Differentiation Structured Worksheet 1 Solutions

1. Differentiate $x \sin \frac{1}{x}$ with respect to x, and hence evaluate the derivative of this function at $x = \frac{6}{\pi}$ and $x = -2$.
(You should be able to give exact expressions and do so without the use of a calculator!)

Solution We begin by using the **Product** Rule. We can write $x \sin \frac{1}{x}$ as $f(x)g(x)$ where $f(x) = \boldsymbol{x}$ and $\boldsymbol{g(x)} = \sin \frac{1}{x}$. By the **Product** Rule,

$$\frac{d}{dx}\big(f(x)g(x)\big) = \boldsymbol{f'(x)g(x) + g'(x)f(x)}.$$

Now $f'(x) = \mathbf{1}$, while to find $g'(x)$ we need the **Chain** Rule. We write

$$g(x) = \sin\big(u(x)\big) \text{ where } u(x) = \boldsymbol{\frac{1}{x}}.$$

So

$$\begin{aligned} g'(x) &= \frac{d}{dx}\big(\sin u(x)\big) \\ &= \frac{d}{du}(\sin u)\frac{du}{dx} \\ &= \cos u \times \boldsymbol{\frac{-1}{x^2}} \\ &= \boldsymbol{-\frac{1}{x^2}} \cos \frac{1}{x}. \end{aligned}$$

Hence

$$\frac{d}{dx}\left(x \sin \frac{1}{x}\right) = \boldsymbol{\sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}}.$$

By substituting in, the derivative at $x = \frac{6}{\pi}$ is $\boldsymbol{\frac{1}{2} - \frac{\pi}{4\sqrt{3}}}$, while the derivative at $x = -2$ is $\boldsymbol{\frac{1}{2} \cos \frac{1}{2} - \sin \frac{1}{2}}$.

(To get these expressions, we have used the facts that $\sin \frac{\pi}{6} = \frac{1}{2}$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\cos\left(-\frac{1}{2}\right) = \cos \frac{1}{2}$ and $\sin\left(-\frac{1}{2}\right) = -\sin \frac{1}{2}$. Make sure that you understand why each of these is true.)

2. Differentiate the function

$$\frac{(2x+1)^3}{(3x-2)^5}$$

with respect to x, simplifying your answer as far as possible. Hence find the derivative of this function at $x = 0$. (Once again it should be possible to give an exact answer without using a calculator.)

Solution We begin by using the **Quotient** Rule. We write the function as $\frac{f(x)}{g(x)}$, where $f(x) = \mathbf{(2x+1)^3}$ and $g(x) = \mathbf{(3x-2)^5}$. By the **Quotient** Rule,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\boldsymbol{f'(x)g(x) - g'(x)f(x)}}{\boldsymbol{g(x)^2}}.$$

To find $f'(x)$ and $g'(x)$ we need the **Chain** Rule. We have that $f(x) = \big(u(x)\big)^3$, where $u(x) = \mathbf{2x+1}$, and $g(x) = \big(v(x)\big)^5$, where $v(x) = \mathbf{3x+2}$. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}\Big(\big(u(x)\big)^3\Big) & \text{and} \qquad g'(x) &= \frac{d}{dx}\Big(\big(v(x)\big)^5\Big) \\ &= \frac{d}{du}(u^3)\frac{du}{dx} & &= \frac{d}{dv}(v^5)\frac{dv}{dx} \\ &= 3u^2 \times \mathbf{2} & &= \mathbf{5v^4} \times 3 \\ &= \mathbf{6}(2x+1)^{\mathbf{2}} & &= \mathbf{15}(3x+2)^{\mathbf{4}}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{\mathbf{6(2x+1)^2(3x-2)^5 - 15(3x-2)^4(2x+1)^3}}{\mathbf{(3x-2)^{10}}} \\ &= \frac{3(2x+1)^2(3x-2)^4\mathbf{(6x-4-10x-5)}}{(3x-2)^{10}} \\ &= \mathbf{-}\frac{3(2x+1)^2\mathbf{(4x+9)}}{(3x-2)^6}. \end{aligned}$$

Hence the derivative at $x = 0$ is $\mathbf{-\frac{27}{64}}$.