(Y) \arrow[r] \arrow[d, "\rho_1"'] & H_n(\mathbb{K}[0]) \arrow[d, "\rho_2"] \\ H_n(Y,\mathbb{K}) \arrow[r] & H_n(\mathbb{K}[0],\mathbb{K}) \end{tikzcd}$$

It is clear that if $n > 0$, then ρ_1 is zero by Lemma 1.1.16. Hence, Y has no mod $\mathfrak{m}$ detectable homology. □

Remark 3.6.8. The converse of Theorem 3.6.7 fails to hold. Consider the chain complex $Y = R^2[0]$. Then it is obvious Y has no homology detected by mod $\mathfrak{m}$ homology. However, Y is not $H^\star$-monogenic since $H^0(Y,\mathbb{K}) = \mathbb{K}^2$ and $\mathcal{A}^0 = \mathbb{K}$.

Now we give some examples of $H^\star$-monogenic chain complexes.

Example 3.6.9. The chain complex $R[0]$ is an $H^\star$-monogenic since $H^\star(R[0],\mathbb{K}) = \mathbb{K} = \mathcal{A}^\star/I$ where I is the ideal of the augmentation map $\lambda^\star\colon \mathcal{A}^\star \longrightarrow \mathbb{K}$.

Example 3.6.10. Since $H^\star(\mathbb{K}[0],\mathbb{K}) = \mathcal{A}^\star$, then the chain complex $\mathbb{K}[0]$ is clearly an $H^\star$-monogenic .

We present now a less obvious example.

Example 3.6.11. Consider the polynomial ring $\mathbb{R}[X,Y]$ in two variables over the real numbers and the maximal ideal $\mathfrak{m} = (X^2+1, Y)$. Let $R = \mathbb{R}[X,Y]_{(X^2+1,Y)}$ be the localization of $\mathbb{R}[X,Y]$ at the prime ideal (X^2+1,Y). Then it is clear that the residue field $\mathbb{K} \cong \mathbb{C}$. Now let $M = R/(Y)R$. Consider the chain complex $M[0]$. Now to calculate $H^\star(\mathbb{C}[0],\mathbb{C})$, we resolve $\mathbb{C}[0]$ with the following minimal free chain complex

$$0 \longrightarrow R \xrightarrow{f} R \oplus R \xrightarrow{g} R \longrightarrow 0.$$

where $g(1,0) = X^2 + 1$, $g(0,1) = Y$ and $f(1) = (Y, -X^2 - 1)$. Therefore,

$$H^i(\mathbb{C}[0], \mathbb{C}) = \begin{cases} \mathbb{C} & i = 0, 2, \\ \mathbb{C}^2 & i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\mathcal{A}^\star = E_{\mathbb{C}}(e_1, e_2)$

Similarly, we resolve $M[0]$ with the following minimal free chain complex

$$0 \longrightarrow R \xrightarrow{h} R \longrightarrow 0.$$

where $h(1) = Y$. So

$$H^i(M[0], \mathbb{C}) = \begin{cases} \mathbb{C} & i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $H^\star(M[0], \mathbb{C}) = \mathcal{A}^\star / \mathcal{A}^\star e_1$. Hence, $M[0]$ is $H^\star$-monogenic.

We now give an example that motivates the characterization of minimal atomic chain complexes.

Example 3.6.12. Let M be an R-module which is not a cyclic and consider the chain complex $Y = M[0]$. Then it is clear that Y is atomic by Definition 3.5.3. But there is no reason why Y is irreducible. For example, the chain complex $Y = R[0] \oplus R[0]$ is obviously an atomic chain complex but Remark 3.6.2 shows that Y is not irreducible.

Observe that the chain complex $R[0] \oplus R[0]$ in Remark 3.6.2 is an example of an atomic chain complex that is not irreducible as well as not minimal atomic.

We come now to the following important result which proves that minimal atomic chain complexes and irreducible chain complexes are the same.

Theorem 3.6.13. *A chain complex Y is irreducible if and only if it is minimal atomic.*

Proof. Assume that Y is irreducible. We show that Y is minimal atomic. Y is atomic by Corollary 3.5.8. Let $\alpha\colon X \longrightarrow Y$ be a d-monomorphism with X atomic.

It is clear that α is a d-isomorphism since Y is irreducible. Hence, Y is minimal atomic.

Conversely, assume that Y is minimal atomic. We prove that Y is irreducible. Let $\alpha\colon Z \longrightarrow Y$ be a d-monomorphism. Then there is a d-monomorphism $\beta\colon X \longrightarrow Z$ such that X is atomic. Thus, the composite $\alpha\beta$ is d-monomorphism with X atomic. Hence, $\alpha\beta$ is d-isomorphism since Y is minimal atomic. This implies that α induces an epimorphism on homology modules. Therefore α is d-isomorphism. Hence, Y is irreducible. $\square$

Example 3.6.14. Example 3.6.3, Example 3.6.4, Example 3.6.5 and Example 3.6.6 are examples of minimal atomic chain complexes using Theorem 3.6.13.

Theorem 3.6.15. *A chain complex Y with $H_0(Y)$ a cyclic R-module is minimal atomic if and only if $Y\{n\}$ is minimal atomic for each $n \geq 0$.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
H_i(Y) \arrow[r] \arrow[d, "\rho_1"] & H_i(Y\{n\}) \arrow[d, "\rho_2"] \\
H_i(Y,\mathbb{K}) \arrow[r] & H_i(Y\{n\},\mathbb{K})
\end{tikzcd}$$

Now assume that Y is minimal atomic. Then ρ_1 is zero since Y is irreducible by Theorem 3.6.13 and thus it has no homology detected by mod $\mathfrak{m}$ homology by Theorem 3.6.1. Since the top horizontal map is an epimorphism for all i, in fact it is an isomorphism for all $i \leq n$ by Theorem 3.3.1, we have that ρ_2 is zero by Lemma 1.1.16. Hence, $Y\{n\}$ is minimal atomic for each $n \geq 0$. Conversely, assume that $Y\{n\}$ is minimal atomic, that is, $\rho_2 = 0$. By Lemma 1.1.16, it suffices to show that the bottom horizontal map is a monomorphism. But $H_i(Y,\mathbb{K}) \cong H_i(Y\{n\},\mathbb{K})$ for all $i \leq n$ by the Derived Whitehead Theorem. Therefore, induction shows that ρ_1 is zero. Hence, Y is minimal atomic. $\square$

Definition 3.6.16. The *k-invariants of Y detect its homology* if each k-invariant $k^{n+2}\colon Y\{n\}[1] \longrightarrow H_{n+1}(Y)[-n-1]$, $n \geq 0$ of a Postnikov tower $\Big\{Y\{n\}\Big\}$ induces an epimorphism

$$H_{n+1}(Y\{n\}[1],\mathbb{K}) \longrightarrow H_{n+1}(H_{n+1}(Y)[-n-1],\mathbb{K}) \cong H_{n+1}(Y) \otimes_R \mathbb{K}.$$

Theorem 3.6.17. *A chain complex Y with $H_0(Y)$ a cyclic R-module is irreducible if and only if the k-invariants of Y detect its homology.*

Proof. We show that the chain complex Y is irreducible if and only if each k-invariant $k^{n+2}\colon Y\{n\}[1] \longrightarrow H_{n+1}(Y)[-n-1]$, $n \geq 0$ of a Postnikov tower $\Big\{Y\{n\}\Big\}$ induces an epimorphism

$$H_{n+1}(Y\{n\}[1], \mathbb{K}) \longrightarrow H_{n+1}(H_{n+1}(Y)[-n-1], \mathbb{K}) \cong H_{n+1}(Y) \otimes_R \mathbb{K}.$$

Consider the following distinguished triangle

$$Y\{n\}[1] \xrightarrow{k^{n+2}} H_{n+1}(Y)[-n-1] \xrightarrow{\gamma} Y\{n+1\} \xrightarrow{\beta_{n+1}} Y\{n\}.$$

Then we have the following commutative diagram

$$\begin{tikzcd}
Y\{n\}[1] \arrow[r] \arrow[d, "k^{n+2}"'] & Y\{n\}[1] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[d, "k^{n+2}\otimes\mathrm{id}"'] \\
H_{n+1}(Y)[-n-1] \arrow[r] \arrow[d, "\gamma"'] & H_{n+1}(Y)[-n-1] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[d, "\gamma\otimes\mathrm{id}"'] \\
Y\{n+1\} \arrow[r] \arrow[d, "\beta_{n+1}"'] & Y\{n+1\} \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[d, "\beta_{n+1}\otimes\mathrm{id}"'] \\
Y\{n\} \arrow[r] & Y\{n\} \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]
\end{tikzcd}$$

Therefore, we have the following commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H_{n+1}(Y\{n\}[1]) \arrow[r] \arrow[d, "k^{n+2}_\star"'] & H_{n+1}(Y\{n\}[1] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \arrow[d, "(k^{n+2}\otimes\mathrm{id})_\star"'] \\
H_{n+1}(H_{n+1}(Y)[-n-1]) \arrow[r, "\rho_1"] \arrow[d, "\gamma_\star"', "\cong"] & H_{n+1}(H_{n+1}(Y)[-n-1] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \arrow[d, "(\gamma\otimes\mathrm{id})_\star"'] \\
H_{n+1}(Y\{n+1\}) \arrow[r, "\rho_2"] \arrow[d] & H_{n+1}(Y\{n+1\} \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

in which the columns are exact. Note that ρ_1 is just reduction mod $\mathfrak{m}$ where

$$H_{n+1}(H_{n+1}(Y)[-n-1] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) = H_{n+1}(Y) \otimes_R \mathbb{K}.$$

Therefore, if $(k^{n+2} \otimes \mathrm{id})_\star$ is an epimorphism for each $n \geq 0$, then ρ_2 is zero for each $n \geq 0$ since $(\gamma \otimes \mathrm{id})_\star$ is zero. Thus, Y is minimal atomic by Theorem 3.6.15 and hence irreducible by Theorem 3.6.13.

Conversely, if Y is irreducible, hence has no mod $\mathfrak{m}$ detectable homology by Theorem 3.6.1, then Y is minimal atomic and thus $Y\{n\}$ is minimal atomic. Therefore, ρ_2 is zero and it follows that $(\gamma \otimes \mathrm{id})_\star$ is zero. Hence, $(k^{n+2} \otimes \mathrm{id})_\star$ is an epimorphism. □

3.7 Nuclear chain complexes

In this section, for $n \geq 0$, the $n+1$-skeleton, $Y^{[n+1]}$, of a chain complex Y is defined to be the mapping cone of a map $\partial_n\colon J_n \longrightarrow Y^{[n]}$, where J_n is a finite direct sum of copies of R.

Definition 3.7.1. A *nuclear* chain complex is a free chain complex Y in which $Y_0 = R$ and

$$\mathrm{Ker}(\partial_{n\star}\colon H_n(\oplus R[-n]) \longrightarrow H_n(Y^{[n]})) \subset \mathfrak{m}\ H_n(\oplus R[-n])$$

for each n.

Observe that Y is nuclear if and only if each n-skeleton $Y^{[n]}$ for $n \geq 0$ is nuclear.

We now give some examples of nuclear chain complexes.

Example 3.7.2. $R[0]$ is an example of a nuclear chain complex.

Example 3.7.3. Consider the following Koszul chain complex Y

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

where R concentrated in degrees 1 and 0 and $x \in \mathfrak{m}$ is a nonzero divisor on R. Then it is clear that $H_0(Y) = R/xR$ is a cyclic R-module and $\mathrm{Ker}(\partial_{0\star}\colon H_0(R[0]) \longrightarrow H_0(Y^{[0]})) = 0$. Therefore, Y is nuclear.

Example 3.7.4. The chain complex Y in Example 3.6.6 is also nuclear.

Definition 3.7.5. A *core* of a chain complex Y is a nuclear chain complex X together with a d-monomorphism $\alpha\colon X \longrightarrow Y$.

Proposition 3.7.6. *A nuclear chain complex is atomic.*

Proof. Let Y be a nuclear chain complex and let $\alpha\colon Y \longrightarrow Y$ be a morphism that induces an isomorphism on H_0. We must show that α is a d-isomorphism or equivalently, $Y^{[n]} \longrightarrow Y^{[n]}$ is a d-isomorphism for each n. Since α induces an isomorphism on H_0, we see that $\alpha_\star\colon H_i(Y^{[0]}) \longrightarrow H_i(Y^{[0]})$ an isomorphism for all i. Assume inductively that $\alpha\colon Y^{[n]} \longrightarrow Y^{[n]}$ is a d-isomorphism. Now we claim that $\alpha\colon Y^{[n+1]} \longrightarrow Y^{[n+1]}$ is a d-isomorphism. It suffices to show that $H_q(Y^{[n+1]}) \longrightarrow H_q(Y^{[n+1]})$ is an isomorphism for $q = n$ and $q = n+1$. We have that

$$\oplus R[-n] \xrightarrow{\partial_n} Y^{[n]} \longrightarrow Y^{[n+1]} \longrightarrow \oplus R[-n-1]$$

is a distinguished triangle. We see that α induces the following commutative diagram.

$$\begin{tikzcd}
\oplus R[-n] \arrow[r, "\partial_n"] \arrow[d, dotted, "\exists\ f"'] & Y^{[n]} \arrow[r] \arrow[d] & Y^{[n+1]} \arrow[r] \arrow[d] & \oplus R[-n-1] \arrow[d, dotted, "f[-1]"] \\
\oplus R[-n] \arrow[r, "\partial_n"] & Y^{[n]} \arrow[r] & Y^{[n+1]} \arrow[r] & \oplus R[-n-1]
\end{tikzcd}$$

There results the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & H_{n+1}(Y^{[n+1]}) \arrow[r] \arrow[d] & H_n(\oplus R[-n]) \arrow[r] \arrow[d, "f_\star"'] & H_n(Y^{[n]}) \arrow[r] \arrow[d, "\cong"'] & H_n(Y^{[n+1]}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_{n+1}(Y^{[n+1]}) \arrow[r] & H_n(\oplus R[-n]) \arrow[r] & H_n(Y^{[n]}) \arrow[r] & H_n(Y^{[n+1]}) \arrow[r] & 0
\end{tikzcd}$$

It suffices to prove that $f_\star\colon H_n(\oplus R[-n]) \longrightarrow H_n(\oplus R[-n])$ is an isomorphism by the Five Lemma. We have the following commutative diagram with exact rows.

$$\begin{tikzcd}
H_n(\oplus R[-n]) \arrow[r, "\partial_{n\star}"] \arrow[d] & H_n(Y^{[n]}) \arrow[r] \arrow[d, "\cong"] & H_n(Y^{[n+1]}) \arrow[r] \arrow[d] & 0 \\
H_n(\oplus R[-n]) \arrow[r, "\partial_{n\star}"] & H_n(Y^{[n]}) \arrow[r] & H_n(Y^{[n+1]}) \arrow[r] & 0
\end{tikzcd}$$

The right vertical arrow is an epimorphism by diagram chasing. Therefore, it is an isomorphism since epimorphic endomorphism of a finitely generated module over a commutative ring R is an isomorphism by Theorem 1.1.13. This implies that right vertical arrow is an isomorphism in the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}\partial_{n\star} \arrow[r, "i"] \arrow[d] & H_n(\oplus R[-n]) \arrow[r] \arrow[d, "f_\star"'] & \operatorname{Im}\partial_{n\star} \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \operatorname{Ker}\partial_{n\star} \arrow[r, "i"] & H_n(\oplus R[-n]) \arrow[r] & \operatorname{Im}\partial_{n\star} \arrow[r] & 0 \end{tikzcd}$$

After tensoring with $\mathbb{K}$, the inclusion i becomes 0 since

$$\operatorname{Ker}(\partial_{n\star}\colon H_n(\oplus R[-n]) \longrightarrow H_n(Y^{[n]})) \subset \mathfrak{m}\ H_n(\oplus R[-n]).$$

Therefore, $f_\star \otimes \mathrm{id}_{\mathbb{K}}$ is an isomorphism. This implies that $f_\star$ is an isomorphism. Hence, Y is an atomic. □

Remark 3.7.7. The converse of Proposition 3.7.6 does not hold in general. Consider the following chain complex Y

$$0 \longrightarrow I \xrightarrow{i} R \longrightarrow 0$$

in which I is an ideal of R in degree 1 and i is the inclusion map. We see that Y has no mod $\mathfrak{m}$ detectable homology and since $H_0(Y) = R/I$ is a cyclic R-module, Y is irreducible by Theorem 3.6.1. Hence, Y is atomic by Theorem 3.6.13. However, Y is not nuclear since it is not free chain complex.

The following result shows that a core of a chain complex whose zero homology is cyclic exists.

Theorem 3.7.8. *Let Y be a chain complex with $H_0(Y)$ a cyclic R-module. Then there is a core $\alpha\colon X \longrightarrow Y$.*

Proof. We have that $H_0(Y)$ is a cyclic R-module. We may change Y up to q-isomorphism to assume that $Y_0 = R$. Let $X_0 = R$ and define $\alpha_0\colon R \longrightarrow R$ by $1 \longmapsto 1$. Assume inductively that we have constructed $X^{[n]}$ and $\alpha_n\colon X^{[n]} \longrightarrow Y$ that induces monomorphism on homology modules in dimension less than n. Choose a minimal (finite) set of generators for the kernel of $\alpha_{n\star}\colon H_n(X^{[n]}) \longrightarrow H_n(Y)$. Let J_n be the sum of a copy of R for each chosen generator, and let

$$\partial_n\colon J_n = \oplus R[-n] \longrightarrow X^{[n]}$$

represent the chosen generators. Define $X^{[n+1]}$ to be the mapping cone of ∂_n,

$$\oplus R[-n] \xrightarrow{\partial_n} X^{[n]} \longrightarrow X^{[n+1]}.$$

We see that the composite

$$\oplus R[-n] \xrightarrow{\partial_n} X^{[n]} \longrightarrow Y$$

is zero. Notice that for Y there is a distinguished triangle

$$Y \xrightarrow{\text{id}} Y \longrightarrow 0 \longrightarrow Y[-1].$$

Now consider the following commutative diagram of solid lines.

$$\begin{tikzcd} \oplus R[-n] \arrow[r] \arrow[d] & X^{[n]} \arrow[r] \arrow[d] & X^{[n+1]} \arrow[d, dotted, "\alpha_{n+1}"] \\ 0 \arrow[r] & Y \arrow[r] & Y \end{tikzcd}$$

Then there exists $\alpha_{n+1}\colon X^{[n+1]} \longrightarrow Y$ making the diagram commute. Note that the morphism $X^{[n]} \longrightarrow X^{[n+1]}$ induces an isomorphism on H_i for $i < n$ and an epimorphism on H_n. By construction, we deduce that α_{n+1} induces a monomorphism on H_i for $i \leq n$. On passage to colimit, we obtain $\alpha\colon X \longrightarrow Y$ that induces monomorphism on all homology modules. The minimality of the chosen set of generators ensures that

$$\operatorname{Ker}(\partial_{n\star}\colon H_n(\oplus R[-n]) \longrightarrow H_n(X^{[n]})) \subset \mathfrak{m}\, H_n(\oplus R[-n])$$

holds which means that X is nuclear. Hence, there is a core $\alpha\colon X \longrightarrow Y$ □

We now give a proof of Theorem 3.5.7, which shows that the core of any chain complex always exists without restriction to cyclicity of the zeroth homology.

Proof of Theorem 3.5.7. Let $u \in H_0(Y) = Y_0/B_0(Y)$ such that $0 \neq \bar{u} \in H_0(Y, \mathbb{K})$. Lift u to an element $\tilde{u} \in Y_0$. Then it is clear that

$$\langle u \rangle = R/I \subset H_0(Y) = Y_0/B_0(Y)$$

for some ideal $I \in R$. Let $X_0 = R$. Define $\alpha_0\colon R \longrightarrow Y_0$ by $1 \longmapsto \tilde{u}$. Assume inductively that we have constructed $X^{[n]}$ and $\alpha_n\colon X^{[n]} \longrightarrow Y$ that induces monomorphism on homology modules in dimension less than n. Choose a minimal finite set of generators for the kernel of $\alpha_{n\star}\colon H_n(X^{[n]}) \longrightarrow H_n(Y)$. Now we continue as in the proof of Theorem 3.7.8 to end up with a nuclear chain complex X and a d-monomorphism $\alpha\colon X \longrightarrow Y$ Therefore, X is atomic by Proposition 3.7.6. Hence, there is a d-monomorphism $\alpha\colon X \longrightarrow Y$ such that X is atomic with $H_0(X)$ a cyclic R-module. □

The proof of Proposition 3.7.6 can be adapted to show the following result.

Proposition 3.7.9. *Let X and Y be nuclear chain complexes and let $\alpha\colon X \longrightarrow Y$ be a core of Y. Then α is a d-isomorphism.*

Proof. It is obvious that $H_0(X^{[0]}) \longrightarrow H_0(Y^{[0]})$ is an isomorphism. Thus, $H_k(X^{[0]}) \longrightarrow H_k(Y^{[0]})$ is an isomorphism for all k. Now assume that $\alpha\colon X^{[n]} \longrightarrow Y^{[n]}$ is a d-isomorphism. We show that $\alpha\colon X^{[n+1]} \longrightarrow Y^{[n+1]}$ is a d-isomorphism. It suffices to show that $H_q(X^{[n+1]}) \longrightarrow H_q(Y^{[n+1]})$ is an isomorphism for $q = n$ and $q = n + 1$. There is a commutative diagram of distinguished triangles.

$$\begin{tikzcd}
J_n = \oplus R[-n] \arrow[r, "j_n"] \arrow[d, dotted, "\exists\ f"] & X^{[n]} \arrow[r] \arrow[d, "\alpha"'] & X^{[n+1]} \arrow[d, "\alpha"'] \\
K_n = \oplus R[-n] \arrow[r, "k_n"] & Y^{[n]} \arrow[r] & Y^{[n+1]}
\end{tikzcd}$$

There results the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & H_{n+1}(X^{[n+1]}) \arrow[r] \arrow[d] & H_n(J_n) \arrow[r] \arrow[d, "f_\star"] & H_n(X^{[n]}) \arrow[r] \arrow[d, "\cong"] & H_n(X^{[n+1]}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_{n+1}(Y^{[n+1]}) \arrow[r] & H_n(K_n) \arrow[r] & H_n(Y^{[n]}) \arrow[r] & H_n(Y^{[n+1]}) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, it suffices to show that $f_\star$ is an isomorphism. We have the following commutative diagram with exact rows.

$$\begin{tikzcd}
H_n(J_n) \arrow[r] \arrow[d] & H_n(X^{[n]}) \arrow[r] \arrow[d, "\cong"] & H_n(X^{[n+1]}) \arrow[r] \arrow[d] & 0 \\
H_n(K_n) \arrow[r] & H_n(Y^{[n]}) \arrow[r] & H_n(Y^{[n+1]}) \arrow[r] & 0
\end{tikzcd}$$

The right vertical arrow is an epimorphism by diagram chasing. Consider the following diagram

$$\begin{tikzcd} H_n(X^{[n+1]}) \arrow[r, "\cong"] \arrow[d] & H_n(X) \arrow[d] \\ H_n(Y^{[n+1]}) \arrow[r, "\cong"] & H_n(Y) \end{tikzcd}$$

We see that the right vertical arrow is monomorphism. Thus, the left vertical arrow is monomorphism, hence isomorphism. Thus, the right vertical arrow is an isomorphism in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} j_{n\star} \arrow[r, "i_1"] \arrow[d] & H_n(J_n) \arrow[r] \arrow[d, "f_\star"] & \operatorname{Im} j_{n\star} \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \operatorname{Ker} k_{n\star} \arrow[r, "i_2"] & H_n(K_n) \arrow[r] & \operatorname{Im} k_{n\star} \arrow[r] & 0 \end{tikzcd}$$

We see that the maps i_1 and i_2 become 0 after tensoring with $\mathbb{K}$. Therefore $f_\star \otimes \mathrm{id}_{\mathbb{K}}$ is an isomorphism. This implies that $f_\star$ is an isomorphism. Hence, α is a d-isomorphism. □

In Proposition 3.7.6, we showed that a nuclear chain complex is atomic and now with the aid of Proposition 3.7.9, we give the following strong result.

Theorem 3.7.10. *A nuclear chain complex is minimal atomic.*

Proof. Let Y be a nuclear chain complex. We prove that Y is minimal atomic. Y is atomic by Proposition 3.7.6. Let $\alpha\colon X \longrightarrow Y$ be a d-monomorphism where X is atomic. We show that α is d-isomorphism. Let $\beta\colon Z \longrightarrow X$ be a core of X. Therefore, the composite $\alpha\beta\colon Z \longrightarrow Y$ is a core of Y. Hence, $\alpha\beta$ is d-isomorphism by Proposition 3.7.9. Thus, α must induce an epimorphism on homology and so it is an isomorphism. Therefore, α is a d-isomorphism. Hence, Y is minimal atomic. □

Theorem 3.7.11. *The following conditions on a chain complex Y are equivalent.*

(i) *Y is minimal atomic.*

(ii) *Any core of Y is a d-isomorphism.*

(iii) *Y is d-isomorphic to a nuclear chain complex.*

Proof. First we show that (i) implies (ii). So assume that Y is minimal atomic. Let $\alpha\colon X \longrightarrow Y$ be a core of Y. Then α is d-isomorphism since Y is minimal atomic. Hence, (i) implies (ii). Next we show that (ii) implies (iii). Assume any core of Y is d-isomorphism. Let $\alpha\colon X \longrightarrow Y$ be a core of Y. That is, X is nuclear and α induces monomorphism on homology groups. Thus, α is a d-isomorphism. Therefore, Y is d-isomorphic to a nuclear chain complex. Hence, (ii) implies (iii). Now we show that (iii) implies (i). Assume that Y is d-isomorphic to a nuclear chain complex. That is, there exists a nuclear chain complex X and a d-isomorphism $\alpha\colon X \longrightarrow Y$. We have that X is nuclear and thus minimal atomic by Theorem 3.7.10. We claim that Y is minimal atomic. We show that Y is atomic. Let $\theta\colon Y \longrightarrow Y$ be a morphism that induces an isomorphism on H_0. We show that θ is d-isomorphism. Consider the following diagram of solid arrows.

$$\begin{tikzcd} X \arrow[r, dotted, "\psi"] \arrow[d, "\alpha"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\theta"'] & Y \end{tikzcd}$$

We have that α is d-isomorphism. Thus, there exists a morphism $\psi\colon X \longrightarrow X$ which induces an isomorphism on H_0. But X is atomic. Thus, ψ is a d-isomorphism. Hence, θ is a d-isomorphism. Therefore, Y is atomic. Let $\beta\colon Z \longrightarrow Y$ be a d-monomorphism, where Z is atomic. We show that β is d-isomorphism. Consider the following diagram.

$$\begin{tikzcd} Z \arrow[dr, "\beta"] & \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

Therefore there exists $\gamma\colon Z \longrightarrow X$ which is d-monomorphism. But X is minimal atomic. Therefore, γ is d-isomorphism. Thus, β is d-isomorphism. Hence, Y is minimal atomic, showing that (iii) implies (i). □

Now we have the following lemma which characterizes minimal chain complexes.

Lemma 3.7.12. *A chain complex (Y, d) is minimal if and only if the inclusion of skeleta $i\colon Y^{[n]} \longrightarrow Y^{[n+1]}$ induces an isomorphism*

$$i_n\colon H_n(Y^{[n]}, \mathbb{K}) \longrightarrow H_n(Y^{[n+1]}, \mathbb{K}) = H_n(Y, \mathbb{K})$$

for each n.

Proof. We may change Y up to q-isomorphism to suppose that Y is a chain complex of projective modules. Assume that Y is minimal, that is, $d_n \otimes \mathrm{id}_{\mathbb{K}} = 0$ for each n. We always have that $i_n \colon H_n(Y^{[n]}, \mathbb{K}) \longrightarrow H_n(Y^{[n+1]}, \mathbb{K})$ is an epimorphism. Since Y is minimal, we have that

$$i_n \colon H_n(Y^{[n]}, \mathbb{K}) = Y_n \otimes_R \mathbb{K} \longrightarrow H_n(Y^{[n+1]}, \mathbb{K}) = Y_n \otimes_R \mathbb{K}$$

is an epimorphism and hence i_n is an isomorphism for each n. Conversely, assume that i_n is an isomorphism for each n. We show that Y is minimal and we do that by induction. At 0, we have that

$$H_0(Y^{[0]}, \mathbb{K}) = Y_0 \otimes_R \mathbb{K} \xrightarrow{\cong} H_0(Y^{[1]}, \mathbb{K}) = (Y_0 \otimes_R \mathbb{K}) / \operatorname{Im}(d_1 \otimes \mathrm{id}_{\mathbb{K}}).$$

Therefore, $\operatorname{Im}(d_1 \otimes \mathrm{id}_{\mathbb{K}}) = 0$. Thus, $d_1 \otimes \mathrm{id}_{\mathbb{K}} = 0$. Assume that $d_n \otimes \mathrm{id}_{\mathbb{K}} = 0$. We claim that $d_{n+1} \otimes \mathrm{id}_{\mathbb{K}} = 0$. We have that

$$H_n(Y^{[n]}, \mathbb{K}) = Y_n \otimes_R \mathbb{K} \xrightarrow{\cong} H_n(Y^{[n+1]}, \mathbb{K}) = (Y_n \otimes_R \mathbb{K}) / \operatorname{Im}(d_{n+1} \otimes \mathrm{id}_{\mathbb{K}}).$$

Therefore, $\operatorname{Im}(d_{n+1} \otimes \mathrm{id}_{\mathbb{K}}) = 0$. Thus, $d_{n+1} \otimes \mathrm{id}_{\mathbb{K}} = 0$. Hence, Y is minimal. $\square$

Lemma 3.7.13. *Let Y be a chain complex with $H_0(Y)$ a cyclic R-module. Then Y is nuclear if and only if $\rho \colon H_n(Y^{[n]}) \longrightarrow H_n(Y^{[n]}, \mathbb{K})$ is zero for $n > 0$.*

Proof. First note that we have the following distinguished triangle

$$\oplus R[-n] \xrightarrow{\alpha} Y^{[n]} \xrightarrow{\beta} Y^{[n+1]} \xrightarrow{\gamma} \oplus R[-n-1].$$

Thus, we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & H_{n+1}(Y^{[n+1]}) \arrow[r, "\gamma_\star"] \arrow[d, "\rho_1"'] & H_n(\oplus R[-n]) \arrow[r, "\alpha_\star"] \arrow[d, "\rho_2"'] & H_n(Y^{[n]}) \arrow[d, "\rho"'] \\
0 \arrow[r] & H_{n+1}(Y^{[n+1]}, \mathbb{K}) \arrow[r] & H_n(\oplus R[-n], \mathbb{K}) \arrow[r] & H_n(Y^{[n]}, \mathbb{K})
\end{tikzcd}$$

Now assume that Y is nuclear. That is, $\operatorname{Ker}(\alpha_\star) \subset \mathfrak{m}\, H_n(\oplus R[-n])$ for each n. But $\operatorname{Ker}(\alpha_\star) = \operatorname{Im}(\gamma_\star)$. Let $y \in H_{n+1}(Y^{[n+1]})$. Therefore, $\rho_2(\gamma_\star(y)) = 0$. Thus, $\rho_1(y) = 0$. Hence, $\rho \colon H_n(Y^{[n]}) \longrightarrow H_n(Y^{[n]}, \mathbb{K})$ is zero for $n > 0$. Conversely, assume that $\rho \colon H_n(Y^{[n]}) \longrightarrow H_n(Y^{[n]}, \mathbb{K})$ is zero for $n > 0$. Then $\operatorname{Im}(\gamma_\star) \subset \operatorname{Ker}(\rho_2)$. But $\operatorname{Ker}(\rho_2) = \mathfrak{m}\, H_n(\oplus R[-n])$. Thus, $\operatorname{Ker}(\alpha_\star) \subset \mathfrak{m}\, H_n(\oplus R[-n])$. Hence, Y is nuclear. $\square$

Remark 3.7.14. Note that ρ_2 is an epimorphism. Therefore, when Y is nuclear, $H_n(\oplus R[-n], \mathbb{K}) \longrightarrow H_n(Y^{[n]}, \mathbb{K})$ is zero. This implies that Y is minimal by Lemma 3.7.12.

Theorem 3.7.15. *Let Y be a chain complex with $H_0(Y)$ a cyclic R-module. Then Y is nuclear if and only if it satisfies*

(i) *Y has no mod $\mathfrak{m}$ detectable homology,*

(ii) *Y is minimal chain complex.*

Proof. Assume that Y is nuclear. Remark 3.7.14 shows that Y is minimal chain complex. Consider the following commutative diagram.

$$\begin{tikzcd} H_n(Y^{[n]}) \arrow[r] \arrow[d, "\rho_1"'] & H_n(Y) \arrow[d, "\rho_2"] \\ H_n(Y^{[n]}, \mathbb{K}) \arrow[r] & H_n(Y, \mathbb{K}) \end{tikzcd}$$

The top arrow is an epimorphism. Since Y is nuclear, we have ρ_1 is zero for $n > 0$ by Lemma 3.7.13. Thus, ρ_2 is zero by Lemma 1.1.16. Hence, Y has no mod $\mathfrak{m}$ detectable homology.

Conversely, assume that (i) and (ii) hold. Thus, the bottom arrow is an isomorphism and ρ_2 is zero for $n > 0$ in the above commutative diagram. Thus, ρ_1 is zero for $n > 0$. Hence, Y is nuclear by Lemma 3.7.13. □

Example 3.7.16. Using Theorem 3.7.15, the chain complex Y in Example 3.7.3 is nuclear since if we tensor Y with $\mathbb{K}$, we will have that $x \otimes \mathrm{id}_{\mathbb{K}}$ is zero since $x \in \mathfrak{m}$ and thus Y is minimal. Also, notice that $H_0(Y) = R/xR$ is a cyclic R-module, $H_1(Y) = 0$ since X is a nonzero divisor and $H_1(Y, \mathbb{K}) = \mathbb{K}$ and thus the reduction map ρ is just a zero map. Therefore, Y has no mod $\mathfrak{m}$ detectable homology.

Example 3.7.17. The chain complex Y in Example 3.6.6 is nuclear since Y has no mod $\mathbb{C}$ detectable homology and is minimal.

Example 3.7.18. Let $R = E_{\mathbb{C}}(x)$. Let Y be the following chain complex

$$\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow 0,$$

that is, R in each degree with multiplication by x as differential. Then it is clear that

$$H_i(Y) = \begin{cases} \mathbb{C} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, Y has no mod $\mathbb{C}$ detectable homology. Since $x \otimes \mathrm{id}_{\mathbb{C}} = 0$, Y is minimal. Hence, Y is nuclear by Theorem 3.7.15.

Example 3.7.19. Note that Example 3.6.4 is not nuclear since it is not minimal. Therefore, we deduce that minimal projective resolution of a cyclic R-module is nuclear.

Now we give the following description of minimal atomic chain complexes.

Theorem 3.7.20. *The following conditions on a chain complex Y with $H_0(Y)$ a cyclic R-module are equivalent.*

(i) *Y is minimal atomic.*

(ii) *Any d-isomorphism $\alpha\colon X \longrightarrow Y$ from a minimal chain complex X to Y is a core of Y.*

(iii) *A minimal chain complex d-isomorphic to Y is nuclear.*

Proof. We prove that (i) implies (ii). Assume that Y is minimal atomic. Let $\alpha\colon X \longrightarrow Y$ be a d-isomorphism from a minimal chain complex X to Y. We show that X is nuclear. We have that X is minimal atomic, hence irreducible by Theorem 3.6.13, since α is d-isomorphism. Thus, X has no mod $\mathfrak{m}$ detectable homology by Theorem 3.6.1. Hence, X is nuclear by Theorem 3.7.15. It is clear that (ii) implies (iii). Next we show that (iii) implies (i). Let X be a minimal chain complex d-isomorphic to Y. Assume X is nuclear. Then X is minimal atomic by Theorem 3.7.10. Hence, Y is minimal atomic. $\square$

Chapter 4

The Adams Spectral Sequence For Chain Complexes

In this chapter, we assume that R is a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{K}$. Consider the derived category $\mathcal{D}_{+(fg)}(R)$ of bounded below chain complexes of finite type. Assume that all chain complexes are connective.

In this chapter, we set up the Adams spectral sequence for chain complexes in $\mathcal{D}_{+(fg)}(R)$ and discuss its convergence and give some examples.

4.1 Setting up the spectral sequence

Before we start we need to set up some notation. We will use subscripts to denote different chain complexes, rather than the modules in a single chain complex, unless otherwise stated. That is, when we write Y_n, we mean a chain complex not the module in degree n of a chain complex. Likewise we will use superscripts to denote different chain complexes rather than the modules in a single cochain complex. That is, when we write Y^n, we mean a chain complex not the module in degree n of a cochain complex.

Definition 4.1.1. A *mod $\mathfrak{m}$ Adams resolution* for a chain complex Y is a diagram

$$\begin{tikzcd} Y = Y_0 \arrow[d, "\alpha_0"'] & Y_1 \arrow[l, "\beta_0"'] \arrow[d, "\alpha_1"'] & Y_2 \arrow[l, "\beta_1"'] \arrow[d, "\alpha_2"'] & \cdots \arrow[l, "\beta_2"'] \\ L_0 & L_1 & L_2 & \end{tikzcd}$$

where each $L_s = Y_s \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$, $\alpha_s{}^\star$ is onto and each

$$Y_{s+1} \xrightarrow{\beta_s} Y_s \xrightarrow{\alpha_s} L_s \longrightarrow Y_{s+1}[-1]$$

is a distinguished triangle.

Lemma 4.1.2. *Let Y be a chain complex. Then Y admits a mod $\mathfrak{m}$ Adams resolution.*

Proof. Let $Y_0 = Y$. Consider the canonical morphism

$$Y_0 \cong Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} R[0] \xrightarrow{\mathrm{id}\otimes\eta} Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] = L_0.$$

Let $\alpha_0 = \mathrm{id}\otimes\eta$. Form the following distinguished triangle

$$Y_1 \xrightarrow{\beta_0} Y_0 \xrightarrow{\alpha_0} L_0 \longrightarrow Y_1[-1].$$

Now we claim that $\alpha_0{}^\star$ is onto. Note that

$$\begin{aligned} H^n(L_0, \mathbb{K}) &= H^n(Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0], \mathbb{K}) \\ &= H^n(\mathrm{RHom}_R(Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0], \mathbb{K}[0])) \\ &= \mathrm{Hom}_{\mathbb{K}}(H_n(Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0], \mathbb{K}), \mathbb{K}). \end{aligned}$$

Let $P \longrightarrow \mathbb{K}[0]$ be a minimal projective resolution and $Q \longrightarrow Y_0$ be a minimal projective resolution. Then

$$\begin{aligned} Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] &\cong (Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \\ &\cong (Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} (P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]). \end{aligned}$$

since $Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ and $P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$. Note that the degree n part of the chain complex $Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ is $Q_n \otimes_R \mathbb{K}$ which is free over $\mathbb{K}$ and $d(Q_n \otimes_R \mathbb{K}) = 0$

since Q is a minimal projective resolution. Using the Künneth formula for complexes Theorem 1.2.23, we see that

$$\mathrm{Tor}_1^{\mathbb{K}}(H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]), H_j(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])) = 0$$

since $H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])$ is free over $\mathbb{K}$ for each i. It follows that

$$H_n((Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{\mathbb{K}}{\otimes}} (P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0])) \cong \bigoplus_{i=0}^{n} H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \otimes_{\mathbb{K}} H_{n-i}(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]).$$

Therefore,

$$\begin{aligned}
H^n(L_0, \mathbb{K}) &= H^n(Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0], \mathbb{K}) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(H_n(Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]), \mathbb{K}) \\
&= \mathrm{Hom}_{\mathbb{K}}(\bigoplus_{i=0}^{n} H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \otimes_{\mathbb{K}} H_{n-i}(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]), \mathbb{K}) \\
&= \bigoplus_{i=0}^{n} \mathrm{Hom}_{\mathbb{K}}(H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0], \mathbb{K}) \otimes_{\mathbb{K}} H_{n-i}(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]), \mathbb{K}) \\
&\cong \bigoplus_{i=0}^{n} \mathrm{Hom}_{\mathbb{K}}(H_i(Q \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]), \mathbb{K}) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathbb{K}}(H_{n-i}(P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0], \mathbb{K})) \\
&\cong \bigoplus_{i=0}^{n} H^i(Q, \mathbb{K}) \otimes_{\mathbb{K}} H^{n-i}(P, \mathbb{K}) \\
&\cong \bigoplus_{i=0}^{n} H^i(Y_0, \mathbb{K}) \otimes_{\mathbb{K}} H^{n-i}(\mathbb{K}[0], \mathbb{K}).
\end{aligned}$$

Thus, we deduce that

$$H^\star(L_0, \mathbb{K}) \cong H^\star(Y_0, \mathbb{K}) \otimes_{\mathbb{K}} \mathcal{A}^\star$$

where $\mathcal{A}^\star$ is the Steenrod algebra. Now it is clear that ${\alpha_0}^\star$ is onto. We can deduce that L_0 is a connective chain complex of finite type. From the following homology long exact sequence,

$$\cdots \longrightarrow H_n(Y_1) \longrightarrow H_n(Y_0) \longrightarrow H_n(L_0) \longrightarrow H_{n-1}(Y_1) \longrightarrow \cdots$$

we deduce that Y_1 is a connective chain complex of finite type. The lemma now follows by induction. $\square$

Now we give the main theorem of this chapter.

Theorem 4.1.3. *Let Y be a connective chain complex. Then there exists a spectral sequence $\{E_r, d_r\}$ with the following properties.*

(i) $d_r \colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$ *for all* r, s, t.

(ii) $E_2^{s,t} \cong \mathrm{Ext}_{\mathcal{A}^\star}^{s,t}(H^\star(Y,\mathbb{K}),\mathbb{K})$.

(iii) $E_{r+1}^{s,t} \subset E_r^{s,t}$ *for* $r > s$ *and* $\cap_{r>s} E_r^{s,t} = E_\infty^{s,t}$.

(iv) *There is a associated decreasing filtration*

$$H_{t-s}(Y) = F^0 H_{t-s}(Y) \supset F^1 H_{t-s}(Y) \supset \cdots \supset F^s H_{t-s}(Y) \supset \cdots .$$

Remark 4.1.4. The above construction of Adams spectral sequence is natural. That is, if $\alpha \colon Y \longrightarrow Y'$ is a morphism, then we have the following commutative diagram

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d, "\alpha_1"'] & Y_0 \arrow[r] \arrow[d, "\alpha_0"'] & Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[r] \arrow[d, "\alpha_0 \otimes \mathrm{id}_{\mathbb{K}}"'] & Y_1[-1] \arrow[d, "\alpha_1[-1]"'] \\
Y_1' \arrow[r] & Y_0' \arrow[r] & Y_0' \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[r] & Y_1'[-1]
\end{tikzcd}$$

since the middle square is commutative. By induction, we construct morphisms $\alpha_n \colon Y_n \longrightarrow Y_n'$ for $n \geq 0$. Therefore, we have a morphism of spectral sequences of Y and Y'. In particular, if $\alpha \colon Y \longrightarrow Y'$ is an isomorphism, then we get an isomorphism of the spectral sequences of Y and Y'. Consequently, we deduce that the filtration of $H_\star(Y)$ is independent of Adams resolution.

Remark 4.1.5. Theorem 4.1.3 does not say that the Adams spectral sequence converges. We need to have some conditions which guarantee the convergence. We will discuss this in detail after the proof of the theorem.

Proof. Consider the following distinguished triangle

$$Y_{s+1} \xrightarrow{\beta_s} Y_s \xrightarrow{\alpha_s} L_s \longrightarrow Y_{s+1}[-1].$$

Then we have the following homology long exact sequence

$$\cdots \longrightarrow H_{t-s}(Y_{s+1}) \xrightarrow{\beta_{s\star}} H_{t-s}(Y_s) \xrightarrow{\alpha_{s\star}} H_{t-s}(L_s) \longrightarrow \cdots$$

Define $D_1^{s,t} = H_{t-s}(Y_s)$ and $E_1^{s,t} = H_{t-s}(L_s)$. Thus, we have the following exact couple

$$\begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

where

$$i_1\colon D_1^{s+1,t+1} \longrightarrow D_1^{s,t}$$

$$j_1\colon D_1^{s,t} \longrightarrow E_1^{s,t}$$

and

$$k_1\colon E_1^{s,t} \longrightarrow D_1^{s+1,t}$$

This exact couple determines a spectral sequence $\{E_r, d_r\}$ where $E_{r+1} = H(E_r, d_r)$ and $d_r\colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$. Thus, we have proved (i). Now notice that we have the following short exact sequence

$$0 \longrightarrow H^\star(Y_{s+1}[-1], \mathbb{K}) \longrightarrow H^\star(L_s, \mathbb{K}) \xrightarrow{\alpha_s{}^\star} H^\star(Y_s, \mathbb{K}) \longrightarrow 0$$

since $\alpha_s{}^\star$ is onto for each s. Gluing these together, we get the following long exact sequence

$$\cdots \longrightarrow H^\star(L_2[-2], \mathbb{K}) \longrightarrow H^\star(L_1[-1], \mathbb{K}) \longrightarrow H^\star(L_0, \mathbb{K}) \longrightarrow H^\star(Y, \mathbb{K}) \longrightarrow 0.$$

which is a free $\mathcal{A}^\star$-resolution for $H^\star(Y, \mathbb{K})$ since each $H^\star(L_s[-s], \mathbb{K})$ is free $\mathcal{A}^\star$-module and the maps are $\mathcal{A}^\star$-maps. Thus, we have a resolution which is needed to identify E_2.

$$\begin{aligned} E_1^{s,t} &= H_{t-s}(L_s) \\ &= H_{t-s}(Y_s, \mathbb{K}) \\ &\cong \mathrm{Hom}_{\mathbb{K}}(H^{t-s}(Y_s, \mathbb{K}), \mathbb{K}) \\ &\cong \mathrm{Hom}_{\mathbb{K}}^{t-s}(H^\star(Y_s, \mathbb{K}), \mathbb{K}) \\ &\cong \mathrm{Hom}_{\mathcal{A}^\star}^{t-s}(H^\star(Y_s, \mathbb{K}) \bigotimes_{\mathbb{K}} \mathcal{A}^\star, \mathbb{K}) \\ &\cong \mathrm{Hom}_{\mathcal{A}^\star}^{t-s}(H^\star(L_s, \mathbb{K}), \mathbb{K}) \\ &\cong \mathrm{Hom}_{\mathcal{A}^\star}^{t}(H^\star(L_s[-s], \mathbb{K}), \mathbb{K}). \end{aligned}$$

The boundary d_1 is induced by the morphism

$$L_s \longrightarrow Y_{s+1} \longrightarrow L_{s+1}$$

where $L_s \longrightarrow Y_{s+1}$ has degree -1. Now we have the following commutative diagram, in which the vertical maps are induced by the morphisms $L_s \longrightarrow L_{s+1}$ of degree -1.

$$\begin{tikzcd}
H_{t-s+1}(L_{s-1}) \arrow[r, "\cong"] \arrow[d, "d_1"'] & \mathrm{Hom}^t_{\mathcal{A}^\star}(H^\star(L_{s-1}[-s+1],\mathbb{K}),\mathbb{K}) \arrow[d] \\
H_{t-s}(L_s) \arrow[r, "\cong"] \arrow[d, "d_1"'] & \mathrm{Hom}^t_{\mathcal{A}^\star}(H^\star(L_s[-s],\mathbb{K}),\mathbb{K}) \arrow[d] \\
H_{t-s-1}(L_{s+1}) \arrow[r, "\cong"] & \mathrm{Hom}^t_{\mathcal{A}^\star}(H^\star(L_{s+1}[-s-1],\mathbb{K}),\mathbb{K})
\end{tikzcd}$$

Therefore,

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}^\star}(H^\star(Y,\mathbb{K}),\mathbb{K}).$$

Hence, we have proved (ii). We have $E_2^{s,t} = 0$ if $s < 0$ and thus $E_r^{s,t} = 0$ if $s < 0$. Since d_r has bidegree $(r, r-1)$, no differential map into $E_r^{s,t}$ if $r > s$. Therefore, there is a monomorphism $E_{r+1}^{s,t} \longrightarrow E_r^{s,t}$ when $r > s$ and thus

$$\bigcap_{r>s} E_r^{s,t} = E_\infty^{s,t}.$$

Filter $H_{t-s}(Y)$ by letting

$$F^s H_{t-s}(Y) = \mathrm{Im}(H_{t-s}(Y_s) \longrightarrow H_{t-s}(Y)).$$

Then it is clear that we have the following decreasing filtration

$$H_{t-s}(Y) = F^0 H_{t-s}(Y) \supset F^1 H_{t-s}(Y) \supset \cdots \supset F^s H_{t-s}(Y) \supset \cdots .$$

Hence, we have proved the theorem. $\square$

Now we will discuss convergence. Note that using Lemma 1.4.8, we have for each r the following short exact sequence

$$0 \longrightarrow D^{s,\star}/\,\mathrm{Ker}(i^r\colon D^{s,\star} \longrightarrow D^{s-r,\star}) + iD^{s+1,\star} \xrightarrow{\bar{j}} E_{r+1}^{s,\star} \xrightarrow{\bar{k}}$$
$$\mathrm{Im}(i^r\colon D^{s+r+1,\star} \longrightarrow D^{s+1,\star}) \cap \mathrm{Ker}(i\colon D^{s+1,\star} \longrightarrow D^{s,\star}) \longrightarrow 0.$$

Now letting r go to infinity, we see that the left hand term stabilizes when $r = s$ since $i^s\colon D^{s,\star} \longrightarrow D^{0,\star}$. Therefore, we have the following short exact sequence

$$0 \longrightarrow D^{s,\star}/\operatorname{Ker}(i^s\colon D^{s,\star} \longrightarrow D^{0,\star}) + iD^{s+1,\star} \xrightarrow{\bar{j}} E_\infty^{s,\star} \xrightarrow{\bar{k}}$$
$$\bigcap_r \operatorname{Im}(i^r\colon D^{s+r+1,\star} \longrightarrow D^{s+1,\star}) \cap \operatorname{Ker}(i\colon D^{s+1,\star} \longrightarrow D^{s,\star}) \longrightarrow 0.$$

Lemma 4.1.6. *There are monomorphisms*

$$0 \longrightarrow F^s H_{t-s}(Y)/F^{s+1}H_{t-s}(Y) \longrightarrow E_\infty^{s,t}.$$

Proof. It suffices to show that

$$F^s H_{t-s}(Y)/F^{s+1}H_{t-s}(Y) \cong D^{s,\star}/\operatorname{Ker}(i^s\colon D^{s,\star} \longrightarrow D^{0,\star}) + iD^{s+1,\star}.$$

Note that $F^s H_{t-s}(Y) = i^s D^{s,t}$ and $F^{s+1}H_{t-s}(Y) = i(i^s D^{s+1,t+1})$. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} i^s + iD^{s+1,\star} \arrow[r] \arrow[d, "i^s"] & D^{s,\star} \arrow[r] \arrow[d, "i^s"] & D^{s,\star}/\operatorname{Ker} i^s + iD^{s+1,\star} \arrow[r] \arrow[d, "\bar{i^s}"] & 0 \\
0 \arrow[r] & i^{s+1}D^{s+1,\star} \arrow[r] & i^s D^{s,\star} \arrow[r] & F^s/F^{s+1} \arrow[r] & 0
\end{tikzcd}$$

in which each row is exact. We see that the middle vertical map i^s is epimorphism and thus $\bar{i^s}$ is epimorphism. Next we show that $\bar{i^s}$ is also a monomorphism. Let $\bar{a} = a + (\operatorname{Ker} i^s + iD^{s+1,\star})$ and $\bar{b} = b + (\operatorname{Ker} i^s + iD^{s+1,\star})$ be in $D^{s,\star}/\operatorname{Ker} i^s + iD^{s+1,\star}$. Assume that $\bar{i^s}(\bar{a}) = \bar{i^s}(\bar{b})$. We claim that $\bar{a} = \bar{b}$. We have that

$$i^s(a) + i^{s+1}D^{s+1,\star} = i^s(b) + i^{s+1}D^{s+1,\star}.$$

Then $i^s(a-b) \in i^{s+1}D^{s+1,\star}$. This implies that either $i^s(a-b) \in i^{s+1}D^{s+1,\star} = i^s(iD^{s+1,\star})$, that is, $a - b \in iD^{s+1,\star}$, or $i^s(a-b) = 0$, that is, $a - b \in \operatorname{Ker} i^s$. In both cases, we have that $\bar{a} = \bar{b}$. Hence, there are monomorphisms

$$0 \longrightarrow F^s H_{t-s}(Y)/F^{s+1}H_{t-s}(Y) \longrightarrow E_\infty^{s,t}. \qquad \square$$

Remark 4.1.7. Let

$$C^{s,t} = \operatorname{Coker}[F^s H_{t-s}(Y)/F^{s+1}H_{t-s}(Y) \longrightarrow E_\infty^{s,t}]$$

where $s \geq 0$ and $D^{t-s} = \bigcap_{s\geq 0} F^s H_{t-s}(Y)$. If the $C^{s,t} = 0$ for all s, t, then we can define a new filtration

$$H_{t-s}(Y)/D^{t-s} = \bar{F}^0 H_{t-s}(Y) \supset \bar{F}^1 H_{t-s}(Y) \supset \ldots \supset \bar{F}^s H_{t-s}(Y) \supset \ldots$$

with $\bar{F}^s H_{t-s}(Y) = F^s H_{t-s}(Y)/D^{t-s}$ for all $s \geq 0$. Then we would still have

$$\bar{F}^s H_{t-s}(Y)/\bar{F}^{s+1} H_{t-s}(Y) \cong F^s H_{t-s}(Y)/F^{s+1} H_{t-s}(Y) \cong E_\infty^{s,t}.$$

for all s, t but in addition $\cap \bar{F}^s H_{t-s}(Y) = 0$. Thus, if the $C^{s,t}$ vanish, we can say that the Adams spectral sequence converges to $H_\star(Y)/D^\star$.

Theorem 4.1.8. *If* $\operatorname{holim}_s Y_s = 0$*, then the Adams spectral sequence converges to* $H_\star(Y)$.

Proof. First note that $\lim^1{}_r E_r^{s,t} = \{0\}$ for all s and t since $E_2^{s,t}$ is finitely generated over $\mathbb{K}$ for each s and t. Hence, the Adams spectral sequence converges to $H_\star(Y)$ by Theorem 1.4.12. □

4.2 Homology Localization and Local Homology

In this section, we define localizations of chain complexes with respect to a homology theory and in particular we define the localization of chain complexes with respect to the homology theory $\mathbb{K}_\star(-) = H_\star(-, \mathbb{K})$. Then we define local homology of chain complexes.

A *homology theory* on the derived category $\mathcal{D}_{+(fg)}(R)$ is a functor $S_\star$ from $\mathcal{D}_{+(fg)}(R)$ to the category of graded R-modules determined by the recipe

$$S_\star(Y) = H_\star(S \overset{\mathrm{L}}{\underset{R}{\otimes}} Y)$$

for some object S in $\mathcal{D}_{+(fg)}(R)$. A chain complex Y is called $S_\star$-*acyclic* if $S_\star(Y) = 0$. A morphism $\alpha\colon X \longrightarrow Y$ is called an $S_\star$-*equivalence* if $\alpha_\star\colon S_\star(X) \cong S_\star(Y)$. We define a chain complex Z to be $S_\star$-*local* if each $S_\star$-equivalence $\alpha\colon X \longrightarrow Y$ induces a bijection

$$\alpha^\star\colon \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, Z)_\star \cong \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(X, Z)_\star$$

or equivalently if $\mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(X, Z)_\star = 0$ for each $S_\star$-acyclic chain complex X. An $S_\star$-localization of Y is an $S_\star$-equivalence $Y \longrightarrow Y_S$ with the property that Y_S is $S_\star$-local.

In [11], it was proved that the localization of a chain complex Y with respect to the homology theory $\mathbb{K}_\star$ is $Y_{\mathbb{K}} = \mathrm{RHom}_R(K, Y)$ where K is the chain complex constructed as follows. Let $r_1, \ldots, r_n$ be the generators of the maximal ideal $\mathfrak{m}$ and consider the following chain complexes

$$0 \longrightarrow R \longrightarrow R[1/r_i] \longrightarrow 0,$$

where R is in degree 0. Then

$$K = \bigotimes_i (0 \longrightarrow R \longrightarrow R[1/r_i] \longrightarrow 0).$$

Furthermore in [11], it was proved that K is isomorphic to a chain complex of free R-modules which is concentrated between dimensions $(-n)$ and 0.

Definition 4.2.1. Let M be an R-module. Then the *local homology* of M at $\mathfrak{m}$ is denoted $H_\star^{\mathfrak{m}}(M)$ and defined by the formula

$$H_n^{\mathfrak{m}}(M) = H_n(\mathrm{RHom}_R(K, M))$$

For the following definition see [11].

Definition 4.2.2. Let Y be an object of $\mathcal{D}(R)$. Then the *derived local homology* of Y at $\mathfrak{m}$ is

$$H^{\mathfrak{m}}(Y) = \mathrm{RHom}_R(K, Y)$$

There is a third quadrant spectral sequence

$$E_2^{s,t} = \bigoplus_{p+q=t} \mathrm{Ext}_R^s(H_p(K), H_q(Y)) \Longrightarrow H_{-t-s}(H^{\mathfrak{m}}(Y)).$$

Now we give some elementary properties of $\mathbb{K}_\star$-localizations whose proofs are straightforward.

Lemma 4.2.3. *If*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[-1]$$

is a distinguished triangle and any two of X, Y, Z are $\mathbb{K}_\star$-local, then so is the third.

Lemma 4.2.4. *A direct summand of a $\mathbb{K}_\star$-local chain complex is $\mathbb{K}_\star$-local.*

Lemma 4.2.5. *The product of a set of $\mathbb{K}_\star$-local chain complexes is $\mathbb{K}_\star$-local.*

Lemma 4.2.6. *If*

$$X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots$$

is a sequence of chain complexes such that X_0, X_1, X_2,... are $\mathbb{K}_\star$-local, then the homotopy inverse limit of this sequence is $\mathbb{K}_\star$-local.

Recall that it was proven in Section 3.4 that $\mathbb{K}[0]$ is a commutative monoid in $\mathcal{D}_{+(fg)}(R)$ with product

$$\phi\colon \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \longrightarrow \mathbb{K}[0]$$

and unit map

$$\eta\colon R[0] \longrightarrow \mathbb{K}[0].$$

Let Y be a chain complex. Consider the chain complex $Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$. Note that we have the following morphism

$$\phi_Y\colon (Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong Y \overset{\mathrm{L}}{\underset{R}{\otimes}} (\mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) \xrightarrow{\mathrm{id}\otimes\phi} Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$$

Then we can see that the following diagrams are commutative in $\mathcal{D}_{+(fg)}(R)$.

$$\begin{tikzcd}
Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[r, "\mathrm{id}\otimes\phi"] \arrow[d, "\phi_Y\otimes\mathrm{id}"'] & Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[d, "\phi_Y"] \\
Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[r, "\phi_Y"] & Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]
\end{tikzcd}$$

$$\begin{tikzcd}
Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} R[0] \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[d, "\cong"'] & Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[d, "\phi_Y"] \\
Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[r, "\mathrm{id}"] & Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]
\end{tikzcd}$$

Therefore, the chain complex $Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ is a module over $\mathbb{K}[0]$ in $\mathcal{D}_{+(fg)}(R)$.

Now we give the following result which is needed later.

Lemma 4.2.7. *For any chain complex Y, the chain complex $Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ is $\mathbb{K}_\star$-local.*

Proof. Let X be a chain complex such that $H_\star(X \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]) = 0$, that is, $X \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \cong 0$. Let $\alpha\colon X \longrightarrow Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ be a morphism. Then the morphism α can be factored as

$$\begin{tikzcd}
X \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[r, "\alpha\otimes\mathrm{id}"] & Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0] \arrow[d, "\mathrm{id}\otimes\phi"] \\
X \overset{\mathrm{L}}{\underset{R}{\otimes}} R[0] \cong X \arrow[u, "\mathrm{id}\otimes\eta"] \arrow[r, "\alpha"] & Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]
\end{tikzcd}$$

Therefore, α is trivial. Hence, $Y \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ is $\mathbb{K}_\star$-local. $\square$

Definition 4.2.8. The $\mathbb{K}[0]$-*nilpotent chain complexes* form the smallest class C of chain complexes in $\mathcal{D}_{+(fg)}(R)$ such that:

(i) $\mathbb{K}[0] \in C$,

(ii) If $X \in C$ and $Y \in \mathcal{D}_{+(fg)}(R)$, then $X \overset{\mathrm{L}}{\underset{R}{\otimes}} Y \in C$,

(iii) If

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[-1]$$

is a distinguished triangle and two of X, Y, Z are in C, then so the third,

(iv) If $Y \in C$ and X is a direct summand of Y, then $X \in C$.

We filter the class C as follows. Let C_0 consist of all chain complexes $Y \cong \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} X$ for some chain complex X and given C_{n-1} with $n-1 \geq 0$ let C_n consist of all chain complexes Y such that either Y is a direct summand of a member of C_{n-1} or there is a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[-1]$$

with $X, Z \in C_{n-1}$. Now we use this filtration to prove the following result.

Lemma 4.2.9. *If Y is $\mathbb{K}[0]$-nilpotent, then Y is $\mathbb{K}_\star$-local.*

Proof. We prove this lemma by induction using the above filtration. Let $Y \in C_0$. Then $Y \cong \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} X$ for some chain complex X. So Y is $\mathbb{K}_\star$-local by Lemma 4.2.7. Assume that every $\mathbb{K}[0]$-nilpotent chain complex in C_{n-1} is $\mathbb{K}_\star$-local. We claim that every $\mathbb{K}[0]$-nilpotent chain complex in C_n is $\mathbb{K}_\star$-local. Let $Y \in C_n$. Then Y is either a direct summand of a member of C_{n-1}, that is, a direct summand of $\mathbb{K}_\star$-local chain complex and thus Y is $\mathbb{K}_\star$-local by Lemma 4.2.4 or there is a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[-1]$$

with X, Z $\mathbb{K}_\star$-local in C_{n-1}. Therefore, Y is $\mathbb{K}_\star$-local by Lemma 4.2.3. $\square$

Definition 4.2.10. A $\mathbb{K}[0]$-*nilpotent resolution* of a chain complex Y is a tower $\{W_s\}_{s\geq 1}$ such that:

(i) W_s is $\mathbb{K}[0]$-nilpotent for each $s \geq 1$.

(ii) For each $\mathbb{K}[0]$-nilpotent chain complex N, the map

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(W_s, N)_\star \longrightarrow \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, N)_\star$$

is isomorphism.

4.3 $\mathbb{K}[0]$-Nilpotent Completion

In this section, we define $\mathbb{K}[0]$-nilpotent completion of a chain complex and show that the Adams spectral sequence for a chain complex Y converges strongly to the homology of the $\mathbb{K}[0]$-nilpotent completion of Y.

Note that there is no reason why $\operatorname{holim}_s Y_s = 0$. Following Bousfield [7], we define chain complexes Y^s by the following distinguished triangles

$$Y_s \longrightarrow Y_0 \longrightarrow Y^s \longrightarrow Y_s[-1].$$

Now we construct a morphism $Y^{s+1} \longrightarrow Y^s$. Using the octahedral axiom for the composite

$$Y_{s+1} \longrightarrow Y_s \longrightarrow Y_0,$$

we obtain the following commutative diagram.

$$\begin{tikzcd}
Y_{s+1} \arrow[r, "\beta_s"] \arrow[d, "\mathrm{id}"'] & Y_s \arrow[r, "\alpha_s"] \arrow[d] & L_s \arrow[r] \arrow[d] & Y_{s+1}[-1] \arrow[d, "\mathrm{id}"] \\
Y_{s+1} \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & Y^{s+1} \arrow[r] \arrow[d] & Y_{s+1}[-1] \arrow[d] \\
0 \arrow[r] \arrow[d] & Y^s \arrow[r, "\mathrm{id}"] \arrow[d] & Y^s \arrow[r] \arrow[d] & 0 \arrow[d] \\
Y_{s+1}[-1] \arrow[r] & Y_s[-1] \arrow[r] & L_s[-1] \arrow[r] & Y_{s+1}[-2]
\end{tikzcd}$$

in which each row and column is a distinguished triangle. In particular, the column

$$L_s \longrightarrow Y^{s+1} \longrightarrow Y^s \longrightarrow L_s[-1]$$

is a distinguished triangle for each $s \geq 1$. Also, we have the following commutative diagrams

$$\begin{tikzcd}
Y_{s+1} \arrow[r] \arrow[d, "\beta_s"'] & Y_0 \arrow[r] \arrow[d, "\mathrm{id}"'] & Y^{s+1} \arrow[r] \arrow[d] & Y_{s+1}[-1] \arrow[d] \\
Y_s \arrow[r] & Y_0 \arrow[r] & Y^s \arrow[r] & Y_s[-1]
\end{tikzcd}$$

and

$$\begin{tikzcd}
L_s \arrow[r] \arrow[d, "\mathrm{id}"'] & Y^{s+1} \arrow[r] \arrow[d] & Y^s \arrow[r] \arrow[d] & L_s[-1] \arrow[d] \\
L_s \arrow[r] & Y_{s+1}[-1] \arrow[r] & Y_s[-1] \arrow[r] & L_s[-1]
\end{tikzcd}$$

Moreover, we get the following commutative diagram

$$\begin{tikzcd}
Y \arrow[d] & Y \arrow[l, "\mathrm{id}"'] \arrow[d] & Y \arrow[l, "\mathrm{id}"'] \arrow[d] & \cdots \arrow[l, "\mathrm{id}"'] \\
Y^1 & Y^2 \arrow[l] & Y^3 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Therefore, we get the following tower

$$\begin{tikzcd}
Y^1[1] \arrow[d] & Y^2[1] \arrow[l] \arrow[d] & Y^3[1] \arrow[l] \arrow[d] & \cdots \arrow[l] \\
L_1 & L_2 & L_3 &
\end{tikzcd}$$

in which each triangle

$$Y^{s+1}[1] \longrightarrow Y^s[1] \longrightarrow L_s \longrightarrow Y^{s+1}$$

is a distinguished triangle. Now using the above tower, we can construct Adams spectral sequence as derived in the proof of Theorem 4.1.3. We note that the above structures are natural in Y.

Now let $\mathbb{K}^\wedge Y$ be the homotopy inverse limit of the tower $\{Y^s\}$. So there is a morphism $Y \longrightarrow \mathbb{K}^\wedge Y$. We call $Y \longrightarrow \mathbb{K}^\wedge Y$ the $\mathbb{K}[0]$-*nilpotent completion* of Y. Since $Y^0 = 0$, we deduce the Adams spectral sequence now conditionally converges to $H_\star(\mathbb{K}^\wedge Y)$. Filter $H_{t-s}(\mathbb{K}^\wedge Y)$ by

$$F^s H_{t-s}(\mathbb{K}^\wedge Y) = \operatorname{Ker}(H_{t-s}(\mathbb{K}^\wedge Y) \longrightarrow H_{t-s}(Y^s)).$$

Also, note that $E_{r+1}^{s,t} \subset E_r^{s,t}$ for $r > s$. Since $\lim^1{}_r E_r^{s,t} = \{0\}$ for all s and t, we have the main result of this section using Theorem 1.4.13.

Theorem 4.3.1. *The Adams spectral sequence converges strongly to $H_\star(\mathbb{K}^\wedge Y)$.*

Lemma 4.3.2. *Let Y be a chain complex. Then $\mathbb{K}^\wedge Y$ is $\mathbb{K}_\star$-local.*

Proof. We show that $\mathbb{K}^\wedge Y$ is $\mathbb{K}_\star$-local. It suffices to show that Y^s is $\mathbb{K}_\star$-local for each s using Lemma 4.2.6. We prove it by induction. For $s = 1$, we have $Y^1 \cong L_0 = Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$. But Y^1 is $\mathbb{K}_\star$-local by Lemma 4.2.7. Assume that Y^s is $\mathbb{K}_\star$-local. We prove that Y^{s+1} is $\mathbb{K}_\star$-local. We have the following distinguished triangle

$$L_s \longrightarrow Y^{s+1} \longrightarrow Y^s \longrightarrow L_s[-1].$$

But $L_s = Y_s \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ is $\mathbb{K}_\star$-local by Lemma 4.2.7 and Y^s is $\mathbb{K}_\star$-local by the assumption. Hence, Y^{s+1} is $\mathbb{K}_\star$-local by Lemma 4.2.3. Hence, $\mathbb{K}^\wedge Y$ is $\mathbb{K}_\star$-local. □

Therefore, there is a unique morphism $\beta\colon Y_{\mathbb{K}} \longrightarrow \mathbb{K}^\wedge Y$ such that the composite

$$Y \longrightarrow Y_{\mathbb{K}} \longrightarrow \mathbb{K}^\wedge Y$$

is $Y \longrightarrow \mathbb{K}^\wedge Y$.

Remark 4.3.3. We see that if $H_\star(Y, \mathbb{K}) \xrightarrow{\cong} H_\star(\mathbb{K}^\wedge Y, \mathbb{K})$, then $Y_{\mathbb{K}} \underset{\cong}{\xrightarrow{\beta}} \mathbb{K}^\wedge Y$ by the Derived Whitehead Theorem 3.2.5.

Lemma 4.3.4. *Let Y be a chain complex. Then the tower $\{Y^s\}_{s\geq 1}$ is a $\mathbb{K}[0]$-nilpotent resolution of Y.*

Proof. First we show that each Y^s is $\mathbb{K}[0]$-nilpotent by induction. When $s = 1$, $Y^1 \cong L_0 = Y_0 \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ is $\mathbb{K}[0]$-nilpotent. Assume that Y^s is $\mathbb{K}[0]$-nilpotent. Consider the following distinguished triangle

$$L_s \longrightarrow Y^{s+1} \longrightarrow Y^s \longrightarrow L_s[-1].$$

We see that Y^{s+1} is a $\mathbb{K}[0]$-nilpotent since $L_s = Y_s \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ and Y^s are $\mathbb{K}[0]$-nilpotent. Therefore, Y^s is a $\mathbb{K}[0]$-nilpotent chain complex for each $s \geq 1$ and hence (i) is satisfied. Next we show (ii) holds. We have the following distinguished triangle

$$Y_s \longrightarrow Y \longrightarrow Y^s \longrightarrow Y_s[-1]$$

for each $s \geq 1$. Let N be a $\mathbb{K}[0]$-nilpotent chain complex. Then by Remark 2.3.6, we have the following long exact sequence

$$\cdots \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_s, N)_{n-1} \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^s, N)_n$$
$$\longrightarrow \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, N)_n \longrightarrow \operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_s, N)_n \longrightarrow \cdots$$

Note that

$$\operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, N)_n = \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, N)_n.$$

It suffices to show that $\operatorname{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_s, N)_\star = 0$. We prove this by induction using the filtration of the class C of $\mathbb{K}[0]$-nilpotent chain complexes. If $N \in C_0$, then $N \cong \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} X$ for some chain complex X. So

$$\mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_s, \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} X)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_{s+1}, \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} X)_n$$

is trivial map for each n since any morphism $\varphi \colon Y_s \longrightarrow \mathbb{K}[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} X$ factors through L_s. Assume that $\mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_s, N)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_{s+1}, N)_n$ is zero for each $N \in C_{n-1}$. Now let $A \in C_n$ such that $W \cong A \oplus B$ where $W \in C_{n-1}$. Since A is a direct summand of W, $\mathrm{id} \colon A \longrightarrow A$ factors through W. Therefore, any morphism

$$\mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_s, A)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_{s+1}, A)_n$$

factors through W and so it is trivial since

$$\mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_s, W)_n \longrightarrow \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_{s+1}, W)_n$$

is trivial by the assumption. While if there is a distinguished triangle

$$X \xrightarrow{\varphi} N \longrightarrow Z \longrightarrow X[-1]$$

with $X, Z \in C_{n-1}$. Then we have the following commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^s, X)_n \arrow[r, "\cong"] \arrow[d] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, X)_n \arrow[d] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^s, N)_n \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, N)_n \arrow[d] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^s, Z)_n \arrow[r, "\cong"] \arrow[d] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, Z)_n \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

in which each column is exact. Using Five Lemma, we deduce that

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^s, N)_n \cong \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, N)_n.$$

for each n. Hence, the tower $\{Y^s\}_{s\geq 1}$ is a $\mathbb{K}[0]$-nilpotent resolution of Y. $\square$

Now we recall the definition of a pro-category and some related results needed later. Let $\mathcal{C}$ be a category. A pro-object (tower) in $\mathcal{C}$ is a sequence of objects $X_i \in \mathcal{C}$ for $i > 0$ together with maps $X_{i+1} \longrightarrow X_i$ for $i > 0$. We can think of a pro-object $X = \{X_i\}_{i\in\mathbb{Z}^+}$ as an inverse system of objects of $\mathcal{C}$. Pro-objects of $\mathcal{C}$ form a category Tower-$\mathcal{C}$ where

$$\operatorname{Hom}(X, Y) = \lim_j(\operatorname*{colim}_i \operatorname{Hom}(X_i, Y_j)).$$

There is a canonical functor

$$\mathcal{C} \longrightarrow \text{Tower-}\mathcal{C}$$

taking the object Y to the constant tower $\{Y\}$. In this way $\mathcal{C}$ becomes a full subcategory of Tower-$\mathcal{C}$.

The following lemma is proved in [2, App.3.2]

Lemma 4.3.5. *Let X and Y be pro-objects in $\mathcal{C}$. Then a morphism $f\colon X \longrightarrow Y$ can be represented up to isomorphism by an inverse system of maps $\{f_i\colon X_i \longrightarrow Y_i\}$. This representation is called level representation.*

Remark 4.3.6. A pro-isomorphism $f\colon X \longrightarrow Y$ between two pro-objects X, Y amounts to the following: for each s there exists a $t > s$ and a morphism $h_{ts}\colon Y_t \longrightarrow X_s$ such that the following diagram

$$\begin{tikzcd} X_t \arrow[r] \arrow[d] & Y_t \arrow[d] \arrow[dl] \\ X_s \arrow[r] & Y_s \end{tikzcd}$$

is commutative. In effect, the maps h_{ts} represent the inverse of f. See [17, Lemma 3.2].

Therefore, it follows that if $\{X_{j(i)}\}$ is a cofinal subtower of $\{X_i\}$, then $\{X_{j(i)}\} \cong \{X_i\}$ in Tower-$\mathcal{C}$.

Definition 4.3.7. A morphism $f\colon \{X_s\} \longrightarrow \{Y_s\}$ in Tower-$\mathcal{D}_{+(fg)}(R)$ is called a *q-isomorphism* if the induced morphism $f_\star\colon \{H_i X_s\} \longrightarrow \{H_i Y_s\}$ is a pro-isomorphism in Tower-R-**mod** for each i, where R-**mod** is the category of R-modules.

The following lemmas are analogous to [7, Lemma 5.10, Lemma 5.11, Proposition 5.8].

Lemma 4.3.8. *If $\{W_s\}$ is a $\mathbb{K}[0]$-nilpotent resolution of the chain complex Y in $\mathcal{D}_{+(fg)}(R)$, then there exists a unique pro-isomorphism $e\colon \{Y^s\} \longrightarrow \{W_s\}$ in Tower-$\mathcal{D}_{+(fg)}(R)$ such that*

$$\begin{tikzcd} \{Y\} \arrow[r, "\mathrm{id}"] \arrow[d] & \{Y\} \arrow[d] \\ \{Y^s\} \arrow[r, "e"] & \{W_s\} \end{tikzcd}$$

commutes.

Lemma 4.3.9. *Let $f\colon \{X_s\} \longrightarrow \{Y_s\}$ be a q-isomorphism in Tower-$\mathcal{D}_{+(fg)}(R)$. If X_∞, $Y_\infty \in \mathcal{D}_{+(fg)}(R)$ are homotopy limits of $\{X_s\}$, $\{Y_s\}$ respectively, then there* *exists an isomorphism* $u\colon X_\infty \cong Y_\infty$ *such that*

$$\begin{tikzcd} \{X_\infty\} \arrow[r, "\{u\}"] \arrow[d] & \{Y_\infty\} \arrow[d] \\ \{X_s\} \arrow[r, "f"] & \{Y_s\} \end{tikzcd}$$

commutes in Tower-$\mathcal{D}_{+(fg)}(R)$.

Lemma 4.3.10. *Let* Y *be in* $\mathcal{D}_{+(fg)}(R)$*. Let* $\{X_s\}$ *be a* $\mathbb{K}[0]$*-nilpotent resolution of* Y *with homotopy limit* X_∞*. Then* $X_\infty \cong \mathbb{K}^\wedge Y$.

4.4 Convergence

In this section, we study the convergence of the Adams spectral sequence. We are still assuming R is a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{K}$.

We start by recalling some important results needed to prove the main theorem of this section.

It is known that $\mathfrak{m} = \mathfrak{m}^1 \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$. If N is an R-module, then $N \supset \mathfrak{m}N \supset \mathfrak{m}^2 N \supset \cdots$. For each $i \leq j$, there is a natural R-linear map $\phi_i^j\colon N/\mathfrak{m}^j N \longrightarrow N/\mathfrak{m}^i N$. The family of quotient modules $N/\mathfrak{m}^i N$ and maps ϕ_i^j for $i \leq j$ is an inverse system indexed by the positive integers. Recall the $\mathfrak{m}$-*adic completion* of N, denoted $\hat{N}$, is $\lim_i N/\mathfrak{m}^i N$.

Let $L_n^{\mathfrak{m}}$ denote the nth left derived functor of the $\mathfrak{m}$-adic completion. Then we have the following result which is proved in [14, Proposition 1.1].

Theorem 4.4.1. *There are short exact sequences*

$$0 \longrightarrow \lim_s{}^1 \operatorname{Tor}_{n+1}^R(R/\mathfrak{m}^s, N) \longrightarrow L_n^{\mathfrak{m}}(N) \longrightarrow \lim_s \operatorname{Tor}_n^R(R/\mathfrak{m}^s, N) \longrightarrow 0.$$

The following result is important and is proved in [14, Proposition 1.5].

Theorem 4.4.2. *If* N *is a finitely generated* R*-module, then*

(i) $L_0^{\mathfrak{m}}(N) \cong \hat{N}$.

(ii) The tower $\{\operatorname{Tor}_n^R(R/\mathfrak{m}^s, N)\}$ *is pro-zero, that is,* $L_n^{\mathfrak{m}}(N) = 0$ *for* $n > 0$.

In Theorem 4.4.2, if $N = R/\mathfrak{m}$, then we have the following result.

Corollary 4.4.3.

$$L_0^{\mathfrak{m}}(R/\mathfrak{m}) \cong R/\mathfrak{m},$$

and for $n > 0$*,*

$$\lim_s \operatorname{Tor}_n^R(R/\mathfrak{m}^s, R/\mathfrak{m}) = 0.$$

Moreover, for $n > 0$*,*

$$\operatorname*{colim}_s \operatorname{Ext}_R^n(R/\mathfrak{m}^s, R/\mathfrak{m}) = 0.$$

We end this review by giving the following important theorem.

Theorem 4.4.4. *Let* F *be an inverse system of* R*-modules for which* $\lim^s F = 0$ *if* $s > 0$*. If* N *is an* R*-module which admits a resolution by finitely generated free modules, then there is a second quadrant spectral sequence*

$$E^2_{s,t} = \lim^{(\text{-s})} \operatorname{Tor}_t^R(N, F) \Longrightarrow \operatorname{Tor}_{s+t}^R(N, \lim F).$$

Now consider the chain complex $R[0]$. We have the following tower, called $\mathfrak{m}$ *adic tower* in [4],

$$\cdots \longrightarrow R/\mathfrak{m}^3[0] \longrightarrow R/\mathfrak{m}^2[0] \longrightarrow R/\mathfrak{m}[0].$$

induced by the tower

$$\cdots \longrightarrow R/\mathfrak{m}^3 \longrightarrow R/\mathfrak{m}^2 \longrightarrow R/\mathfrak{m}.$$

Lemma 4.4.5. *Let*

$$\cdots \xrightarrow{f_4} X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

be a tower of chain complexes in $\mathbf{Ch}(R)$ *such that* $\lim^1 X_i = 0$*. If we consider the image of this tower in* $\mathcal{D}(R)$*, then* $\lim X_i \cong \operatorname{holim} X_i$*.*

Proof. First note that we have the following short exact sequence

$$0 \longrightarrow \lim X_i \longrightarrow \prod X_i \xrightarrow{\operatorname{id} - f} \prod X_i \longrightarrow 0.$$

Since every short exact sequence in $\mathbf{Ch}(R)$ gives rise to a distinguished triangle in $\mathcal{D}(R)$, we have the following distinguished triangle

$$\lim X_i \longrightarrow \prod X_i \xrightarrow{\mathrm{id}-f} \prod X_i \longrightarrow \lim X_i[-1].$$

Now consider the image of this tower in $\mathcal{D}(R)$. Then we have the following distinguished triangle

$$\operatorname{holim} X_i \longrightarrow \prod X_i \xrightarrow{\mathrm{id}-f} \prod X_i \longrightarrow \operatorname{holim} X_i[-1].$$

Note that we have the following commutative diagram in $\mathcal{D}(R)$.

$$\begin{tikzcd}
\lim X_i \arrow[r] \arrow[d, "\phi"'] & \prod X_i \arrow[r, "\mathrm{id}-f"] \arrow[d, "\mathrm{id}"'] & \prod X_i \arrow[r] \arrow[d, "\mathrm{id}"'] & \lim X_i[-1] \arrow[d, "\phi[-1]"'] \\
\operatorname{holim} X_i \arrow[r] & \prod X_i \arrow[r, "\mathrm{id}-f"] & \prod X_i \arrow[r] & \operatorname{holim} X_i[-1]
\end{tikzcd}$$

where the morphism ϕ exists since the middle square commutes. ϕ is an isomorphism by the Five Lemma. $\square$

The first substantial result of this section is the following.

Lemma 4.4.6. *The tower $\{R/\mathfrak{m}^s[0]\}_{s\geq 1}$ is a $\mathbb{K}[0]$-nilpotent resolution of the chain complex $R[0]$.*

Proof. We verify (i) and (ii) of Definition 4.2.10. First we verify (i) using induction. $R/\mathfrak{m}[0]$ is $\mathbb{K}[0]$-nilpotent. Assume that $R/\mathfrak{m}^s[0]$ is $\mathbb{K}[0]$-nilpotent. We claim $R/\mathfrak{m}^{s+1}[0]$ is $\mathbb{K}[0]$-nilpotent. We have the following distinguished triangle

$$\mathfrak{m}^s/\mathfrak{m}^{s+1}[0] \longrightarrow R/\mathfrak{m}^{s+1}[0] \longrightarrow R/\mathfrak{m}^s[0] \longrightarrow \mathfrak{m}^s/\mathfrak{m}^{s+1}[-1].$$

But $\mathfrak{m}^s/\mathfrak{m}^{s+1}[0]$ is a $\mathbb{K}$-module by Lemma 3.1.10. So $\mathfrak{m}^s/\mathfrak{m}^{s+1}[0] \cong \oplus R[0] \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$. Hence, it is $\mathbb{K}[0]$-nilpotent. Therefore, $R/\mathfrak{m}^{s+1}[0]$ is $\mathbb{K}[0]$-nilpotent.

Next we verify (ii). That is, for each $\mathbb{K}[0]$-nilpotent chain complex N, we show that for each n,

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N)_n \longrightarrow \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N)_n$$

First if $N = \mathbb{K}[0]$, then using Corollary 4.4.3,

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N)_n = \begin{cases} \mathbb{K} & n = 0, \\ 0 & n \neq 0. \end{cases}$$

But

$$\operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N)_n = \begin{cases} \mathbb{K} & n = 0, \\ 0 & n \neq 0. \end{cases}$$

Hence, (ii) holds for $\mathbb{K}[0]$. Assume inductively that for each n,

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], \oplus_{i=0}^{m-1}\mathbb{K}[0])_n \cong \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], \oplus_{i=0}^{m-1}\mathbb{K}[0])_n$$

We show (ii) holds for $\oplus_{i=0}^{m}\mathbb{K}[0]$. Note that we have the following commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], \mathbb{K}[0])_n \arrow[r, "\cong"] \arrow[d] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], \mathbb{K}[0])_n \arrow[d] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], \oplus_{i=0}^{m}\mathbb{K}[0])_n \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], \oplus_{i=0}^{m}\mathbb{K}[0])_n \arrow[d] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], \oplus_{i=0}^{m-1}\mathbb{K}[0])_n \arrow[r, "\cong"] \arrow[d] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], \oplus_{i=0}^{m-1}\mathbb{K}[0])_n \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

in which each column is exact. Using the Five Lemma, we deduce that for each n,

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], \oplus_{i=0}^{m}\mathbb{K}[0])_n \cong \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], \oplus_{i=0}^{m}\mathbb{K}[0])_n.$$

Now let $N \in C_0$. Then $N \cong X \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ for some $X \in \mathcal{D}_{+(fg)}(R)$. Let $P \longrightarrow X$ be a minimal projective resolution. Then $N \cong P \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathbb{K}[0]$ with 0 differential such that the degree i part is a finitely generated $\mathbb{K}$-vector space. We use induction to show that for each n,

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N)_n \cong \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N)_n.$$

It is clear that for each n,

$$\operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], \oplus\mathbb{K}[-m])_n \cong \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], \oplus\mathbb{K}[-m])_n.$$

Assume inductively that for each n,

$$\operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N^{[m-1]})_n \cong \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N^{[m-1]})_n.$$

Then we have the following commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N^{[m-1]})_n \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N^{[m-1]})_n \arrow[d] \\
\mathrm{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N^{[m]})_n \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N^{[m]})_n \arrow[d] \\
\mathrm{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], \oplus\mathbb{K}[-m])_n \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], \oplus\mathbb{K}[-m])_n \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

in which each column is exact. Using the Five Lemma, we deduce that

$$\operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N^{[m]})_n \cong \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N^{[m]})_n.$$

Hence, for each n,

$$\operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N)_n \cong \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N)_n.$$

Assume inductively that for each n,

$$\operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N)_n \cong \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N)_n.$$

for each $N \in C_{i-1}$.

Now let $N \in C_i$. Suppose first that $W = N \oplus B$ where $W \in C_{i-1}$. In this case, $e\colon W \longrightarrow W$ is idempotent where

$$e\colon W \longrightarrow N \longrightarrow W.$$

Then using Proposition 2.2.25,

$$N \cong \operatorname{hocolim}(W \xrightarrow{e} W \xrightarrow{e} W \xrightarrow{e} \cdots).$$

Therefore, we have the following commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], W)_n \arrow[r, "\cong"] \arrow[d, "e_\star"'] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], W)_n \arrow[d, "e_\star"'] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], W)_n \arrow[r] \arrow[d, "e_\star"'] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], W)_n \arrow[d, "e_\star"'] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], W)_n \arrow[r, "\cong"] \arrow[d, "e_\star"'] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], W)_n \arrow[d, "e_\star"'] \\
\vdots & \vdots
\end{tikzcd}$$

in which $e_\star$ is idempotent. It follows that the colimits of the two sequences are the same, that is, for each n,

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N)_n \cong \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N)_n.$$

Otherwise there is a distinguished triangle

$$X \longrightarrow N \longrightarrow Y \longrightarrow X[-1]$$

with X and Y are in C_{i-1}. In this case, we have the following diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], X)_n \arrow[r, "\cong"] \arrow[d] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], X)_n \arrow[d] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N)_n \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N)_n \arrow[d] \\
\operatorname{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], Y)_n \arrow[r, "\cong"] \arrow[d] & \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], Y)_n \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

in which each column is exact. Using the Five Lemma, we deduce that for each n,

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R/\mathfrak{m}^s[0], N)_n \cong \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(R[0], N)_n.$$

Hence, (ii) is satisfied. $\square$

Since the $\mathfrak{m}$-adic tower $\{R/\mathfrak{m}^s[0]\}$ is $\mathbb{K}[0]$-nilpotent resolution of $R[0]$, then using Lemma 4.3.8, we have the following result.

Proposition 4.4.7. *There exists a unique pro-isomorphism* $\{R[0]^s\} \longrightarrow \{R/\mathfrak{m}^s[0]\}$ *in Tower-*$\mathcal{D}_{+(fg)}(R)$ *such that*

$$\begin{tikzcd} \{R[0]\} \arrow[r, "\mathrm{id}"] \arrow[d] & \{R[0]\} \arrow[d] \\ \{R[0]^s\} \arrow[r] & \{R/\mathfrak{m}^s[0]\} \end{tikzcd}$$

commutes.

Using Lemma 4.3.10, we have the following important result.

Theorem 4.4.8. *The* $\mathfrak{m}$*-adic tower* $\{R/\mathfrak{m}^s[0]\}$ *has homotopy limit*

$$\operatorname*{holim}_s R/\mathfrak{m}^s[0] \cong \mathbb{K}^\wedge R[0].$$

Now we give the main theorem of this section.

Theorem 4.4.9. *The natural map* $R[0] \longrightarrow \mathbb{K}^\wedge R[0]$ *induces an isomorphism*

$$H_\star(R[0], \mathbb{K}) \cong H_\star(\mathbb{K}^\wedge R[0], \mathbb{K})$$

and therefore

$$\mathrm{RHom}_R(K, R[0]) \cong \mathbb{K}^\wedge R[0].$$

Proof. It suffices to show that

$$H_\star(R[0], \mathbb{K}) \longrightarrow H_\star(\operatorname*{holim}_s R/\mathfrak{m}^s[0], \mathbb{K})$$

is an isomorphism. Using Theorem 4.4.4, we see that the spectral sequence collapses to give

$$\begin{aligned} \lim_s \mathrm{Tor}^R_n(R/\mathfrak{m}^s[0], \mathbb{K}[0]) &\cong \mathrm{Tor}^R_n(\lim_s R/\mathfrak{m}^s[0], \mathbb{K}[0]) \\ &\cong \mathrm{Tor}^R_n(\operatorname*{holim}_s R/\mathfrak{m}^s[0], \mathbb{K}[0]). \end{aligned}$$

Using Corollary 4.4.3, we have that

$$\lim_s \mathrm{Tor}^R_n(R/\mathfrak{m}^s[0], \mathbb{K}[0]) = \begin{cases} \mathbb{K} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Hence,

$$H_i(R[0], \mathbb{K}) \longrightarrow H_i(\operatorname*{holim}_s R/\mathfrak{m}^s[0], \mathbb{K})$$

is an isomorphism for each i. Therefore,

$$H_\star(R[0], \mathbb{K}) \cong H_\star(\mathbb{K}^\wedge R[0], \mathbb{K})$$

Using Remark 4.3.3, we have that

$$\mathrm{RHom}_R(K, R[0]) \cong \mathbb{K}^\wedge R[0]. \qquad \square$$

We can generalize the previous results for the chain complex $\oplus_{i=0}^n R[0]$.

Lemma 4.4.10. *The tower $\{\oplus_{i=0}^n R/\mathfrak{m}^s[0]\}_{s\geq 1}$ is a $\mathbb{K}[0]$-nilpotent resolution of the chain complex $\oplus_{i=0}^n R[0]$.*

Proof. It is the same proof as the proof of Lemma 4.4.6. $\square$

Then we have the following result.

Proposition 4.4.11. *There exists a unique pro-isomorphism*

$$\{\oplus_{i=0}^n R[0]^s\} \longrightarrow \{\oplus_{i=0}^n R/\mathfrak{m}^s[0]\}$$

in Tower-$\mathcal{D}_{+(fg)}(R)$ such that

$$\begin{array}{ccc} \{\oplus_{i=0}^n R[0]\} & \xrightarrow{\mathrm{id}} & \{\oplus_{i=0}^n R[0]\} \\ \downarrow & & \downarrow \\ \{\oplus_{i=0}^n R[0]^s\} & \longrightarrow & \{\oplus_{i=0}^n R/\mathfrak{m}^s[0]\} \end{array}$$

commutes.

Theorem 4.4.12. *The tower $\{\oplus_{i=0}^n R/\mathfrak{m}^s[0]\}$ has homotopy limit*

$$\operatorname*{holim}_s \oplus_{i=0}^n R/\mathfrak{m}^s[0] \cong \mathbb{K}^\wedge \oplus_{i=0}^n R[0].$$

Theorem 4.4.13. *The natural map $\oplus_{i=0}^n R[0] \longrightarrow \mathbb{K}^\wedge \oplus_{i=0}^n R[0]$ induces an isomorphism*

$$H_\star(\oplus_{i=0}^n R[0], \mathbb{K}) \cong H_\star(\mathbb{K}^\wedge \oplus_{i=0}^n R[0], \mathbb{K})$$

and therefore

$$\mathrm{RHom}_R(K, \oplus_{i=0}^n R[0]) \cong \mathbb{K}^\wedge \oplus_{i=0}^n R[0].$$

Proof. It is the same proof as the proof of Theorem 4.4.9. □

Now let Y be a bounded chain complex consisting of finitely generated free R-modules in each degree. Then we can generalize Theorem 4.4.13 by induction on the skeletons of Y as follows.

First we prove an analogous result to Lemma 4.4.10. In the following Lemma, there is an exception to our convention. That is, Y_i means the degree i part of the chain complex Y.

Lemma 4.4.14. *The tower $\{Y \otimes_R R/\mathfrak{m}^s[0]\}_{s\geq 1}$ is a $\mathbb{K}[0]$-nilpotent resolution of the chain complex Y.*

Proof. We prove this lemma by induction on s. When $s = 1$, we see that the chain complex $Y \otimes_R R/\mathfrak{m}[0]$ is $\mathbb{K}[0]$-nilpotent. Assume inductively that $Y \otimes_R R/\mathfrak{m}^s[0]$ is $\mathbb{K}[0]$-nilpotent. We show that $Y \otimes_R R/\mathfrak{m}^{s+1}[0]$ is $\mathbb{K}[0]$-nilpotent. Now we induct on the skeletons of Y. We can show that $Y^{[0]} \otimes_R R/\mathfrak{m}^{s+1}[0]$ is $\mathbb{K}[0]$-nilpotent by induction as in the proof of Lemma 4.4.6. Assume that $Y^{[i]} \otimes_R R/\mathfrak{m}^{s+1}[0]$ is $\mathbb{K}[0]$-nilpotent. We show that $Y^{[i+1]} \otimes_R R/\mathfrak{m}^{s+1}[0]$ is $\mathbb{K}[0]$-nilpotent. We have the following distinguished triangle

$$\begin{aligned} Y^{[i]} \otimes_R R/\mathfrak{m}^{s+1}[0] \to Y^{[i+1]} \otimes_R R/\mathfrak{m}^{s+1}[0] \to Y_{i+1}[-i-1] \otimes_R R/\mathfrak{m}^{s+1}[0] \\ \to Y^{[i]}[-1] \otimes_R R/\mathfrak{m}^{s+1}[0]. \end{aligned}$$

Therefore, $Y^{[i+1]} \otimes_R R/\mathfrak{m}^{s+1}[0]$ is $\mathbb{K}[0]$-nilpotent. Hence, $Y \otimes_R R/\mathfrak{m}^{s+1}[0]$ is $\mathbb{K}[0]$-nilpotent.

Next we show that

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y \otimes_R R/\mathfrak{m}^s[0], N)_\star \longrightarrow \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, N)_\star$$

is an isomorphism for each $\mathbb{K}[0]$-nilpotent chain complex N. We induct on the skeletons of Y. First note that

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[0]} \otimes_R R/\mathfrak{m}^s[0], N)_\star \longrightarrow \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[0]}, N)_\star$$

is an isomorphism by Lemma 4.4.10. Assume inductively that

$$\operatorname*{colim}_s \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i]} \otimes_R R/\mathfrak{m}^s[0], N)_\star \longrightarrow \operatorname{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i]}, N)_\star$$

is an isomorphism. We show that

$$\operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i+1]} \otimes_R R/\mathfrak{m}^s[0], N)_\star \longrightarrow \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i+1]}, N)_\star$$

is an isomorphism. We have the following commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\operatorname{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_{i+1}[-i-1] \otimes_R R/\mathfrak{m}^s[0], N)_n \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y_{i+1}[-i-1], N)_n \arrow[d] \\
\operatorname{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i+1]} \otimes_R R/\mathfrak{m}^s[0], N)_n \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i+1]}, N)_n \arrow[d] \\
\operatorname{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i]} \otimes_R R/\mathfrak{m}^s[0], N)_n \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i]}, N)_n \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

in which each column is exact. Using the Five Lemma, we deduce that for each n,

$$\operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i+1]} \otimes_R R/\mathfrak{m}^s[0], N)_n \cong \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y^{[i+1]}, N)_n.$$

Therefore,

$$\operatorname*{colim}_s \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y \otimes_R R/\mathfrak{m}^s[0], N)_\star \cong \mathrm{Hom}_{\mathcal{D}_{+(fg)}(R)}(Y, N)_\star.$$

Hence, $\{Y \otimes_R R/\mathfrak{m}^s[0]\}_{s\geq 1}$ is $\mathbb{K}[0]$-nilpotent resolution for Y. □

Then the following result is similar to Proposition 4.4.11.

Proposition 4.4.15. *There exists a unique pro-isomorphism*

$$\{Y^s\} \longrightarrow \{Y \otimes_R R/\mathfrak{m}^s[0]\}$$

in Tower-$\mathcal{D}_{+(fg)}(R)$ such that

$$\begin{tikzcd}
\{Y\} \arrow[r, "\mathrm{id}"] \arrow[d] & \{Y\} \arrow[d] \\
\{Y^s\} \arrow[r] & \{Y \otimes_R R/\mathfrak{m}^s[0]\}
\end{tikzcd}$$

commutes.

Also, the following result is similar to Theorem 4.4.12 .

Theorem 4.4.16. *The tower* $\{Y \otimes_R R/\mathfrak{m}^s[0]\}$ *has homotopy limit*

$$\operatorname*{holim}_s Y \otimes_R R/\mathfrak{m}^s[0] \cong \mathbb{K}^\wedge Y.$$

Now we give our main result of this chapter. Note that in the following Theorem, there is an exception to our convention. That is, Y_i means the degree i part of the chain complex Y.

Theorem 4.4.17. *The natural map* $Y \longrightarrow \mathbb{K}^\wedge Y$ *induces an isomorphism*

$$H_\star(Y, \mathbb{K}) \cong H_\star(\mathbb{K}^\wedge Y, \mathbb{K})$$

and therefore

$$\mathrm{RHom}_R(K, Y) \cong \mathbb{K}^\wedge Y.$$

Proof. We induct on the skeletons of Y. By Theorem 4.4.13, we have that

$$H_\star(Y^{[0]}, \mathbb{K}) \cong H_\star(\mathbb{K}^\wedge Y^{[0]}, \mathbb{K}).$$

Assume inductively that

$$H_\star(Y^{[i]}, \mathbb{K}) \cong H_\star(\mathbb{K}^\wedge Y^{[i]}, \mathbb{K}).$$

We claim that

$$H_\star(Y^{[i+1]}, \mathbb{K}) \cong H_\star(\mathbb{K}^\wedge Y^{[i+1]}, \mathbb{K}).$$

The following degreewise short exact sequence of inverse systems of chain complexes

$$0 \to \{Y^{[i]} \otimes_R R/\mathfrak{m}^s[0]\} \to \{Y^{[i+1]} \otimes_R R/\mathfrak{m}^s[0]\} \to \{Y_{i+1}[-i-1] \otimes_R R/\mathfrak{m}^s[0]\} \to 0$$

gives rise to the following short exact sequence of chain complexes

$$0 \to \lim_s Y^{[i]} \otimes_R R/\mathfrak{m}^s[0] \to \lim_s Y^{[i+1]} \otimes_R R/\mathfrak{m}^s[0] \to \lim_s Y_{i+1}[-i-1] \otimes_R R/\mathfrak{m}^s[0] \to 0.$$

since

$$\lim_s{}^1\{Y^{[i]} \otimes_R R/\mathfrak{m}^s[0]\} = 0.$$

Note that in $\mathcal{D}_{+(fg)}(R)$, we have the following commutative diagram

$$\begin{tikzcd}
Y^{[i]} \arrow[r] \arrow[d] & \lim_s Y^{[i]} \otimes_R R/\mathfrak{m}^s[0] \arrow[d] \\
Y^{[i+1]} \arrow[r] \arrow[d] & \lim_s Y^{[i+1]} \otimes_R R/\mathfrak{m}^s[0] \arrow[d] \\
Y_{i+1}[-i-1] \arrow[r] \arrow[d] & \lim_s Y_{i+1}[-i-1] \otimes_R R/\mathfrak{m}^s[0] \arrow[d] \\
Y^{[i]}[-1] \arrow[r] & \lim_s Y^{[i]} \otimes_R R/\mathfrak{m}^s[0][-1]
\end{tikzcd}$$

in which each column is a distinguished triangle. Then we have the following commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H_n(Y^{[i]}, \mathbb{K}) \arrow[r, "\cong"] \arrow[d] & H_n(\lim_s Y^{[i]} \otimes_R R/\mathfrak{m}^s[0], \mathbb{K}) \arrow[d] \\
H_n(Y^{[i+1]}, \mathbb{K}) \arrow[r] \arrow[d] & H_n(\lim_s Y^{[i+1]} \otimes_R R/\mathfrak{m}^s[0], \mathbb{K}) \arrow[d] \\
H_n(Y_{i+1}[-i-1], \mathbb{K}) \arrow[r, "\cong"] \arrow[d] & H_n(\lim_s Y_{i+1}[-i-1] \otimes_R R/\mathfrak{m}^s[0], \mathbb{K}) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

in which each column is a long exact sequence. Using the Five Lemma, we see that

$$H_n(Y^{[i+1]}, \mathbb{K}) \longrightarrow H_n(\lim_s Y^{[i+1]} \otimes_R R/\mathfrak{m}^s[0], \mathbb{K})$$

is an isomorphism for each n. But

$$H_n(\lim_s Y^{[i+1]} \otimes_R R/\mathfrak{m}^s[0], \mathbb{K}) \cong H_n(\operatorname{holim}_s Y^{[i+1]} \overset{\mathrm{L}}{\underset{R}{\otimes}} R/\mathfrak{m}^s[0], \mathbb{K}).$$

is an isomorphism for each n by Lemma 4.4.5. Therefore,

$$H_\star(Y^{[i+1]}, \mathbb{K}) \cong H_\star(\mathbb{K}^\wedge Y^{[i+1]}, \mathbb{K}).$$

Hence,

$$\mathrm{RHom}_R(K, Y) \cong \mathbb{K}^\wedge Y.$$

□

4.5 Examples

In this section, we present some examples.

Example 4.5.1. Let F be an arbitrary field. Let $R = F[X]_{(X)}$ be the localization of the polynomial algebra $F[X]$ at the maximal ideal (X). Note that R is a local noetherian ring with maximal ideal $\mathfrak{m} = (X)R$ and residue field $\mathbb{K} \cong F$. By Proposition 3.1.9, we have that $\mathcal{A}^\star = E_F(e)$ where $|e| = 1$. Consider the chain complex $R[0]$. We have that $H^\star(R[0], F) \cong F$. We can deduce that the following sequence

$$\cdots \xrightarrow{d_{n+1}} u^n\mathcal{A}^\star \xrightarrow{d_n} \cdots \xrightarrow{d_2} u\mathcal{A}^\star \xrightarrow{d_1} \mathcal{A}^\star$$

is an $\mathcal{A}^\star$-free minimal resolution of F, where $u^n\mathcal{A}^\star$ is the free $\mathcal{A}^\star$-module on one generator u^n of degree n with $d_n(u^n) = u^{n-1}e$. Therefore,

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}^\star}(F,F) \cong \begin{cases} F & s = t, \\ 0 & s \neq t. \end{cases}$$

Note that we have the following commutative diagram

$$\begin{tikzcd}
0 & F \arrow[l] & \mathcal{A}^\star \arrow[l] & u\mathcal{A}^\star \arrow[l, "d_1"'] \arrow[dl, "\epsilon"'] \arrow[d, "\mathrm{id}"] & u^2\mathcal{A}^\star \arrow[l, "d_2"'] \arrow[d, "\mathrm{id}"] & \cdots \arrow[l] \\
& 0 & F[-1] \arrow[l] & \mathcal{A}^\star[-1] \arrow[l, "\epsilon[-1]"'] & u\mathcal{A}^\star[-1] \arrow[l, "d_1[-1]"'] \arrow[d, "\epsilon[-1]"'] & \cdots \arrow[l] \\
& & & & F[-2] &
\end{tikzcd}$$

where $\epsilon \in \mathrm{Hom}^0_{\mathcal{A}^\star}(u\mathcal{A}^\star, F[-1])$. We can see that

$$0 \neq \epsilon^2 = \epsilon[-1]\,\mathrm{id} \in \mathrm{Hom}^0_{\mathcal{A}^\star}(u^2\mathcal{A}^\star, F[-2]).$$

Similarly, we can deduce that $0 \neq \epsilon^n \in \mathrm{Hom}^0_{\mathcal{A}^\star}(u^n\mathcal{A}^\star, F[-n])$. The spectral sequence collapses and thus

$$E_\infty^{s,t} \cong \begin{cases} F & s = t, \\ 0 & s \neq t. \end{cases}$$

Now we show that ϵ in $\mathrm{Ext}^{1,1}_{\mathcal{A}^\star}(F,F)$ detects the map $X\colon R \longrightarrow R$. Note that the following sequence

$$R[0] \xrightarrow{X} R[0] \longrightarrow \mathrm{cone}(X) \longrightarrow R[-1]$$

is a distinguished triangle. We can deduce that

$$H^i(\mathrm{cone}(X), F) \cong \begin{cases} F & i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have the following extension

$$0 \longrightarrow H^\star(R[-1], F) \longrightarrow H^\star(\mathrm{cone}(X), F) \longrightarrow H^\star(R[0], F) \longrightarrow 0.$$

Thus, this sequence identifies the only possible extension that corresponds to ϵ. Hence, $H_0(F^\wedge R[0]) \cong F[[X]]$.

Now let $R = F[X_1, X_2]_{(X_1,X_2)}$. Then $\mathcal{A}^\star = E_F(e_1, e_2)$ where $|e_1| = |e_2| = 1$ by Proposition 3.1.9. Note that

$$\mathcal{A}^\star \cong E_F(e_1) \otimes_F E_F(e_2).$$

Using Theorem 1.2.10, we have that

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}^{s,t}_{\mathcal{A}^\star}(F, F) \\ &\cong \mathrm{Ext}^{s_1,t_1}_{E_F(e_1)}(F, F) \otimes_F \mathrm{Ext}^{s_2,t_2}_{E_F(e_2)}(F, F). \end{aligned}$$

where $s_1 + s_2 = s$ and $t_1 + t_2 = t$. Therefore, we can deduce that $H_0(F^\wedge R[0]) \cong F[[X_1, X_2]]$.

Inductively, we can show that if $R = F[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$, then $H_0(F^\wedge R[0]) \cong F[[X_1, \ldots, X_n]]$

Example 4.5.2. Let $R = F[X]/(X^2)$ where F is a field. First note that R is a noetherian local ring with maximal ideal $\mathfrak{m} = (X)/(X^2)$ and residue field $\mathbb{K} \cong F$. We calculate $\mathcal{A}^\star$. We construct an R-free minimal resolution of F

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} F \longrightarrow 0.$$

Let $P_0 = R$ with $\epsilon(X) = 0$. Then $\mathrm{Ker}(\epsilon) = \langle X \rangle$. Let $P_1 = R$ with $d_1(1) = X$. Then $\mathrm{Ker}(d_1) = \langle X \rangle$. Let $P_2 = R$ with $d_2(1) = X$. Continuing this way, we deduce that the following

$$\cdots \xrightarrow{X} R \xrightarrow{X} R \xrightarrow{X} R$$

is a minimal R-free resolution of F. For each $n \geq 0$,

$$\mathcal{A}^\star \cong \operatorname{Ext}_R^n(F[0], F[0]) \cong F.$$

Now we determine the ring structure of $\mathcal{A}^\star$. Let a be the augmentation in $\operatorname{Hom}_R(P_1, F)$. Note that we have the following commutative diagram

$$\begin{tikzcd}
0 & F \arrow[l] & P_0 \arrow[l, "a"'] & P_1 \arrow[l, "X"'] \arrow[d, "\mathrm{id}"] \arrow[dll, "a"'] & P_2 \arrow[l, "X"'] \arrow[d, "\mathrm{id}"] & \cdots \arrow[l] \\
& 0 & F \arrow[l] & P_0 \arrow[l, "a"'] & P_1 \arrow[l, "X"'] \arrow[d, "a"] & \cdots \arrow[l] \\
& & & & F &
\end{tikzcd}$$

Then $0 \neq a^2 = a\,\mathrm{id} \in \operatorname{Hom}_R(P_2, F)$. It is clear that $0 \neq a^n \in \operatorname{Hom}_R(P_n, F)$. Hence, $\mathcal{A}^\star \cong F[a]$, $|a| = 1$.

Consider the chain complex $R[0]$. We have that $H^\star(R[0], F) \cong F$. Note that the following sequence

$$0 \longrightarrow u\mathcal{A}^\star \xrightarrow{\partial} \mathcal{A}^\star \longrightarrow F \longrightarrow 0$$

is an $\mathcal{A}^\star$-free minimal resolution of F, where $u\mathcal{A}^\star$ is the free $\mathcal{A}^\star$-module on one generator u of degree one with $\partial(u) = a$. Therefore,

$$E_2^{s,t} = \begin{cases} F & s = t = 0, \\ F & s = t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We see that the spectral sequence collapses and thus

$$E_\infty^{s,t} = \begin{cases} F & s = t = 0, \\ F & s = t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $b \in \operatorname{Hom}^1_{\mathcal{A}^\star}(u\mathcal{A}^\star, F)$. We can show that b detects the map $X\colon R \longrightarrow R$ as proved in Example 4.5.1. Hence, $H_0(F^\wedge R[0]) \cong R$.

Now assume that $R = F[X_1, X_2]/(X_1^2, X_2^2)$. Note that

$$R \cong F[X_1]/(X_1^2) \otimes_F F[X_2]/(X_2^2).$$

Since

$$\mathrm{Ext}^n_{F[X_1]/(X_1^2)}(F,F) \otimes_F \mathrm{Ext}^n_{F[X_2]/(X_2^2)}(F,F) \longrightarrow \mathrm{Ext}^{2n}_R(F,F)$$

is an isomorphism by Theorem 1.2.10, we can deduce that

$$\mathcal{A}^\star \cong F[a_1, a_2], \qquad |a_1| = |a_2| = 1.$$

Since $\mathcal{A}^\star \cong F[a_1] \otimes_F F[a_2]$ and

$$\mathrm{Ext}^{s_1,t_1}_{F[a_1]}(F,F) \otimes_F \mathrm{Ext}^{s_2,t_2}_{F[a_2]}(F,F) \longrightarrow \mathrm{Ext}^{s_1+s_2,t_1+t_2}_{\mathcal{A}^\star}(F,F)$$

is an isomorphism by Theorem 1.2.10, we can deduce that $H_0(F^\wedge R[0]) \cong R$.

Using induction, if $R = F[X_1, \ldots, X_n]/(X_1^2, \ldots, X_n^2)$, then

$$\mathcal{A}^\star \cong F[a_1, \ldots, a_n], \qquad |a_1| = \ldots = |a_n| = 1.$$

and $H_0(F^\wedge R[0]) \cong R$.

Example 4.5.3. Let $R = \mathbb{Z}/(p)[X]/X^{p^i}$. First note that R is a noetherian local ring with maximal ideal $\mathfrak{m} = (X)/(X^{p^i})$ and residue field $\mathbb{K} \cong \mathbb{Z}/(p)$. We calculate $\mathcal{A}^\star$. We construct an R-free minimal resolution of $\mathbb{Z}/(p)$

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathbb{Z}/(p) \longrightarrow 0.$$

Let $P_0 = R$ with $\epsilon(X) = 0$. Then $\operatorname{Ker}\epsilon = \langle X \rangle$. Let $P_1 = R$ with $d_1(1) = X$. Then $\operatorname{Ker} d_1 = \langle X^{p^i-1} \rangle$. Let $P_2 = R$ with $d_2(1) = X^{p^i-1}$. Then $\operatorname{Ker} d_2 = \langle X \rangle$. Let $P_3 = R$ with $d_3(1) = \langle X \rangle$. Then $\operatorname{Ker} d_3 = \langle X^{p^i-1} \rangle$. Let $P_4 = R$ with $d_4(1) = X^{p^i-1}$. Continuing this way, we deduce that for each $n \geq 0$,

$$\mathcal{A}^n \cong \mathrm{Ext}^n_R(\mathbb{Z}/(p)[0], \mathbb{Z}/(p)[0]) \cong \mathbb{Z}/(p).$$

Now we determine the ring structure of $\mathcal{A}^\star$. Let a be the augmentation in $\mathrm{Hom}_R(P_1, \mathbb{Z}/(p))$ and b the augmentation in $\mathrm{Hom}_R(P_2, \mathbb{Z}/(p))$. We show that $a^2 = 0$. We have the following commutative diagram

$$\begin{tikzcd}
0 & \mathbb{Z}/(p) \arrow[l] & P_0 \arrow[l] & P_1 \arrow[l] \arrow[dl, "a"'] \arrow[d, "\mathrm{id}"] & P_2 \arrow[l, "X^{p^i-1}"'] \arrow[d, "X^{p^i-2}"] & \cdots \arrow[l] \\
 & 0 & \mathbb{Z}/(p) \arrow[l] & P_0 \arrow[l, "a"'] & P_1 \arrow[l, "X"'] \arrow[d, "a"] & \cdots \arrow[l] \\
 & & & & \mathbb{Z}/(p) &
\end{tikzcd}$$

Hence, $a^2 = aX^{p^i-2} = 0$. We show that $b^n \neq 0$. We have the following commutative diagram

$$\begin{tikzcd}
0 & \mathbb{Z}/(p) \arrow[l] & P_0 \arrow[l] & P_1 \arrow[l] & P_2 \arrow[l] \arrow[dl, "b"'] \arrow[d, "\mathrm{id}"] & P_3 \arrow[l, "X"'] \arrow[d, "\mathrm{id}"] & P_4 \arrow[l, "X^{p^i-1}"'] \arrow[d, "\mathrm{id}"] & \cdots \arrow[l] \\
& & 0 & \mathbb{Z}/(p) \arrow[l] & P_0 \arrow[l, "b"'] & P_1 \arrow[l, "X"'] & P_2 \arrow[l, "X^{p^i-1}"'] \arrow[d, "b"] & \cdots \arrow[l] \\
& & & & & & \mathbb{Z}/(p) &
\end{tikzcd}$$

Thus, $0 \neq b^2 = b\,\mathrm{id} \in \mathrm{Hom}_R(P_4, \mathbb{Z}/(p))$. Similarly, we can show that $b^n \neq 0$. We show that $0 \neq ba$ and $ab = ba$. We have the following commutative diagram

$$\begin{tikzcd}
0 & \mathbb{Z}/(p) \arrow[l] & P_0 \arrow[l] & P_1 \arrow[l] \arrow[dl, "a"'] \arrow[d, "\mathrm{id}"] & P_2 \arrow[l, "X^{p^i-1}"'] \arrow[d, "X^{p^i-2}"] & P_3 \arrow[l, "X"'] \arrow[d, "\mathrm{id}"] & \cdots \arrow[l] \\
& 0 & \mathbb{Z}/(p) \arrow[l] & P_0 \arrow[l, "a"'] & P_1 \arrow[l, "X"'] & P_2 \arrow[l, "X^{p^i-1}"'] \arrow[d, "b"] & \cdots \arrow[l] \\
& & & & & \mathbb{Z}/(p) &
\end{tikzcd}$$

Hence, $0 \neq ba = b\,\mathrm{id} \in \mathrm{Hom}_R(P_3, \mathbb{Z}/(p))$. Also, we have the following commutative diagram

$$\begin{tikzcd}
0 & \mathbb{Z}/(p) \arrow[l] & P_0 \arrow[l] & P_1 \arrow[l] & P_2 \arrow[l] \arrow[dl, "b"'] \arrow[d, "\mathrm{id}"] & P_3 \arrow[l, "X"'] \arrow[d, "\mathrm{id}"] & \cdots \arrow[l] \\
& & 0 & \mathbb{Z}/(p) \arrow[l] & P_0 \arrow[l, "b"'] & P_1 \arrow[l, "X"'] \arrow[d, "a"] & \cdots \arrow[l] \\
& & & & & \mathbb{Z}/(p) &
\end{tikzcd}$$

Hence, $0 \neq ab = a\,\mathrm{id} \in \mathrm{Hom}_R(P_3, \mathbb{Z}/(p))$. Moreover, $ab = ba$. Also, we can show that $0 \neq ab^n \in \mathrm{Hom}_R(P_{2n+1}, R)$. Therefore, we deduce that

$$\mathcal{A}^\star \cong \mathbb{Z}/(p)[a,b]/a^2 \qquad |a| = 1, |b| = 2.$$

Consider the chain complex $R[0]$. We have $H^\star(R[0], \mathbb{Z}/(p)) \cong \mathbb{Z}/(p)$. We construct a $\mathcal{A}^\star$-free minimal resolution of $\mathbb{Z}/(p)$

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z}/(p) \longrightarrow 0.$$

Let $P_0 = \mathcal{A}^\star$ with $d_0(a) = d_0(b) = 0$. Then $\mathrm{Ker}\, d_0 = \langle a, b \rangle$. Let $P_1 = u\mathcal{A}^\star \oplus v\mathcal{A}^\star$ where $u\mathcal{A}^\star$ is the free $\mathcal{A}^\star$-module on a generator u of degree one and $v\mathcal{A}^\star$ is the free

$\mathcal{A}^\star$-module on a generator v of degree two with $d_1(u) = a$ and $d_1(v) = b$. We can deduce that $\operatorname{Ker} d_1 = \langle au, -av + bu \rangle$. Let $P_2 = u^2\mathcal{A}^\star \oplus uv\mathcal{A}^\star$ where $u^2\mathcal{A}^\star$ is the free $\mathcal{A}^\star$-module on a generator u^2 of degree two and $uv\mathcal{A}^\star$ is the free $\mathcal{A}^\star$-module on a generator uv of degree three with $d_2(u^2) = au$ and $d_2(uv) = -av + bu$. We can deduce that $\operatorname{Ker} d_2 = \langle au^2, -auv + bu^2 \rangle$. Let $P_3 = u^3\mathcal{A}^\star \oplus u^2v\mathcal{A}^\star$ where $u^3\mathcal{A}^\star$ is the free $\mathcal{A}^\star$-module on a generator u^3 of degree three and $u^2v\mathcal{A}^\star$ is the free $\mathcal{A}^\star$-module on a generator u^2v of degree four with $d_3(u^3) = au^2$ and $d_3(u^2v) = -auv + bu^2$. We can deduce that $\operatorname{Ker} d_3 = \langle au^3, -au^2v + bu^3 \rangle$. Let $P_4 = u^4\mathcal{A}^\star \oplus u^3v\mathcal{A}^\star$ with $d_4(u^4) = au^3$ and $d_4(u^3v) = -au^2v + bu^3$. Continuing this way, we can deduce that the following sequence

$$\cdots \xrightarrow{d_{n+1}} u^n\mathcal{A}^\star \oplus u^{n-1}v\mathcal{A}^\star \xrightarrow{d_n} \cdots \xrightarrow{d_3} u^2\mathcal{A}^\star \oplus uv\mathcal{A}^\star \xrightarrow{d_2} u\mathcal{A}^\star \oplus v\mathcal{A}^\star \xrightarrow{d_1} \mathcal{A}^\star$$

is a minimal $\mathcal{A}^\star$-free resolution of $\mathbb{Z}/(p)$. Therefore,

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/(p) & s = t, \\ \mathbb{Z}/(p) & t - s = 1, s > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we have the following commutative diagram

$$\begin{tikzcd}
0 & \mathbb{Z}/(p) \arrow[l] & \mathcal{A}^\star \arrow[l] & u\mathcal{A}^\star \oplus v\mathcal{A}^\star \arrow[l] \arrow[dl, "\alpha"'] \arrow[d, "(\mathrm{id},0)"] & u^2\mathcal{A}^\star \oplus uv\mathcal{A}^\star \arrow[l, "d_2"'] \arrow[d, "(\mathrm{id},\mathrm{id})"] & \cdots \arrow[l] \\
& 0 & \mathbb{Z}/(p)[-1] \arrow[l] & \mathcal{A}^\star[-1] \arrow[l, "d_0[-1]"'] & u\mathcal{A}^\star[-1] \oplus v\mathcal{A}^\star[-1] \arrow[l, "d_1[-1]"'] \arrow[d, "\alpha[-1]"] & \cdots \arrow[l] \\
& & & & \mathbb{Z}/(p)[-2] &
\end{tikzcd}$$

where $\alpha = (d_0, 0) \in \operatorname{Hom}^0_{\mathcal{A}^\star}(P_1, \mathbb{Z}/(p)[-1])$. We see that $0 \neq \alpha^2 \in \operatorname{Hom}^0_{\mathcal{A}^\star}(P_2, \mathbb{Z}/(p)[-2])$. Similarly, we can show that $0 \neq \alpha^n \in \operatorname{Hom}^0_{\mathcal{A}^\star}(P_n, \mathbb{Z}/(p)[-n])$. Let $\beta = (0, d_0) \in \operatorname{Hom}^0_{\mathcal{A}^\star}(P_1, \mathbb{Z}/(p)[-2])$. Then we can show that $\beta^2 = 0$. Also, note that we have the following commutative diagram

$$\begin{tikzcd}
0 & \mathbb{Z}/(p) \arrow[l] & \mathcal{A}^\star \arrow[l] & u\mathcal{A}^\star \oplus v\mathcal{A}^\star \arrow[l] \arrow[dl, "\alpha"'] \arrow[d, "(\mathrm{id},0)"] & u^2\mathcal{A}^\star \oplus uv\mathcal{A}^\star \arrow[l, "d_2"'] \arrow[d, "(\mathrm{id},\mathrm{id})"] & \cdots \arrow[l] \\
& 0 & \mathbb{Z}/(p)[-1] \arrow[l] & \mathcal{A}^\star[-1] \arrow[l, "d_0[-1]"'] & u\mathcal{A}^\star[-1] \oplus v\mathcal{A}^\star[-1] \arrow[l, "d_1[-1]"'] \arrow[d, "\beta[-1]"] & \cdots \arrow[l] \\
& & & & \mathbb{Z}/(p)[-3] &
\end{tikzcd}$$

Hence, $0 \neq \beta\alpha \in \mathrm{Hom}^0_{\mathcal{A}^\star}(P_2, \mathbb{Z}/(p)[-3])$. Similarly, we can show that $0 \neq \beta\alpha^{n-2} \in \mathrm{Hom}^0_{\mathcal{A}^\star}(P_{n-1}, \mathbb{Z}/(p)[-n])$. We can show that α detects the map $X \colon R \longrightarrow R$. Therefore, for $r < p^i - 1$, d_r must be zero and $d^{1,s}_{p^i-1}$ is an isomorphism for each $s > 0$. Hence, $H_0(\mathbb{Z}/(p)^\wedge R[0]) \cong R$.

References

[1] J.F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.

[2] M. Artin and B. Mazur. *Etale homotopy.* Lecture Notes in Mathematics, No. 100. Springer-Verlag, Berlin, 1969.

[3] M.F. Atiyah and I.G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[4] A. Baker and A. Lazarev. On the Adams spectral sequence for R-modules. *Algebr. Geom. Topol.*, 1:173–199 (electronic), 2001.

[5] A.J. Baker and J.P. May. Minimal atomic complexes. *Topology*, 43(3):645–665, 2004.

[6] J.M. Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.

[7] A.K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[8] H. Cartan and S. Eilenberg. *Homological algebra.* Princeton University Press, Princeton, N. J., 1956.

[9] J.F. Davis and P. Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001.

[10] W.G. Dwyer. Localizations. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 3–28. Kluwer Acad. Publ., Dordrecht, 2004.

[11] W.G. Dwyer and J.P.C. Greenlees. Complete modules and torsion modules. *Amer. J. Math.*, 124(1):199–220, 2002.

[12] B. Farb and R.K. Dennis. *Noncommutative algebra*, volume 144 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.

[13] S.I. Gelfand and Y.I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[14] J.P.C. Greenlees and J.P. May. Derived functors of I-adic completion and local homology. *J. Algebra*, 149(2):438–453, 1992.

[15] J.P.C. Greenlees and J.P. May. Completions in algebra and topology. In *Handbook of algebraic topology*, pages 255–276. North-Holland, Amsterdam, 1995.

[16] P. Hu, I. Kriz, and J.P. May. Cores of spaces, spectra, and E_∞ ring spectra. *Homology Homotopy Appl.*, 3(2):341–354, 2001.

[17] D.C. Isaksen. Strict model structures for pro-categories. In *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)*, volume 215 of *Progr. Math.*, pages 179–198. Birkhäuser, Basel, 2004.

[18] C.U. Jensen. *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 254.

[19] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[20] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[21] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.

[22] J. McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[23] J.W. Milnor and J.C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[24] R.E. Mosher and M.C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row Publishers, New York, 1968.

[25] A. Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[26] D.G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[27] D.C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.

[28] P. Roberts. *Homological invariants of modules over commutative rings*, volume 72 of *Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics]*. Presses de l'Université de Montréal, Montreal, Que., 1980.

[29] J-E Roos. Sur les foncteurs dérivés de $\varprojlim$. Applications. *C. R. Acad. Sci. Paris*, 252:3702–3704, 1961.

[30] J.J. Rotman. *An introduction to homological algebra*, volume 85 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1979.

[31] B.E. Shipley. Pro-isomorphisms of homology towers. *Math. Z.*, 220(2):257–271, 1995.

[32] R.M. Switzer. *Algebraic topology—homotopy and homology.* Springer-Verlag, New York, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.

[33] C.A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.

[34] S. Wüthrich. Homology of powers of regular ideals. *Glasg. Math. J.*, 46(3):571–584, 2004.