Representation theory, exercise sheet 3

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Throughout, G denotes a finite group, and K denotes any field.

Section A

1. Let N be a normal subgroup of a finite group G, let χ be an irreducible character of G such that $\langle \operatorname{Res}_N \chi, \mathbf{1}\rangle_N \neq 0$. Show that then $N \leq \ker \chi$, i.e. $\operatorname{Res}_N \chi$ is a bunch of copies of $\mathbf{1}$. Now let ϕ be an irreducible character of N. What can you deduce if you know that $\langle \operatorname{Ind}^G \phi, \mathbf{1}\rangle_G \neq 0$?

2. Let $G = S_4$. For each irreducible character χ of G, compute the characters $\chi^{\otimes 2}$, $S^2\chi$ and $\Lambda^2\chi$, and decompose each of them into irreducibles.

3. Repeat with D_{10}, with $G = C_5 \rtimes C_4$, where C_4 is viewed as $C_4 \cong \operatorname{Aut}(C_5)$, and with just about any finite group that you can think of, until you get bored.

4. Recall that both D_8 and Q_8 each have an irreducible two-dimensional complex representation V. Calculate the characters of $\Lambda^2 V$ and $S^2 V$ in both cases and and compute the inner products of these with $\mathbf{1}$.

Section B

1. Let $\rho_1, \ldots, \rho_n$ be the complete set of irreducible complex representations of G and let $\tau_1, \ldots, \tau_m$ be the complete set of irreducible complex representations of H. Regard them as representations of $G \times H$ by lifting from the quotients $G \cong (G \times H)/H$ and $H \cong (G \times H)/G$. Show that $\rho_i \otimes \tau_j$, $1 \leq i \leq n$, $1 \leq j \leq m$ is the complete set of complex irreducible representations of $G \times H$.

2. (a) Let $x_1, \ldots, x_n$ be commuting indeterminates. Show that the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$

thought of as an element of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is the homogeneous polynomial

$$f(x_1, \ldots, x_n) = \prod_{i<j} (x_j - x_i).$$

In particular, if $\alpha_1, \dots, \alpha_n$ are compelex numbers, then

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{vmatrix} = 0 \iff \alpha_i = \alpha_j \text{ for some } i \neq j.$$

(Hint: first show that any $(x_i - x_j)$ divides f in $\mathbb{C}[x_1, \dots, x_n]$.)

(b) Show that if χ is a faithful character of G (i.e. one with trivial kernel) that takes on exactly n distinct values on G, then any irreducible character of G is a constituent of some χ^i for $0 \leq i < n$. Note that you already know this in the special case when χ is the regular character (for what n?). (Hint: you will need to use the first part of the question.)

3. Let G be a group of order 2^n, $n \in \mathbb{N}$.

   (a) How many self-dual one-dimensional characters does G have?

   (b) Let $G' = [G, G]$ be the derived subgroup. Show that G/G' cannot have order 2, unless $|G| = 2$. (Recall that a p-group always has a normal index p subgroup.)

   (c) Show that if the number of involutions (= elements of order 2) in G is congruent to 1 modulo 4, then either G is cyclic or $[G : G'] = 4$. (Hint: you might find it useful to consider the formula $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$ modulo 8.)

   Remark: The 2-groups satisfying $[G : G'] = 4$ have been classified by O. Taussky: they are either dihedral, or semi-dihedral, or generalised quaternion. In each of these, the number of involutions really is $\equiv 1 \pmod 4$.

Section C

1. (a) Compute the dimension of $S^n V$ and of $\Lambda^n V$ for all n.

   (b) Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of g on V. Compute the eigenvalues of g on $S^n V$ and on $\Lambda^n V$ in terms of the λ_i.

   (c) Let $f(x) = \det(g - xI)$ be the characteristic polynomial of g on V. Express the character $\chi_{\Lambda^n V}(g)$ in terms of the coefficients of f.

   (d) (Harder:) Find a relationship between $\chi_{S^n V}(g)$ and f.