Representation theory, exercise sheet 1

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The questions in Section A are supposed to be trivial, and/or to just cover background material that you are expected to already know. You should be able to do them without looking at your notes. Section B is the crucial part. Doing these questions as we go along is more important (and more rewarding, well yes, also harder!) than turning up for lectures. Section C is for those who enjoy a challenge.

Throughout, G denotes a finite group, and K denotes an arbitrary field.

Section A

1. Define the concepts of

   (a) a representation of G over K, and a homomorphism between two such representations,

   (b) the group algebra $K[G]$,

   (c) a module over $K[G]$, and a homomorphism between two such modules.

   Explain why the first and the third concept are really the same thing. This applies to both the objects and the homomorphisms between them.

2. Let G be abelian, $G \cong \mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_k\mathbb{Z}$. Classify all the complex irreducible representations of G.

3. Prove that the 1-dimensional complex representations of G are precisely the ones that factor through G/G'. (Recall that G' is the unique smallest normal subgroup of G with abelian quotient.)

Section B

1. Let G be cyclic of order $n \geq 2$. Find each complex irreducible representation of G as a subrepresentation of the regular representation. Hence decompose the regular representation explicitly as a direct sum of irreducible representations, by giving a change of basis matrix from the standard basis $v_g, g \in G$ to a new basis, such that with respect to the new basis, all the matrices of the G-action become diagonal.

2. (a) Let G be a finite group and let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a complex irreducible representation. Using Schur's Lemma, show that for any element $z \in Z(G)$ of the centre of G, $\rho(z) = \mathrm{diag}(\lambda, \ldots, \lambda)$ for some $\lambda \in \mathbb{C}^\times$.

(b) Let G be a finite group with trivial centre and let H be a subgroup with non-trivial centre. A representation ρ of G can be restricted to H to obtain a representation of H. Show that any $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ that is injective (we say *faithful*) becomes reducible when restricted to H.

3. You have gone into the very profitable character table repair business. You have bought on the flee market a broken character table, where some rows are missing (here, $\omega = \frac{-1+\sqrt{-3}}{2}$, $\zeta = \frac{-1+\sqrt{-7}}{2}$, and the columns are labelled by the sizes of conjugacy classes):

G	1	3	3	7	7
χ_1	1	1	1	ω	$\overline{\omega}$
χ_2	3	ζ	$\bar{\zeta}$	0	0

Complete the character table and describe the group G, e.g. in terms of generators and relations, or otherwise, so that you can sell the complete thing for profit (or keep it in your shelf as a trophy).

4. For your job as character table repair man, you have bought a group analyser that can detect the order of a group, the conjugacy class sizes, and occasionally find some characters. While walking back home after a night out, you find a mysterious finite group lying on the ground. After putting it into your portable group analyser, you find that it has order 720. The analyser also spits out the conjugacy class sizes and two (not necessarily irreducible) characters to you, as follows:

G	1	15	40	90	45	120	144	120	90	15	40
ψ_1	6	2	0	0	2	2	1	1	0	-2	3
ψ_2	21	1	-3	-1	1	1	1	0	-1	-3	0

Show that the group has an irreducible character of degree 16 and write down its values on the conjugacy classes.

Section C

1. (a) Let $\rho : G \to GL_n(\mathbb{R})$ be a real representation of a finite group. Show that there is a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P\rho(g)P^{-1} \in O(n)$, the group of orthogonal matrices, i.e. those satisfying $M^{\mathrm{Tr}}M = I$, or equivalently, the matrices that preserve the standard inner product on $\mathbb{R}^n$: $\langle v, u\rangle = \langle Mv, Mu\rangle$ for all $v, u \in \mathbb{R}^n$. (Hint: start with some inner product on $\mathbb{R}^n$ and construct a new G-invariant inner product out of it.)

(b) Recall that the group $Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1}\rangle$ has an irreducible complex representation

$$x \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Deduce from the first part of the question that this representation of Q_8 cannot be defined over $\mathbb{R}$, i.e. there is no change of basis on $\mathbb{C}^2$ such that x and y are represented by real matrices with respect to the new basis.

2. (a) Let G be a finite group, and $g \in G$. Prove that g is conjugate to g^{-1} if and only if $\chi(g)$ is real for every irreducible character χ of G. We say that the conjugacy class of g is self-inverse in this case.

   (b) Prove that the number of irreducible characters of G that take only real values (so-called *real characters*) is equal to the number of self-inverse conjugacy classes of G (so-called real classes).

3. (a) Show that the complex conjugate of an irreducible character is also an irreducible character.

   (b) Let M be the character table of a finite group G, thought of as a square matrix. Show that $\det M = \pm \det \overline{M}$. Deduce that $\det M$ is either real or purely imaginary.

   (c) Express $|\det M|$ in terms of group theoretic data.