Differentiation Structured Worksheet 3 Solutions

1. Show that

$$\frac{d}{dx}\big((1+\cos x)^3 \sin x\big) = (1+\cos x)^3(4\cos x - 3).$$

(Hint: you may need to use a famous formula involving trigonometric functions.)

Solution By the **Product** and **Chain** Rules,

$$\begin{aligned}
\frac{d}{dx}\big((1+\cos x)^3 \sin x\big) &= \boldsymbol{-3(1+\cos x)^2 \sin^2 x + (1+\cos x)^3 \cos x} \\
&= \boldsymbol{(1+\cos x)^2\big(\cos x + \cos^2 x - 3\sin^2 x\big)} \\
&= \boldsymbol{(1+\cos x)^2\big(\cos x + \cos^2 x - 3 + 3\cos^2 x\big)} \\
&= \boldsymbol{(1+\cos x)^2\big(4\cos^2 x + \cos x - 3\big)} \\
&= \boldsymbol{(1+\cos x)^2(4\cos x - 3)(\cos x + 1)} \\
&= \boldsymbol{(1+\cos x)^3(4\cos x - 3)}
\end{aligned}$$

To get from the second line to the third line, we used the fact that $\sin^2 x + \cos^2 x = 1$. Note also that on the fourth line we have a quadratic in $\cos x$, which we factorised to get the fifth line.

2. Show that if $f(x) = \tan x$, the Newton Quotient $N(h)$ for f at the point x is given by

$$N(h) = \frac{\tan h}{h}\left(\frac{1+\tan^2 x}{1-\tan x \tan h}\right),$$

and hence deduce that

$$\frac{d}{dx}(\tan x) = 1 + \tan^2 x.$$

Solution The Newton Quotient $N(h)$ is given by

$$\begin{aligned} N(h) &= \frac{1}{h}(\tan(x+h) - \tan x) \\ &= \frac{1}{h}\left(\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x\right) \\ &= \frac{1}{h}\left(\frac{\tan x + \tan h - \tan x + \tan^2 x \tan h}{1 - \tan x \tan h}\right) \\ &= \frac{\tan h}{h}\left(\frac{1+\tan^2 x}{1-\tan x \tan h}\right). \end{aligned}$$

Since $\lim_{h\to 0} \tan h = 0$ and $\lim_{h\to 0} \frac{\tan h}{h} = 1$, we have

$$\lim_{h\to 0} N(h) = 1 \cdot \frac{1+\tan^2 x}{1 - \tan x \cdot 0} = 1 + \tan^2 x.$$

n.b. To prove that $\lim_{h\to 0} \frac{\tan h}{h} = 0$, write

$$\frac{\tan h}{h} = \frac{\sin h}{h} \cdot \frac{1}{\cos h}$$

and use the fact that $\lim_{h\to 0} \frac{\sin h}{h} = 1$.