Representation theory, exercise sheet 2

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Throughout, G denotes a finite group, and K denotes any field.

Section A

1. Explicate Mackey's formula in the case $G = S_3$, $H = U = \langle(1,2)\rangle$, and ρ is the non-trivial one-dimensional representation of H over $\mathbb{C}$. Verify that Mackey is right, by decomposing $\operatorname{Res}^G_U \operatorname{Ind}^G_H \rho$ into irreducible H-representations without using Mackey, and then comparing the result with what Mackey tells you.

2. Let $H' \leq H \leq G$. Prove that for any representation ρ of H', $\operatorname{Ind}^G_H \operatorname{Ind}^H_{H'} \rho = \operatorname{Ind}^G_{H'} \rho$, i.e. induction is transitive.

3. Show that if $H \leq G$, and $\mathbf{1}$ denotes the trivial representation over K, then $\operatorname{Ind}^G_H \mathbf{1}$ is isomorphic to the permutation representation $K[G/H]$.

Section B

1. Prove that if G has an abelian subgroup of index n, then every irreducible representation of G has dimension at most n.

2. A character is called monomial if it is induced from a one-dimensional character. Let χ be a monomial character of G and let $H \leq G$. Show that if $\operatorname{Res}_H \chi$ is irreducible, then it is also monomial.

3. Let $H \leq G$, and let $\rho : H \to \mathrm{GL}(V)$ be a representation of H over K. The following is an equivalent definition of $\operatorname{Ind}^G_H \rho$: define W to be the vector space of V-valued functions on G, satisfying a certain transformation property:

$$W = \{f : G \to V \mid f(hg) = \rho(h) \cdot f(g) \ \ \forall h \in H, g \in G\}.$$

   Let G act on W by $\sigma \cdot f = (g \mapsto f(g\sigma)) \in W$ for $\sigma \in G$, $f \in W$.

   (a) Verify that the above really does define a left action of G on W; in other words, verify firstly that $(\sigma\sigma') \cdot f = \sigma \cdot (\sigma' \cdot f)$, and secondly that if $f \in W$, then $\sigma \cdot f \in W$.

   (b) Show that the resulting representation of G is isomorphic to the induced representation as defined in the lectures.

4. This exercise classifies the irreducible representations of semidirect products by abelian groups. Let $A \lhd G$ be abelian, $H \leq G$, $AH = G$, $A \cap H = \{1\}$, so that $G = A \rtimes H$. Then, H acts on the one-dimensional characters of A by ${}^h\chi(a) = \chi(h^{-1}ah)$. For a one-dimensional character χ of A, denote by S_χ its stabiliser in H. Extend χ to AS_χ by

$$\chi(as) = \chi(a) \text{ for } a \in A,\ s \in S_\chi.$$

   (a) Check that this gives a well-defined one-dimensional character χ of AS_χ, i.e. verify that this really is a group homomorphism $AS_\chi \to \mathbb{C}^\times$.

   Let τ be an irreducible character of $S_\chi \cong AS_\chi/A$, lifted to AS_χ. Define $\rho_{\chi,\tau} = \mathrm{Ind}_{AS_\chi}^G(\tau \otimes \chi)$, the induction from AS_χ to G.

   (b) Prove that for any one-dimensional χ and any irreducible character τ of S_χ, $\rho_{\chi,\tau}$ is an irreducible character of G.

   (c) Show that $\rho_{\chi,\tau} = \rho_{\chi',\tau'}$ if and only if χ and χ' are in the same orbit under the action of H on $\mathrm{Irr}(A)$ and $\tau = \tau'$.

   (d) Show that all irreducible characters of G arise as some $\rho_{\chi,\tau}$.

5. Let $G = D_{2n}$, the dihedral group of order n (corresponding to rotations and reflections of a regular n-gon), let $C \leq G$ be the normal cyclic subgroup of order n (corresponding to the rotations). Using Mackey's formula, determine for which irreducible characters χ of C $\mathrm{Ind}_C^G \chi$ is irreducible. Hence, construct the character table of G.

Section C

1. Show that a simple group cannot have a two-dimensional irreducible representation. (**Hint:** You might find it helpful to consider the determinant of such a representation.)

2. Let χ be a character of G that is constant on $G\backslash\{1\}$. Show that $\chi = a\mathbf{1}_G + b\rho_G$ for some $a, b \in \mathbb{Z}$, where ρ_G is the regular character. Show also that if $\ker\chi \neq G$, then $\chi(1) \geq |G| - 1$.