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Representations of symplectic reflection algebras and resolutions of deformations of symplectic quotient singularities

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Dedicated to Claus Michael Ringel on the occasion of his sixtieth birthday

Abstract. We give an equivalence of triangulated categories between the derived category of finitely generated representations of symplectic reflection algebras associated with wreath products (with parameter $t = 0$) and the derived category of coherent sheaves on a crepant resolution of the spectrum of the centre of these algebras.

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1. Introduction

1.1. In this paper we take a step towards a geometric understanding of the representation theory of certain symplectic reflection algebras (with parameter $t = 0$). A number of papers have shown that this representation theory is closely related to the singularities of the centre of these algebras and to resolutions of these singularities, [9], [4], [12]. Here we make such a relationship precise by proving that the category of finitely generated modules is derived equivalent to the category of coherent sheaves on an appropriate desingularisation.

A long term goal in this project is to find character formulae for simple modules, generalising the work [10] and [13] in which Kostka polynomials appear. A simple consequence of the derived equivalence is a geometric interpretation for the number of simple modules of a symplectic reflection algebra with given central character. A closer analysis will undoubtedly reveal more.

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1.2. Let us summarise our results. Let Γ be a non-trivial finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ and n a positive integer. Let $H_{\mathbf{c}}$ be the symplectic reflection algebra (with parameter $t = 0$) for the wreath product $\Gamma_n = S_n \ltimes \Gamma^n$. (All undefined notation and definitions can be found later in the paper.) The spectrum of the centre of this algebra, $X_{\mathbf{c}} = \operatorname{Spec} Z_{\mathbf{c}}$, is a deformation of the symplectic quotient singularity $\mathbb{C}^{2n}/\Gamma_n$. Our principal result is the following.

Theorem. *There is a crepant resolution $\pi_{\mathbf{c}} : Y_{\mathbf{c}} \longrightarrow X_{\mathbf{c}}$ such that there is an equivalence of triangulated categories*

$$D^b(\operatorname{mod} H_{\mathbf{c}}) \longrightarrow D^b(\operatorname{coh} Y_{\mathbf{c}})$$

between the bounded derived category of finitely generated $H_{\mathbf{c}}$–modules and the bounded derived category of coherent sheaves on $Y_{\mathbf{c}}$.

1.3. In the special case $\mathbf{c} = \mathbf{0}$ the variety $X_{\mathbf{c}}$ is the orbit space $\mathbb{C}^{2n}/\Gamma_n$, so the above theorem includes the results of [21] on Kleinian singularities ($n = 1$) and the observation of [32, Section 4.4] (for general n). The proof we give here, however, is by deformation from the $\mathbf{c} = \mathbf{0}$ case and so depends on these results. To prove the equivalence, we use the methods of [2], which were adapted to a mildly non–commutative situation in [28], together with results from [14] and [15].

1.4. For $x \in X_{\mathbf{c}}$, let $\mathfrak{m}_x$ be the corresponding maximal ideal of $Z_{\mathbf{c}}$. The simple modules of the finite dimensional algebra $H_{\mathbf{c}}/\mathfrak{m}_x H_{\mathbf{c}}$ are the simple $H_{\mathbf{c}}$–modules with central character x. The following corollary is a straightforward consequence of the above theorem.

Corollary. *Let $\pi_{\mathbf{c}} : Y_{\mathbf{c}} \longrightarrow X_{\mathbf{c}}$ be the crepant resolution in Theorem 1.2. For all $x \in X_{\mathbf{c}}$, there is an isomorphism of Grothendieck groups $K(H_{\mathbf{c}}/\mathfrak{m}_x H_{\mathbf{c}}) \cong K(\pi_{\mathbf{c}}^{-1}(x))$.*

1.5. In the case $n = 1$ this recovers known results on the simple modules of deformed preprojective algebras, whilst for $\mathbf{c} = \mathbf{0}$ the content is essentially the (generalised) McKay correspondence, [20].

1.6. Symplectic reflection algebras also exist for finite Coxeter groups, W. However, [12, Theorem 1.1] shows that the only orbits spaces $\mathbb{C}^{2n}/W$ admitting crepant resolutions are for W of type A and B, that is $W = S_n$ or $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. Thus there is no analogue of Theorem 1.2 valid for all finite Coxeter groups; our result is as general as we can expect.

1.7. The paper is organised as follows. In Section 2 we recall the definition and basic properties of symplectic reflection algebras. A discussion of non–commutative crepant resolutions and derived equivalences is given in Section 3. Section 4 presents some results on deformations of semi–small morphisms and their relation to crepant resolutions and derived equivalences. Finally, in Section 5, we prove our main result and discuss the application to counting simple modules.

2. Symplectic reflection algebras

2.1. Let $\tilde{\omega}$ be the standard symplectic form on $\mathbb{C}^2$, Γ a finite subgroup of $SL(2, \mathbb{C})$ and n a positive integer. The wreath product $\Gamma_n \equiv S_n \ltimes \Gamma^n$ acts on $V \equiv (\mathbb{C}^2)^n$, preserving the symplectic form $\omega \equiv \tilde{\omega}^n$.

2.2. Recall that $\gamma \in \Gamma_n$ acting on V is called a *symplectic reflection* if $\dim(1 - \gamma)(V) = 2$. The set of all symplectic reflections is denoted by $\mathcal{S}$. Let $\mathbf{c}$ be a $\mathbb{C}$–valued function on $\mathcal{S}$, constant on conjugacy classes ($\gamma \mapsto c_\gamma$). Given $\gamma \in \mathcal{S}$ define the form ω_γ on V to have radical $\ker(1 - \gamma)$ and to be the restriction of ω on $(1 - \gamma)(V)$.

2.3. The *symplectic reflection algebra* $H_{\mathbf{c}}$ is the $\mathbb{C}$–algebra, defined as the quotient of the skew group ring $TV * \Gamma_n$ by the relations

$$x \otimes y - y \otimes x = \sum_{\gamma \in \mathcal{S}} c_\gamma \omega_\gamma(x, y)\gamma,$$

for all $x, y \in V$.

Remark. Usually, symplectic reflection algebras depend on a further parameter $t \in \mathbb{C}$, [9, Section 1]. The definition above is the case $t = 0$.

2.4. There is an increasing $\mathbb{N}$–filtration on $H_{\mathbf{c}}$, obtained by setting $F^0 H_{\mathbf{c}} = \mathbb{C}\Gamma_n$, $F^1 = V \otimes \mathbb{C}\Gamma_n + \mathbb{C}\Gamma_n$, and $F^i = (F^1)^i$. By the PBW theorem, [9, Theorem 1.3], $\operatorname{gr} H_{\mathbf{c}} \cong \mathbb{C}[V] * \Gamma_n$ (where we have used ω to identify the Γ_n–spaces V and V^*). In particular, a non–zero $\mathbf{c}$ yields a flat family of symplectic reflection algebras $H_{u\mathbf{c}}$ over $\mathbb{C}[u]$.

Each $H_{\mathbf{c}}$ is a prime noetherian ring because its associated graded ring has these properties [9, Theorem 1.3], [25, Theorem 3.17], and [24, Prop. 1.6.6, Theorem 1.6.9].

2.5. Let $Z_{\mathbf{c}}$ denote the centre of $H_{\mathbf{c}}$, and set $X_{\mathbf{c}} = \operatorname{Spec} Z_{\mathbf{c}}$. It is known that $Z_{\mathbf{c}}$ is a finitely generated $\mathbb{C}$-algebra, and that $H_{\mathbf{c}}$ is a finite $Z_{\mathbf{c}}$–module, [9, Theorem 1.5 and Theorem 3.1]. By a lemma of Dixmier, it follows that all simple $H_{\mathbf{c}}$–modules are finite dimensional. In fact $|\Gamma_n|$ is the strict upper bound for the dimension of simple $H_{\mathbf{c}}$–modules, [9, Proposition 3.8]. We therefore have a map

$$\chi : \mathsf{Simp}(H_{\mathbf{c}}) \longrightarrow X_{\mathbf{c}}$$

which sends a simple module S to its central character $\chi(S) \in X_{\mathbf{c}}$.

2.6. The algebra $Z_{\mathbf{c}}$ has a Poisson bracket, making $X_{\mathbf{c}}$ a Poisson variety. The following subsets of $X_{\mathbf{c}}$ are the same:

(1) the locus where the form is non-degenerate;
(2) its non-singular locus, $\mathsf{Sm}(X_{\mathbf{c}})$ [4, Theorem 7.8];

(3) the Azumaya locus of $H_{\mathbf{c}}$, that is $\{\chi(S) : S$ is a simple module of maximal dimension$\}$ [9, Theorem 1.7];
(4) $\{x \in X_{\mathbf{c}} : \chi^{-1}(x)$ is a singleton$\}$ [9, Theorem 3.7].

Moreover, by [9, Theorem 3.7], if $x \in \mathsf{Sm}(X_{\mathbf{c}})$, then the unique simple $H_{\mathbf{c}}$-module having central character x is isomorphic to $\mathbb{C}\Gamma_n$ as a $\mathbb{C}\Gamma_n$-module.

2.7. Let $e \in \mathbb{C}\Gamma_n$ be the symmetrising idempotent $|\Gamma_n|^{-1}\sum_{\gamma\in\Gamma_n}\gamma$. The map $Z_{\mathbf{c}} \longrightarrow eH_{\mathbf{c}}e$, $z \mapsto ze$, is an isomorphism, [9, Theorem 3.1]. Thus, following 2.4, there is a flat family of commutative algebras $Z_{u\mathbf{c}}$ over $\mathbb{C}[u]$. We set $X_{u\mathbf{c}} = \operatorname{Spec} Z_{u\mathbf{c}}$.

2.8. For later use we need the following lemma.

Lemma. *For non–zero* $\mathbf{c}$*, the following sets are the same*

(1) the smooth locus, $\mathsf{Sm}(X_{u\mathbf{c}})$*;*
(2) the Azumaya locus of $H_{u\mathbf{c}}$*;*
(3) the central characters of simple $H_{\mathbf{c}}$*-modules that are isomorphic to the regular representation of* Γ_n *as* Γ_n*-modules.*

Proof. As the generic simple module for $H_{u\mathbf{c}}$ has dimension $|\Gamma_n|$ it follows that the Azumaya locus of $H_{u\mathbf{c}}$ is the union of the Azumaya loci for $H_{\lambda\mathbf{c}}$ for $\lambda \in \mathbb{C}$. This proves the equivalence of (1) and (3).

Since the associated graded ring of $H_{u\mathbf{c}}$ is $\mathbb{C}[V \oplus \mathbb{C}] * \Gamma_n$, it follows from [24, Proposition 1.6.6, Theorem 1.6.9 and Corollary 7.6.18] and [3, Theorem 3.8] that the equivalence of (1) and (2) follows if the non–Azumaya locus has codimension at least 2. This, however, is clear as for each $\lambda \in \mathbb{C}$ the Azumaya locus of $H_{\lambda\mathbf{c}}$ has codimension at least 2 in $X_{\lambda\mathbf{c}}$.

2.9. Observe as a consequence of the above proof that $Z_{u\mathbf{c}}$ is normal. Indeed, $Z_{u\mathbf{c}}$ is Cohen–Macaulay (in fact Gorenstein) since $eH_{u\mathbf{c}}e$ has an associated graded ring isomorphic to $\mathbb{C}[V \oplus \mathbb{C}]^{\Gamma_n}$, [9, Theorem 3.3]. Combining this with the smoothness of $X_{u\mathbf{c}}$ in codimension one, which is proved above, shows that $Z_{u\mathbf{c}}$ is normal, [5, Theorem 2.2.11].

3. Non–commutative crepant resolutions

3.1. Let X be a Gorenstein $\mathbb{C}$–variety. A resolution of singularities $f : Y \longrightarrow X$ is said to be *crepant* if $f^*\omega_X = \omega_Y$. Crepant resolutions are a generalisation of the notion of a minimal resolution in two dimensions. However, crepant resolutions need not exist, and need not be unique when they do exist.

3.2. In the setup of Section 2 we could take $X = X_{\mathbf{c}}$. This is a Gorenstein variety, [9, Theorem 1.5(i)]. Moreover, since the Poisson form on $X_{\mathbf{c}}$ is symplectic on $\mathsf{Sm}(X_{\mathbf{c}})$, the canonical bundle $\omega_{X_{\mathbf{c}}}$ is trivial. Thus any crepant resolution of $X_{\mathbf{c}}$ has trivial canonical class.

Remark. In passing we remark that $Y_{\mathbf{c}}$ is a crepant resolution of $X_{\mathbf{c}}$ if and only if $Y_{\mathbf{c}}$ is a symplectic variety whose form agrees with that of $X_{\mathbf{c}}$ on $\mathsf{Sm}(X_{\mathbf{c}})$ (the case $\mathbf{c} = 0$ is [19, Proposition 3.2]). This follows from [11, Proposition 1.1] once we know that $X_{\mathbf{c}}$ has symplectic singularities, meaning that on some (and hence every) resolution of $X_{\mathbf{c}}$ the form on $\mathsf{Sm}(X_{\mathbf{c}})$ extends to a regular, but not necessarily non–degenerate, 2–form. In this situation, such a resolution of $X_{\mathbf{c}}$ is given by a quiver variety, [26], generalising the $\mathbf{c} = \mathbf{0}$ case in [31, Sections 1.3 and 1.4] and the generic $\mathbf{c}$ case in [9, Section 11].

3.3. For the following definitions see [28, Sections 3 and 4]. Throughout R denotes a commutative noetherian domain over $\mathbb{C}$. A module–finite R–algebra A is *homologically homogeneous* if, for all $\mathfrak{p} \in \operatorname{Spec} R$, $\operatorname{gldim} A_{\mathfrak{p}} = \operatorname{Kdim} R_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ is maximal Cohen–Macaulay. A *non–commutative crepant resolution* of R is a homologically homogeneous R–algebra of the form $A = \operatorname{End}_R(M)$, where M is a reflexive R–module.

A justification for this definition of non–commutative crepant resolution is given in [28, Section 4].

3.4. We recall some material from [22, Sections 3 and 4] and [28, Section 6]. Set $X = \operatorname{Spec} R$. Let A be an R–algebra that is finitely generated as an R–module, and let $(e_i)_{i=1,\dots,p}$ be pairwise orthogonal idempotents in A such that $1 = \sum_i e_i$.

For a map $R \longrightarrow K$ with K a field and V a finite dimensional $A \otimes_R K$–module, we write $\underline{\dim} V = (\dim_K e_i V)_i \in \mathbb{Z}^p$.

Pick $\lambda \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^p, \mathbb{R})$ and let $\alpha = \underline{\dim} V$. We say that a finite dimensional $A \otimes_R K$-module V is *stable* (respectively, *semi-stable*) with respect to λ if $\lambda(\alpha) = 0$ and for every proper $A \otimes_R K$-submodule W of V we have $\lambda(\underline{\dim} W) < 0$ (resp., $\lambda(\underline{\dim} W) \leq 0$).

Our definition of stability is different from that in [28]: where we have $\lambda(\underline{\dim} W) < 0$ Van den Bergh has $\lambda(\underline{\dim} W) > 0$. Of course, the difference is only cosmetic because we can pass back and forth between the two notions by replacing λ by $-\lambda$. The reason for this difference is that later on in Section 5.1 we want our notion of stability to coincide with the notion that Haiman uses in [14].

We say that λ is *generic* (for α) if all semi-stable representations of dimension vector α are stable. There is a generic λ if and only if α is indivisible, meaning that the greatest common divisor of the α_is is 1. The condition $\lambda(\beta) \neq 0$ for all $0 < \beta < \alpha$ ensures λ is generic.

3.5. Let T be an R-scheme. A *family* of A-modules of dimension α parametrised by T is a locally free sheaf $\mathcal{F}$ of $\mathcal{O}_T$-modules together with an R-algebra homomorphism $\phi : A \to \operatorname{End}_T \mathcal{F}$ such that $e_i\mathcal{F}$ has constant rank α_i for all i. We say that $\mathcal{F}$ is *semi-stable* (resp., *stable*) if for every every field K and every morphism $\xi : \operatorname{Spec} K \to T$, $\xi^*\mathcal{F}$ is a semi-stable (resp., stable) $A \otimes_R K$-module. Two families $(\mathcal{F}, \phi)$ and $(\mathcal{F}', \phi')$ are *equivalent* if there is an invertible $\mathcal{O}_T$-module $\mathcal{L}$ and an isomorphism $\psi : \mathcal{F} \to \mathcal{F}' \otimes \mathcal{L}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\phi'"'] & \operatorname{End}\mathcal{F} \arrow[d, "\overline{\psi}"] \\ \operatorname{End}\mathcal{F}' \arrow[r, "\sim", "\mathrm{Id}\otimes\mathcal{L}"'] & \operatorname{End}\mathcal{F}'\otimes\mathcal{L} \end{tikzcd}$$

commutes; in this diagram $\overline{\psi}(\nu) = \psi\nu\psi^{-1}$.

A family $(\mathcal{U}, \rho)$ parametrised by W is *universal* if for every R-scheme T and every family $(\mathcal{F}, \phi)$ over T there is a unique morphism $\xi : T \to W$ such that $(\xi^*\mathcal{U}, \xi^*\rho)$ is equivalent to $(\mathcal{F}, \phi)$; here $\xi^*\rho$ denotes the composition $A \to \operatorname{End}\mathcal{U} \to \operatorname{End}\xi^*\mathcal{U}$.

Proposition 3.6 below says that in suitable situations there is a universal family. We also express this by saying that W is a fine moduli space for families of A-modules of dimension α and call $\mathcal{U}$ the *universal family*.

3.6. Define a functor $\mathbf{R}^s$: R-schemes $\longrightarrow$ Sets by

$$T \mapsto \{\text{equiv. classes of families of } \lambda\text{–stable } A\text{–modules over } T \text{ with dimension } \alpha\}.$$

Proposition ([28, Proposition 6.2.1]). *Suppose that λ is generic for α. Then $\mathbf{R}^s$ is represented by a closed subscheme $W^s \subset \mathbb{P}^N_X$.*

We write $f : W^s \to X$ for the structure morphism, and $\mathcal{B}$ for the universal family of λ-stable A-modules of dimension α.

If U is either an open or closed subscheme of X then the representing scheme for U is $f^{-1}(U)$ and its universal family is $\mathcal{B}\big|_U$ (cf. the sentence after [28, Lemma 6.2.2]).

3.7. It is shown in [28, Lemma 6.2.3] that in the case $A = \operatorname{Mat}_{\sum\alpha_i}(R)$ the map $W^s \longrightarrow X$ is an isomorphism.

3.8. Assume $A = \operatorname{End}_R M$ is non–commutative crepant resolution of R. Suppose we have an R-module decomposition $M = \oplus_{i=1}^p M_i$ and let $e_1, \ldots, e_p$ be the projections onto the M_is viewed as idempotents in A. Define

$$\alpha_i := \operatorname{rank} M_i = \operatorname{rank} e_i M.$$

Suppose that λ is generic for α. Let $f : W^s \longrightarrow X$ and $\mathcal{B}$ be as in 3.6.

Let $U \subset X$ be the locus where M is locally free. It follows from 3.7 that $f^{-1}(U) \longrightarrow U$ is an isomorphism. Let Y be the closure of $f^{-1}(U)$; this is the unique irreducible component of W^s mapping birationally onto X. We continue to denote the restriction of f to Y by f. Let $\mathcal{G}$ be the restriction of $\mathcal{B}$ to Y.

There is a pair of adjoint functors between $D^b(\operatorname{coh} Y)$ and $D^b(\operatorname{mod} A)$:

$$\Phi : D^b(\operatorname{coh} Y) \longrightarrow D^b(\operatorname{mod} A) : C \mapsto \mathbf{R}\Gamma(C \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G})$$

$$\Psi : D^b(\operatorname{mod} A) \longrightarrow D^b(\operatorname{coh} Y) : D \mapsto D \otimes^{\mathbf{L}}_A \mathcal{G}^*.$$

The following theorem follows [2] closely.

Theorem ([28, 6.3.1]). *Assume that for every point $x \in X$,*

$$\dim(Y \times_X Y) \times_X \operatorname{Spec} \mathcal{O}_{X,x} \leq \operatorname{codim} x + 1.$$

Then $f : Y \longrightarrow X$ is a crepant resolution of X and Φ and Ψ are inverse equivalences.

3.9. Let $x \in X$. We write $D^b_x(\operatorname{coh} Y)$ and $D^b_x(\operatorname{mod} A)$ for the full subcategories of $D^b(\operatorname{coh} Y)$ and $D^b(\operatorname{mod} A)$ consisting of complexes supported on $f^{-1}(x)$ and on x respectively. Since Φ and Ψ are functors over X they restrict to equivalences between $D^b_x(\operatorname{coh} Y)$ and $D^b_x(\operatorname{mod} A)$, [2, 9.1] and [28, 6.6].

3.10. Fix a non–zero $\mathbf{c}$. The following lemma allows us to apply the machinery in this section.

Lemma. *The algebra $H_{\mathbf{c}}$ is a non–commutative crepant resolution of $X_{\mathbf{c}}$.*

Proof. By [9, Theorem 15.(ii), (iii)] $H_{\mathbf{c}}e$ is a finitely generated, reflexive, Cohen–Macaulay $H_{\mathbf{c}}$–module. Furthermore, by [9, Theorem 1.5(iv)] we have $H_{\mathbf{c}} \cong \operatorname{End}_{eH_{\mathbf{c}}e}(H_{\mathbf{c}}e)$. By [24, Corollary 6.18], gldim $H_{\mathbf{c}} < \infty$. Thus $H_{\mathbf{c}}$ is a non–commutative crepant resolution of $eH_{\mathbf{c}}e \cong Z_{\mathbf{c}}$ by [28, Lemma 4.2].

Let $H_{u\mathbf{c}}$ be the flat family of symplectic reflection algebras defined in 2.4. The arguments used to prove Lemma 3.10 all extend to $H_{u\mathbf{c}}$, showing that $H_{u\mathbf{c}}$ is a non-commutative crepant resolution of $X_{u\mathbf{c}}$.

3.11. If we set $R = eH_{u\mathbf{c}}e$, $M = H_{u\mathbf{c}}e$, and $A = H_{u\mathbf{c}}$, we are in the situation of section 3.8. Let $(e_i)_{i=1,\dots,p} \in \mathbb{C}\Gamma_n$ be the central orthogonal idempotents corresponding to the irreducible representations, labelled so that e_1 corresponds to the trivial representation. Thus $e_1 = e$. By the PBW theorem for $H_{\mathbf{c}}$ and $H_{u\mathbf{c}}$ we can consider the e_is as elements of $H_{\mathbf{c}}$ and $H_{u\mathbf{c}}$. If we set $M_i = e_iM$ there is a decomposition $H_{u\mathbf{c}}e = M = \oplus_{i=1}^p M_i$ as in Section 3.8. Let $\alpha_i = \operatorname{rank} e_i H_{u\mathbf{c}}e$ and write $\alpha = (\alpha_i)_{1\leq i \leq p} \in \mathbb{Z}^p$.

Let E_i be the simple $\mathbb{C}\Gamma_n$-module corresponding to e_i. As remarked in the proof of [9, Lemma 2.24], $e_iH_{u\mathbf{c}}e \cong \operatorname{Hom}_{\Gamma_n}(E_i, H_{u\mathbf{c}}e)$ so

$$\alpha_i = \operatorname{rank}_{Z_{u\mathbf{c}}} e_i H_{u\mathbf{c}}e = \dim E_i.$$

In particular, $\alpha_1 = 1$ so α is indivisible, and there are many maps $\lambda : \mathbb{Z}^p \to \mathbb{R}$ such that $\lambda(\alpha) = 0$ and $\lambda(\beta) > 0$ for all $0 < \beta < \alpha$. For each such λ there is a moduli space for λ–stable $H_{u\mathbf{c}}$–modules having dimension α (equivalently, that are isomorphic to $\mathbb{C}\Gamma_n$ as $\mathbb{C}\Gamma_n$-modules), and each such moduli space has a unique irreducible component that maps birationally to $X_{u\mathbf{c}}$.

4. Semi–small maps

4.1. A proper birational map $f : Y \longrightarrow X$ between irreducible varieties is *semi-small* if

$$2\,\mathrm{codim}_Y Z \geq \mathrm{codim}_X f(Z)$$

for all irreducible subvarieties $Z \subset Y$. Note that if f is semi–small the above inequality holds for all (not necessarily irreducible) subvarieties of Y. It is a theorem of Verbitsky, [29, Theorem 2.8], and Kaledin, [19, Proposition 4.4], that any crepant resolution of V/Γ_n is semi–small.

4.2. The following result relates semi–small maps to the hypothesis of Theorem 3.8.

Lemma. *Suppose that $f : Y \longrightarrow X$ is a semi–small morphism between irreducible varieties of finite type over a field k. Then*

$$\dim(Y \times_X Y) \times_X \mathrm{Spec}\,\mathcal{O}_{X,x} \leq \mathrm{codim}\,x + 1$$

for every point $x \in X$.

Proof. It is well-known (see, e.g., [7, Proposition 2.1.1 and Remark 2.1.2]) that semi-smallness of f is equivalent to the condition that every irreducible component of $Y \times_X Y$ has dimension at most $\dim X$, so it suffices to prove that if Z is an irreducible variety of dimension $\leq \dim X$ and $g : Z \to X$ a morphism, then $\dim Z \times_X \mathrm{Spec}\,\mathcal{O}_{X,x} \leq \mathrm{codim}\,x$ for every point $x \in X$.

This reduces to the affine case. We need to prove the following: if $R \to S$ is a homomorphism between two domains that are finitely generated k-algebras such that $\mathrm{Kdim}\,S \leq \mathrm{Kdim}\,R$, then $\mathrm{Kdim}\,S \otimes_R R_\mathfrak{p} \leq \mathrm{Kdim}\,R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}\,R$.

Let $\overline{R}$ denote the image of R in S, and let $\mathfrak{q} \in \mathrm{Spec}\,R$ denote the kernel of $R \to S$. The hypotheses on R and S are such that they, their localizations, and the homomorphic images of these are catenary.

Write d for $\mathrm{Kdim}\,S - \mathrm{Kdim}\,\overline{R}$. Since the Krull dimensions of $\overline{R}$ and S are equal to the transcendence degrees over k of their fraction fields, we can write $S = \overline{R}[x_1, \dots, x_d, \dots, x_n]$ where $\{x_1, \dots, x_d\}$ is algebraically independent over $\overline{R}$, and $x_{d+1}, \dots, x_n$ are algebraic over $\overline{R}[x_1, \dots, x_d]$.

Write $\overline{R}_\mathfrak{p}$ for $\overline{R} \otimes_R R_\mathfrak{p}$. Then $S \otimes_R R_\mathfrak{p} = \overline{R}_\mathfrak{p}[x_1, \dots, x_n]$. The Krull dimension of an extension $C[x]$ is equal to either $\mathrm{Kdim}\,C$ or $\mathrm{Kdim}\,C + 1$ depending on whether x is algebraic over C or not, so an induction argument shows that $\mathrm{Kdim}\,S \otimes_R R_\mathfrak{p} \leq \mathrm{Kdim}\,\overline{R}_\mathfrak{p} + d$. Thus

$$\begin{aligned}\mathrm{Kdim}\,S \otimes_R R_\mathfrak{p} &\leq \mathrm{Kdim}\,\overline{R}_\mathfrak{p} + \mathrm{Kdim}\,S - \mathrm{Kdim}\,\overline{R}\\ &\leq \mathrm{Kdim}\,\overline{R}_\mathfrak{p} + \mathrm{Kdim}\,S - \mathrm{Kdim}\,R + \mathrm{ht}\,\mathfrak{q},\end{aligned}$$

where $\mathrm{ht}\,\mathfrak{q}$ denotes the height of $\mathfrak{q}$. But $\mathrm{ht}\,\mathfrak{q} \leq \mathrm{ht}\,\mathfrak{q}R_\mathfrak{p}$ so

$$\begin{aligned}\operatorname{Kdim} S \otimes_R R_{\mathfrak{p}} &\le \operatorname{Kdim} \overline{R}_{\mathfrak{p}} + \operatorname{Kdim} S - \operatorname{Kdim} R + \operatorname{ht} \mathfrak{q} R_{\mathfrak{p}} \\ &= \operatorname{Kdim} R_{\mathfrak{p}} + \operatorname{Kdim} S - \operatorname{Kdim} R,\end{aligned}$$

and the result follows because $\operatorname{Kdim} S \le \operatorname{Kdim} R$.

4.3. For the rest of the section we will be interested in schemes of finite type over $\mathbb{C}$ with a $\mathbb{C}^*$-action. All morphisms will be $\mathbb{C}^*$-equivariant. We always consider $\mathbb{C}$ to have the multiplicative action of $\mathbb{C}^*$. In case $f : X \longrightarrow \mathbb{C}$ is a $\mathbb{C}^*$–equivariant morphism, we will denote the fibres $f^{-1}(s)$ by X_s. Note that $X_s \cong X_1$ for all non–zero $s \in \mathbb{C}$.

We say that an affine variety X has an *expanding* $\mathbb{C}^*$–action if the corresponding $\mathbb{Z}$–grading $\mathbb{C}[X] = \oplus_{i\in\mathbb{Z}}\mathbb{C}[X]_i$ is concentrated in non–negative degrees.

4.4. The following lemma can be compared with [23, II.4.2, Satz 2].

Lemma. *Let X be an irreducible variety and $f : X \longrightarrow \mathbb{C}$ be a $\mathbb{C}^*$-equivariant morphism. Let $Z \subset X_s$ be an irreducible subvariety. Then either $\overline{\mathbb{C}^*Z} \cap X_0$ is empty or*

$$\dim(\overline{\mathbb{C}^*Z} \cap X_0) = \dim Z.$$

Furthermore, if X is an affine variety with expanding $\mathbb{C}^*$–action, then $\overline{\mathbb{C}^*Z} \cap X_0$ is non–empty.

Proof. Suppose $\overline{\mathbb{C}^*Z} \cap X_0$ is non–empty. Since $\overline{\mathbb{C}^*Z}$ is irreducible and $\dim(\overline{\mathbb{C}^*Z}) = \dim Z + 1$, we have $\dim(\overline{\mathbb{C}^*Z} \cap X_0) \le \dim Z$. On the other hand, the dimension of the fibers of the restriction $f|_{\overline{\mathbb{C}^*Z}} : \overline{\mathbb{C}^*Z} \to \mathbb{C}$ is minimal on a dense open set of $\mathbb{C}$. Since the $\mathbb{C}^*$–action identifies the fibres of this map over non–zero elements of $\mathbb{C}$, we see that the minimal fibre dimension is bounded below by $\dim Z$, as required.

Now assume that X is an affine variety with expanding $\mathbb{C}^*$–action. Let I be the ideal of $\mathbb{C}[X]$ annihilating Z. Then the ideal corresponding to $\overline{\mathbb{C}^*Z} \cap X_0$ is $\operatorname{gr} I$, the ideal consisting of leading terms of elements of I, [23, II.4.2, Satz 3]. In particular, as I is proper, so too is $\operatorname{gr} I$. Thus $\overline{\mathbb{C}^*Z} \cap X_0$ is non–empty.

4.5. We need a simple lemma.

Lemma. *Suppose there is a commutative diagram of $\mathbb{C}^*$-equivariant morphisms*

$$\begin{tikzcd} Y \arrow[rr, "\pi"] \arrow[dr] & & X \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

where π is proper. If $Z \subset Y_s$ is an irreducible subvariety, then

$$\pi(\overline{\mathbb{C}^*Z} \cap Y_0) = \overline{\mathbb{C}^*\pi(Z)} \cap X_0. \tag{1}$$

In particular, if $\overline{\mathbb{C}^*Z} \cap Y_0$ is non–empty, then $\dim(\overline{\mathbb{C}^*Z} \cap Y_0) = \dim Z$ *and* $\dim \pi(\overline{\mathbb{C}^*Z} \cap Y_0) = \dim \pi(Z)$.

Proof. First we show that the right-hand side of (1) is contained in the left-hand side. Let $x \in \overline{\mathbb{C}^*\pi(Z)} \cap X_0$. Certainly, $\mathbb{C}^*\pi(Z) = \pi(\mathbb{C}^* Z) \subset \pi(\overline{\mathbb{C}^* Z})$; but the last term is closed because π is proper, so $\overline{\pi(\mathbb{C}^* Z)} \subset \pi(\overline{\mathbb{C}^* Z})$. Hence $x = \pi(y)$ for some $y \in \overline{\mathbb{C}^* Z}$; but $x \in X_0$, so $y \in Y_0$, whence $x \in \pi(\overline{\mathbb{C}^* Z} \cap Y_0)$.

The proof that the left-hand side of (1) is contained in the right-hand side does not depend on the hypothesis that π is proper: For any subsets W and W' of Y, $\pi(W \cap W') \subset \pi(W) \cap \pi(W')$, and $\pi(\overline{W}) \subset \overline{\pi(W)}$. Thus $\pi(\overline{W} \cap W') \subset \pi(\overline{W}) \cap \pi(W') \subset \overline{\pi(W)} \cap \pi(W')$. Now apply this with $W = \mathbb{C}^* Z$ and $W' = Y_0$.

Under the non–emptiness hypothesis, the equality of dimensions follows from Lemma 4.3.

4.6. The following result allows us to deform semi–small morphisms.

Lemma. *Let X be an affine variety with expanding $\mathbb{C}^*$–action and suppose that we have a commutative diagram of $\mathbb{C}^*$-equivariant morphisms*

$$\begin{tikzcd} Y \arrow[rr, "\pi"] \arrow[dr] & & X \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

where π is proper. Assume that $\dim Y_0 = \dim Y_s$ *and* $\dim X_0 = \dim X_s$ *for all $s \in \mathbb{C}$. If $\pi_0 : Y_0 \longrightarrow X_0$ is semismall, so too is $\pi_s : Y_s \longrightarrow X_s$ for all s.*

Proof. Let $Z \subset Y_s$ be an irreducible subvariety. Set $Z_0 := \overline{\mathbb{C}^* Z} \cap Y_0$. As X has an expanding action, $\pi(Z_0) = \overline{\mathbb{C}^*\pi(Z)} \cap X_0$ is non–empty by Lemmas 4.3 and 4.5. Thus the semi–small hypothesis shows

$$2\dim Y_0 - 2\dim Z_0 \geq \dim X_0 - \dim \pi(Z_0). \tag{2}$$

By Lemma 4.5, we may replace $\dim Z_0$ and $\dim \pi(Z_0)$ in this inequality by $\dim Z$ and $\dim \pi(Z)$. Now the lemma follows by replacing $\dim Y_0$ and $\dim X_0$ in this inequality by $\dim Y_s$ and $\dim X_s$.

5. Application

5.1. A representation of $H_{\mathbf{0}} = \mathbb{C}[V] * \Gamma_n$ is called a Γ_n*–constellation* if its restriction to $\mathbb{C}\Gamma_n$ is isomorphic to the regular representation. A constellation M is a *cluster* if it is generated as a $\mathbb{C}[V]$ module by M^{Γ_n}, the copy of the trivial representation it contains.

We write $K(\Gamma_n)$ for the Grothendieck group of $\mathbb{C}\Gamma_n$ and α for the class of regular representation. Let $\lambda : K(\Gamma_n) \longrightarrow \mathbb{R}$ be a linear function such that $\lambda(\alpha) = 0$. Following 3.4, a constellation M is λ (semi–)stable if for every proper $H_{\mathbf{0}}$–submodule $N \subset M$ we have $\lambda(N)(\leq) < 0$. For generic λ there is a moduli space of λ–stable Γ_n–constellations, a projective scheme over Spec $Z_{\mathbf{0}} = V/\Gamma_n$.

If λ is chosen so that $\lambda(\alpha) = 0$ and, for each simple Γ_n-module S,

$$\lambda(S) = \begin{cases} 1 & \text{if } S \text{ is trivial} \\ < 0 & \text{if } S \text{ is not trivial} \end{cases}$$

then a constellation is λ-stable if and only if it is a cluster.

5.2. The following construction was given in [30, Corollaries 3 and 4]. Let X_Γ be the minimal resolution of the Kleinian singularity $\mathbb{C}^2/\Gamma$. We have maps

$$\mathrm{Hilb}^n(X_\Gamma) \longrightarrow \mathrm{Sym}^n(X_\Gamma) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2/\Gamma) \cong V/\Gamma_n,$$

where the first map is the Hilbert–Chow map, [27, Chapter 1], and the second arises from functoriality. Since X_Γ is symplectic, so too is $\mathrm{Hilb}^n(X_\Gamma)$ and thus the composition is a crepant resolution of V/Γ_n, see 3.2.

5.3. Let $\mathrm{Hilb}^{S_n}(X_\Gamma^n)$ denote the S_n–Hilbert scheme of Ito and Nakamura, [18, Introduction and Sect. 8.2]. The following result is due to Haiman: we include our own outline of the proof for the reader's benefit.

Theorem ([15, Section 7.2.3], [16]). *There is an isomorphism between* $\mathrm{Hilb}^n(X_\Gamma)$ *and* $\mathrm{Hilb}^{S_n}(X_\Gamma^n)$. *In particular, there exists* λ *such that* $\mathrm{Hilb}^n(X_\Gamma)$ *is a moduli space of* λ*–stable* Γ_n *constellations.*

Proof. The isomorphism follows from the $n!$–conjecture applied to the smooth surface X_Γ, [14, Sect. 5.2] and [15]. There is a commutative diagram of S_n-equivariant morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{p} & X_\Gamma^n \\ {\scriptstyle q}\big\downarrow & & \big\downarrow \\ \mathrm{Hilb}^{S_n}(X_\Gamma^n) & \longrightarrow & V/\Gamma_n \end{array}$$

in which Σ is the reduced fibre product and q is finite and flat.

It is well-known that X_Γ is a fine moduli space for Γ-clusters on $\mathbb{C}^2$. We write $\mathcal{B}$ for the locally free sheaf on X_Γ that is the universal family of Γ-clusters. The obvious permutation action makes $\mathcal{B}^{\boxtimes n}$ an S_n-equivariant sheaf on X_Γ^n. Let $\mathcal{P}$ denote $q_* p^* \mathcal{B}^{\boxtimes n}$. Because q and p are S_n-equivariant $\mathcal{P}$ is an S_n-equivariant sheaf on $\mathrm{Hilb}^{S_n}(X_\Gamma^n)$. Since the S_n-action on $\mathrm{Hilb}^{S_n}(X_\Gamma^n)$ is trivial this means that S_n acts as automorphisms of $\mathcal{P}$.

The ring homomorphism $\mathbb{C}[x, y] * \Gamma \to \mathrm{End}\,\mathcal{B}$ induces homomorphisms $\mathbb{C}[x, y]^{\otimes n} * \Gamma^n \to \mathrm{End}\, p^*\mathcal{B}^{\boxtimes n}$ and $\mathbb{C}[V] * \Gamma^n \to \mathrm{End}\, q_* p^* \mathcal{B}^{\boxtimes n} = \mathrm{End}\,\mathcal{P}$. Combining the last of these with the S_n-action produces a ring homomorphism $\mathbb{C}[V] * \Gamma_n \to \mathrm{End}\,\mathcal{P}$.

Consider $\lambda : K(\Gamma_n) \to \mathbb{R}$ of the form

$$\lambda(M) = C\rho(M|_{\Gamma^n}) + \sigma(M|_{S_n})$$

where $\rho : K(\Gamma^n) \to \mathbb{R}$ and $\sigma : K(S_n) \to \mathbb{R}$ are such that stable constellations are clusters. It can be shown that for a suitable choice of $C \gg 1$, the geometric fibers $\mathcal{P}(x) := \mathcal{P}/\mathfrak{m}_x\mathcal{P}$ of $\mathcal{P}$ are λ-stable Γ_n–constellations, and hence that $\mathcal{P}$ is a family of λ–stable Γ_n–constellations.

The fixed points subsheaf $\mathcal{P}^{\Gamma_{n-1}}$ is the universal family of $\mathbb{C}[x, y]*\Gamma$-modules whose fibres have n copies of the regular representation of Γ [15, Prop. 7.2.12].

Let M_λ be the moduli space of λ–stable Γ_n–constellations and $\mathcal{S}$ the universal family on it of λ-stable $\mathbb{C}[V] * \Gamma_n$-constellations.

The homomorphisms $\phi : \mathbb{C}[V] * \Gamma_n \to \operatorname{End}\mathcal{P}$ and $\psi : \mathbb{C}[V] * \Gamma_n \to \operatorname{End}\mathcal{S}$ restrict to homomorphisms $\phi' : \mathbb{C}[x, y]*\Gamma \to \operatorname{End}\mathcal{P}^{\Gamma_{n-1}}$ and $\psi' : \mathbb{C}[x, y]*\Gamma \to \operatorname{End}\mathcal{S}^{\Gamma_{n-1}}$.

Since $\mathcal{P}$ is a family of λ–stable Γ_n–constellations, there is a morphism $f : \operatorname{Hilb}^n(X_\Gamma) \longrightarrow M_\lambda$ such that $f^*\mathcal{S} \cong \mathcal{P}$ and $\phi = f^*\psi$. Thus $\phi' = f^*\psi'$. Similarly, by the universal property of $\mathcal{P}^{\Gamma_{n-1}}$, there exists $g : M_\lambda \longrightarrow \operatorname{Hilb}^n(X_\Gamma)$ such that $g^*\mathcal{P}^{\Gamma_{n-1}} \cong \mathcal{S}^{\Gamma_{n-1}}$ and $\phi' = g^*\psi'$.

Both M_λ and $\operatorname{Hilb}^n(X_\Gamma)$ have trivial Γ_n-action so f and g are automatically Γ_n–equivariant. Therefore

$$(gf)^*\mathcal{P}^{\Gamma_{n-1}} \cong f^*(\mathcal{S}^{\Gamma_{n-1}}) \cong \mathcal{P}^{\Gamma_{n-1}}.$$

Notice too that $f^*g^*\psi' = \psi'$. Since $\operatorname{Hilb}^n(X_\Gamma)$ is a fine moduli space with universal family $\mathcal{P}^{\Gamma_{n-1}}$, it follows that $gf = \operatorname{Id}$.

There is a non-empty open subset U of V/Γ_n such that the natural maps $\alpha : \operatorname{Hilb}^n X_\Gamma \to V/\Gamma_n$ and $\beta : M_\lambda \to V/\Gamma_n$ restrict to isomorphisms $\alpha^{-1}(U) \to U$ and $\beta^{-1}(U) \to U$. The closure Y_λ of $\beta^{-1}(U)$ is the unique irreducible component of M_λ that maps birationally to V/Γ_n. Since f and g are morphisms of V/Γ_n-schemes and $fg = \operatorname{Id}$ they restrict to mutually inverse isomorphisms between $\alpha^{-1}(U)$ and $\beta^{-1}(U)$ and hence between their closures. But $\operatorname{Hilb}^n X_\Gamma$ is irreducible, so f and g yield an isomorphism $\operatorname{Hilb}^n X_\Gamma \cong Y_\lambda$.

5.4. As noted in [32, Section 4.4] and [15] the previous theorem, together with the main result in [2], has the following important consequence.

Corollary. *The derived categories* $D^b(\operatorname{coh}\operatorname{Hilb}^n(X_\Gamma))$ *and* $D^b(\operatorname{mod} H_{\mathbf{0}})$ *are equivalent.*

Proof. This follows from Theorem 5.2 since the resolution $\operatorname{Hilb}^n(X_\Gamma) \longrightarrow V/\Gamma_n$ is crepant, hence semismall by 4.1, and so, using Lemma 4.2, satisfies the hypothesis of Theorem 3.8.

5.5. Set $\Theta = \{\mu : K(\Gamma_n) \longrightarrow \mathbb{R} : \mu(\alpha) = 0\}$. Let λ be the element in Θ given by Theorem 5.2. Define $\Theta_\lambda^+ = \{0 \neq M \subset \mathbb{C}\Gamma_n : \lambda(M) > 0\}$ and $\Theta_\lambda^- = \{0 \neq M \subset \mathbb{C}\Gamma_n : \lambda(M) < 0\}$. If M is a proper Γ_n–submodule of the regular representation such that $\lambda(M) = 0$ we can perturb λ to μ so that $\Theta_\lambda^+ \cup \{M\} \subseteq \Theta_\mu^+$ and $\Theta_\lambda^- \subseteq \Theta_\mu^-$. The λ–stable constellations are the same as the μ–stable constellations since λ is generic. Notice that every μ-semistable constellation is stable. Thus, without loss of generality, we may replace λ by μ and assume that $\lambda(M) \neq 0$ for all proper subrepresentations of a λ-stable constellation M.

5.6. Let $H_{u\mathbf{c}}$ be the flat family of symplectic reflection algebras defined in 2.4.

We will now use Van den Bergh's result in Theorem 3.8 to extend Corollary 5.4 to the deformations $Y_\mathbf{c} \to X_\mathbf{c}$ where $Y_\mathbf{c}$ is a suitable moduli space of $H_\mathbf{c}$-modules.

In Section 3.11 α denoted the element of $\mathbb{Z}^p$ defined by

$$\alpha_i = \operatorname{rank}_{Z_{u\mathbf{c}}} e_i H_{u\mathbf{c}} e = \dim E_i$$

where E_i is the irreducible representation of Γ_n corresponding to the central idempotent $e_i \in \mathbb{C}\Gamma_n \subset H_\mathbf{c}$. Therefore under the isomorphism $K(\Gamma_n) \to \mathbb{Z}^p$, $[E_i] \mapsto (\delta_{1i}, \ldots, \delta_{ni})$, we have $[\mathbb{C}\Gamma_n] \mapsto \alpha$. In particular, the use of α in the last few subsections is compatible with the use of α in Section 3.11.

Let $\lambda : K(\Gamma_n) \to \mathbb{R}$ be generic for α. Let W be the moduli space, as constructed in Section 3, of λ–stable $H_{u\mathbf{c}}$–modules isomorphic to $\mathbb{C}\Gamma_n$ (equivalently, of dimension α), and let Y be the irreducible component of W that maps birationally to $\mathsf{Sm}(X_{u\mathbf{c}})$.

There is a natural $\mathbb{C}^*$-equivariant map $W \to \mathbb{C}$ and its restriction to Y fits into the following commutative diagram in the $\mathbb{C}^*$–equivariant category

$$\begin{tikzcd} Y \arrow[rr, "\pi"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & \mathbb{C} & \end{tikzcd}$$

The horizontal arrow is obtained by taking the central character.

Theorem. *Keep the above notation.*

1. *The fibre* $Y_\mathbf{c} := f^{-1}(1)$ *is a crepant resolution of* $X_\mathbf{c} = \operatorname{Spec} Z_\mathbf{c}$.
2. *There is an equivalence of categories between* $D^b(\operatorname{coh} Y_\mathbf{c})$ *and* $D^b(\operatorname{mod} H_\mathbf{c})$.

Proof. For $\tau \in \mathbb{C}$, we write W_τ for the fiber of $W \to \mathbb{C}$ over τ; we also define $Y_\tau = f^{-1}(\tau)$ and $X_\tau = g^{-1}(\tau)$. By Theorem 3.8 and Lemma 4.2, it is enough to show that Y_1 is the irreducible component of W_1 that is birational to $\mathsf{Sm}(X_1)$, and that the restriction of π to a morphism $Y_1 \longrightarrow X_1$ is semi–small.

The variety Y_τ is a moduli space of λ–stable $H_{\tau\mathbf{c}}$–modules of dimension $\alpha = \underline{\dim}(\mathbb{C}\Gamma_n)$. By Lemma 2.8, Y_τ contains the irreducible component of W_τ that maps birationally to $\mathsf{Sm}X_\tau$. In particular, $\dim Y_\tau \geq \dim X_\tau$; in fact, these dimensions are equal because Y is irreducible of dimension $\dim X = \dim X_\tau + 1$. If τ is non–zero, then Y_τ is irreducible since $\overline{\mathbb{C}^* Y_\tau}$ is a closed subset of Y of the same dimension as Y. On the other hand, if $\tau = 0$ then [2, Section 8] combined with Corollary 5.4 shows that this particular irreducible component is a connected component of the moduli space W_0. By 2.9, X is normal, so Zariski's main theorem implies that Y_0 is connected [17, Corollary III.11.4], and we see that Y_0 is also irreducible. We deduce that for any τ, Y_τ is the irreducible component of W_τ that is birational to $\mathsf{Sm}(X_\tau)$.

The semi–smallness follows from Lemma 4.6 since the restriction of π to $Y_0 \longrightarrow X_0$ is semi–small, being the crepant resolution of Theorem 5.2.

5.7. Let $\mathfrak{m}_x$ be the maximal ideal of $Z_{\mathbf{c}}$ corresponding to $x \in X_{\mathbf{c}}$. The simple $H_{\mathbf{c}}$-modules with central character x are precisely the simple modules of the finite dimensional algebra $H_{\mathbf{c}}/\mathfrak{m}_x H_{\mathbf{c}}$.

Corollary. *Let $\pi_{\mathbf{c}} : Y_{\mathbf{c}} \longrightarrow X_{\mathbf{c}}$ be the crepant resolution above. There is an equivalence of triangulated categories between $D^b_x(\mathrm{coh}\, Y_{\mathbf{c}})$ and $D^b_x(\mathrm{mod}\, H_{\mathbf{c}})$. In particular there is an isomorphism between the Grothendieck groups $K(\pi_{\mathbf{c}}^{-1}(x))$ and $K(H_{\mathbf{c}}/\mathfrak{m}_x H_{\mathbf{c}})$.*

Proof. The first sentence has already been noted in 3.9. By devissage, the Grothendieck groups of $D^b_x(\mathrm{coh}\, Y_{\mathbf{c}})$ and $D^b_x(\mathrm{mod}\, H_{\mathbf{c}})$ are isomorphic to $K(\pi_{\mathbf{c}}^{-1}(x))$ and $K(H_{\mathbf{c}}/\mathfrak{m}_x H_{\mathbf{c}})$ respectively, thus confirming the second sentence.

5.8. When $n = 1$ the varieties $X_{\mathbf{c}}$ are deformations of Kleinian singularities $\mathbb{C}^2/\Gamma$ and any crepant resolution coincides with the minimal resolution. Hence, by the McKay correspondence, the K–theory of the fibre $f^{-1}(x)$ is completely determined by the type of orbifold singularity at $x \in X_{\mathbf{c}}$. Indeed, suppose the singularity at $x \in X_{\mathbf{c}}$ is locally of the form $\mathbb{C}^2/G$ for some finite subgroup G of $SL_2(\mathbb{C})$. Then the rank of $K(f^{-1}(x))$ equals the number of irreducible representations of G. On the other hand, the algebras $H_{\mathbf{c}}$ are deformed preprojective algebras. These algebras also depend only on the type of orbifold singularity at $x \in X_{\mathbf{c}}$, since there is a "slice" theorem which reduces the representation theory to the case of the point $0 \in \mathbb{C}^2/G$, [6, Corollary 4.10]. Thus the rank of $K(H_{\mathbf{c}}/\mathfrak{m}_x H_{\mathbf{c}})$ also equals the number of irreducible representations of G, as expected.

Of course, in the case of arbitrary n but $\mathbf{c} = 0$, the corollary (which is an immediate consequence of Haiman's work) recovers the generalised McKay correspondence proved by Kaledin, [20], for the orbifold singularities appearing locally in V/Γ_n.

5.9. Corollary 5.7 gives us a geometric description for the number of simple modules in $H_{\mathbf{c}}/\mathfrak{m}_x H_{\mathbf{c}}$. In practice this is not immediately applicable as we have no geometric understanding of $Y_{\mathbf{c}}$. There is, however, evidence to suggest that the following question has a positive answer:

Question. Let V have Gorenstein singularities and let $v \in V$. Is rank $K(f^{-1}(v))$ independent of the choice of crepant resolution $f : \tilde{V} \longrightarrow V$?

Indeed, [8, Proposition 6.3.2] shows that the mixed Hodge polynomial (of Borel–Moore homology) of $f^{-1}(x)$ is independent of the choice of resolution. Thus if the homology groups of the fibres are spanned by algebraic cycles (as seems reasonable given the results of [20] and the comments below), the answer is "yes". Furthermore, [1, Section 5] conjectures that all crepant resolutions of X have equivalent bounded derived categories of coherent sheaves. Confirmation of this conjecture would also give a positive answer.

As mentioned in Remark 3.2 in the particular case of $X_{\mathbf{c}}$, it is possible to show that there is a crepant resolution which can be described as a quiver variety, [26]. Here the K–theory of the fibres has been studied, and is related to weight spaces of integrable representations of Kac–Moody Lie algebras. Hence it is reasonable to expect the number of simple $H_{\mathbf{c}}/\mathfrak{m}_x H_{\mathbf{c}}$–modules also has this interesting description. We will return to this in future work.

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