ARITHMETIC STATISTICS AND THE COHEN–LENSTRA HEURISTICS

TITLES AND ABSTRACTS

Monday 27 June

10:30 – 12:00: Bjorn Poonen
Introduction to the Cohen-Lenstra heuristics
I will discuss the Cohen-Lenstra heuristics, both in their original form and as reinterpreted later by Friedman and Washington and by Ellenberg and Venkatesh. Also I will discuss Delaunay's heuristics for Shafarevich–Tate groups, and equivalent conjectures of Bhargava, Kane, Lenstra, myself, and Rains.

15:30 – 16:30: Wei Ho
Ranks of elliptic curves: heuristics, theorems, and data (revisited!)
In 2007, Bektemirov, Mazur, Stein, and Watkins published a beautiful paper ("Average ranks of elliptic curves: tension between data and conjecture") that described the state of the art in the study of the distribution of ranks of elliptic curves over Q. The last decade has seen progress on these questions in many directions, perhaps the most surprising being genuine theorems giving upper bounds on the average ranks of elliptic curves over Q (led by work of Bhargava-Shankar). In these theorems, the set of elliptic curves over Q are ordered by height, which is a measure of the size of the coefficients defining each curve, as opposed to by conductor, a more arithmetic invariant used to order curves in most previous work.

In this talk, we will discuss the updated heuristics and conjectures in this field as well as some of these theorems and the main ideas behind their proofs. We will compare the expected behavior with data from a recent database of elliptic curves over Q ordered by height (joint work with Balakrishnan, Kaplan, Spicer, Stein, and Weigandt); perhaps the new data resolves some of the old tension?

17:00 – 18:00: Tim Dokchitser
Overview of Selmer parity
I will give a survey of what we know about the parity of Selmer ranks of elliptic curves and abelian varietes and the existing techniques for controlling it. At the end I will sketch the proof of a recent result with Vladimir Dokchitser, that a positive proportion of hyperelliptic curves have Jacobians with odd (respectively, even) 2-Selmer rank.

Tuesday 28 June

09:30 – 10:00: Mark Shusterman
Elementary approaches to class numbers of quadratic number fields
I intend to present some very elementary approaches to understanding the class group of quadratic number fields. These include mainly the pigeonhole prinicple, and Jacobi sums which are used to represent primes and their powers by quadratic forms. Once we know which primes are represnted by a form, we understand which primes split in a certain class field, so we obtain information about class numbers. Joint work with Ohad Avnery.

10:15 – 11:15: Daniel Kane
Average Phi-Selmer of Elliptic Curves
We consider the average size of the Selmer group associated to a 2-isogony of an elliptic curve averaged over all curves with a rational 2-torsion point sorted by height. Unlike many results in this area, we find that the average size is not constant and instead scales like the square root of the logarithm of the height. Additionally, while this problem can be rephrased in terms of a more or less familiar geometry of numbers problem, the techniques required to solve it are substantially different. This is joint work with Zev Klagsbrun.

11:30 – 12:30: Jürgen Klüners
4-Ranks of Class Groups of Quadratic Number Fields and Applications
In this talk we give an overview about some proven cases of the Cohen-Lenstra-Gerth conjecture concerning 4-ranks. We also consider applications like the solvability of the negative Pell equation and the connection of the Cohen-Lenstra conjecture to the asymptotical behaviour of number fields with dihedral groups as Galois groups.

15:30 – 16:30: Ila Varma
The average size of 3-torsion elements in ray class groups of quadratic fields
In 1971, Davenport and Heilbronn determined the mean number of 3-torsion elements in the class groups of quadratic fields, when ordered by discriminant. I will describe some aspects of the proof of Davenport and Heilbronn's theorem; in particular, they prove a relationship via class field theory between the number of 3-torsion ideal classes of quadratic fields and the number of nowhere totally ramified cubic fields over $\mathbb{Q}$. This argument generalizes to give a relationship between 3-torsion elements of the ray class groups of quadratic fields and certain pairs of cubic fields satisfying explicit ramification conditions. I will illustrate how the combination of this fact with Davenport-Heilbronns asymptotics on the number of cubic fields of bounded discriminant allows one to compute the mean number of 3-torsion elements in ray class groups of quadratic fields with fixed conductor. If time permits, I will discuss the analogous theorems computing the mean size of 2-torsion elements in ray class groups of cubic fields ordered by discriminant, generalizing Bhargavas result confirming the only known case of the Cohen-Martinet-Malle heuristics.

17:00 – 18:00: Cornelius Greither
CL heuristics for Galois modules and Iwasawa modules
For a fixed prime p, the sum $\sum_M |Aut(M)|^{-1}$ over all finite $\mathbb{Z}_p$-modules M (of course: modulo isomorphism) converges to the limit $S = \prod_{k \geqslant 1}(1 - p^{-k})^{-1}$. (This is a fundamental fact underpinning the classical CL heuristics.) In this talk we are concerned with similar sums S_R attached to related but bigger rings R. First we look at group rings $R = \mathbb{Z}_p[G]$ with G a finite abelian p-group. For non-cyclic G, S_R diverges, but there is a variant $^{ct}_{R}$ involving cohomologically trivial modules, where the sum not only converges but even has the same "old" limit S. These c.t. modules do occur in nature as minus parts of class groups of suitable cyclotomic fields. For cyclic G, Wittmann showed that the full sum converges (as conjectured by the speaker)

and calculated its limit. In the last part of the talk, we will discuss the case where R is the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$. Here we adjust the weigth $|Aut(M)|^{-1}$ to $|M|^{-1}|Aut(M)|^{-1}$. This is, in a way, dictated by the arithmetic background: one should only look at totally real base fields since over imaginary fields the standard Iwasawa module X is often infinite (so it is not clear what the heuristics should be), but over totally real base fields Greenberg's conjecture predicts that X is finite, and the CL heuristics in the totally real case involve this adjusted weight factor. We obtain the value of the adjusted sum S'_Λ, and we mention a connection with the almost forgotten Kähler zeta function of Λ.

Wednesday 29 June

09:30 – 10:00: Jack Klys
The distribution of 3-torsion in cyclic cubic fields
We extend the Cohen-Lenstra heuristics to the case of 3-torsion in cyclic cubic fields and prove this case by computing all the moments of the 3-rank and obtain a distribution. We follow the methods from recent work of Fouvry and Klüners where they compute the distribution of 4-torsion in quadratic fields. We express the 3-rank in a cubic field as a character sum, and use analytic methods to compute these quantities. Thus far all extensions of the Cohen–Lenstra heuristics have been restricted to p coprime to the degree, except for the case proved by Fouvry and Klüners which was conjectured by Gerth. The distribution we compute turns out to be closely related to theirs.

10:15 – 11:00: Joseph Gunther
Counting Low-Degree Extensions of Function Fields
Recent work of Bhargava, Shankar, and Wang extended results on counting low-degree S_n-extensions to allow any global field as the base field. Their work uses geometry of numbers for both number fields and function fields. We'll show how, in the function field case, one can instead give algebro-geometric proofs, which shed light on the geometry present in the number field situation as well. This is joint work with Daniel Hast and Vlad Matei.

11:15 – 12:00: Amanda Tucker
The statistics of the genus numbers of cubic fields
The genus number of a number field is the degree of the maximal unramified extension of the number field that is obtained as a compositum of the field with an abelian extension of $\mathbb{Q}$. We will explain our recent proof that 96.2% of cubic fields have genus number one and, if time permits, talk about some applications. This represents joint work with Kevin McGown.

12:15 – 13:00: Zev Klagsbrun
Phi-Selmer groups and the Cohen-Lenstra heuristics
Suppose E is an elliptic curve with a cyclic two-isogeny Φ and S_u is the set of squarefree integers d such that the logarithmic Tamagawa ratio $t(E^d)$ is equal to $-u$. We show that the Φ-Selmer ranks of quadratic twists of E by d in S_u are distributed like u-ranks of random groups as defined in Cohen and Lenstra's original work.

Thursday 30 June

09:30 – 10:00: Adam Morgan
Parity of Selmer ranks in quadratic twist families
We study the parity of 2-Selmer ranks in the family of quadratic twists of a fixed principally polarised abelian variety over a number field. Specifically, we prove results about the proportion of twists having odd (resp. even) 2-Selmer rank. This generalises work of Klagsbrun–Mazur–Rubin for elliptic curves and Yu for Jacobians of hyperelliptic curves. Several differences in the statistics arise due to the possibility that the Shafarevich–Tate group (if finite) may have order twice a square.

10:30 – 11:30: Jack Thorne
2-descent on pointed plane quartics
I will describe a formulation of 2-descent on the Jacobians of pointed plane quartics in terms of arithmetic invariant theory, del Pezzo surfaces, and Heisenberg groups.

12:00 – 12:30: Gunter Malle
A class group heuristic based on the distribution of 1-eigenspaces in matrix groups

15:30 – 16:30: Melanie Matchett Wood
Nonabelian Cohen-Lenstra heuristics
We discuss heuristics for the distribution of Galois groups of the maximal unramified extension of imaginary quadratic fields, including work of Boston–Bush–Hajir, Boston–W., and W. In particular, we will give a conjecture for the average number of unramified G-extensions of an imaginary (or real) quadratic field for any finite group G. This involves "corrections" for roots of unity in the base field. We give results towards these conjectures in the function field case, including joint work with Boston.

17:00 – 18:00: Hendrik Lenstra
Adjusting the Cohen–Martinet hypothesis
Twenty-six years ago, Cohen and Martinet formulated a heuristic hypothesis expressing that class groups of number fields in families are governed by certain probability distributions. In joint work with Alex Bartel, it was discovered that their hypothesis is incorrect as stated. I will describe our efforts to turn it into a believable conjecture.

Friday 01 July

09:30 – 10:15: Frank Thorne
Levels of Distribution for Prehomogeneous Vector Spaces
One important technical ingredient in many arithmetic statistics papers is upper bounds for finite exponential sums which arise as Fourier transforms of characteristic functions of orbits. This is typical in results obtaining power saving error terms, treating "local conditions", and/or applying any sort of sieve.

In my talk I will explain what these exponential sums are, how they arise, and what their relevance is. I will also outline a new method, developed by Takashi Taniguchi and myself, for explicitly evaluating them. We carried this out in full for five prehomogeneous vector spaces, and I will explain some pleasant surprises in our end results.

If there is time (doubtful!) I will also describe a new sieve application, proving that there are "many" quartic field discriminants with at most eight prime factors.

10:30 – 11:30: John Voight
Heuristics for narrow class groups and signature ranks of units in number fields
This is joint work with David S. Dummit. We give heuristics for the size of the narrow class group and the signature rank of the unit group of a number field K. Our model is based on computing coranks of groups of matrices related to the '2-Selmer group' for K, using the Friedman–Washington interpretation of the Cohen–Lenstra heuristics for ordinary class groups. We present some evidence, both theoretical and computational, for these heuristics.

11:45 – 12:45: Manjul Bhargava
Statistics for class groups and ideal groups of orders
The Cohen-Lenstra and Cohen-Martinet heuristics focus on class groups of maximal orders in quadratic and higher degree fields. In this talk, we discuss some new phenomena that occur when one ranges over all orders in such number fields, or over orders satisfying specified local conditions. This is joint work with Ila Varma.