Differentiation Structured Worksheet 2 Solutions

1. Differentiate the function

$$f(x) = \tan\left(\frac{\pi x^2}{2(1+x^2)}\right)$$

with respect to x, and hence evaluate the derivative of f at $x = \sqrt{2}$.
Does $f'(x)$ exist for all real numbers x?
(Hint: Observe that $0 \leqslant x^2 < 1 + x^2$ for all $x \in \mathbb{R}$.)

Solution By the **Chain** Rule,

$$\begin{aligned} f'(x) &= \mathbf{sec^2}\left(\frac{\pi x^2}{2(1+x^2)}\right) \times \frac{d}{dx}\left(\boldsymbol{\frac{\pi x^2}{2(1+x^2)}}\right) \\ &= \mathbf{sec^2}\left(\boldsymbol{\frac{\pi x^2}{2(1+x^2)}}\right) \times \frac{\pi}{2}\left(\boldsymbol{\frac{2x(1+x^2) - 2x.x^2}{(1+x^2)^2}}\right) \\ &\qquad \text{(by the } \textbf{Quotient} \text{ Rule)} \\ &= \frac{\pi}{2}\mathbf{sec^2}\left(\boldsymbol{\frac{\pi x^2}{2(1+x^2)}}\right)\left(\boldsymbol{\frac{2x(1+x^2-x^2)}{(1+x^2)^2}}\right) \\ &= \boldsymbol{\frac{\pi x}{(1+x^2)^2}}\,\mathbf{sec^2}\left(\boldsymbol{\frac{\pi x^2}{2(1+x^2)}}\right). \end{aligned}$$

Hence

$$f'(\sqrt{2}) = \boldsymbol{\frac{\pi\sqrt{2}}{3^2}}\,\mathbf{sec^2}\left(\boldsymbol{\frac{2\pi}{2.3}}\right) \boldsymbol{=} \boldsymbol{\frac{\pi\sqrt{2}}{9\cos^2\left(\frac{\pi}{3}\right)}} \boldsymbol{=} \boldsymbol{\frac{4\pi\sqrt{2}}{9}}.$$

The derivative exists for all real numbers x since for any $x \in \mathbb{R}$,

$$\begin{aligned} 0 \leqslant x^2 < 1 + x^2 &\implies 0 \leqslant \frac{x^2}{1+x^2} < 1 \\ &\implies 0 \leqslant \frac{\pi x^2}{2(1+x^2)} < \frac{\pi}{2} \\ &\implies \cos^2\left(\frac{\pi x^2}{2(1+x^2)}\right) > 0 \\ &\implies f'(x) \text{ exists.} \end{aligned}$$

2. Differentiate the function

$$g(x) = \sqrt{\frac{1-x}{1+x}}$$

with respect to x, simplifying your answer as far as possible.

For which real values of x is $g'(x)$ defined? What about $g(x)$?

Solution There are (at least) two strategies for attempting this question. We can think of $g(x)$ as either $\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$ or $\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}$. In the first case we would use the Chain Rule followed by the Quotient Rule, while in the second case we would use the Quotient Rule followed by (two applications of) the Chain Rule. We will pursue the first strategy, but you should try the other method for comparison—and for the practice!

By the Chain Rule,

$$\begin{aligned}
g'(x) &= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{1-x}{1+x}\right) \\
&= \frac{(1+x)^{\frac{1}{2}}}{2(1-x)^{\frac{1}{2}}}\left(\frac{-(1+x)-1.(1-x)}{(1+x)^2}\right) \quad \text{by the } \textbf{Quotient} \text{ Rule} \\
&= \frac{(1+x)^{\frac{1}{2}}}{2(1-x)^{\frac{1}{2}}} \cdot \frac{-2}{(1+x)^2} \\
&= \frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}}.
\end{aligned}$$

We see that $g'(x)$ is defined if and only if $1-x>0$ and $1+x>0$, i.e. when $x \in (-1,1)$. Meanwhile $g(x)$ is defined if and only if $1-x \geqslant 0$ and $1+x>0$, i.e. when $x \in (-1,1]$. (Note the subtle difference!)