Representation theory, exercise sheet 4

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Throughout, G denotes a finite group.

Section A

1. For some of your favourite groups (e.g. first few symmetric groups, alternating groups, dihedral groups, some semi-direct products/wreath products), compute an expression for the trivial character as a rational linear combination of characters $\mathrm{Ind}_H^G \mathbf{1}$, as H ranges over representatives of conjugacy classes of cyclic subgroups of G.

2. Let $R(G)$ be the subgroup of the ring of class functions consisting of $\mathbb{Z}$-linear combinations of irreducible characters, let $P(G)$ be the subgroup of $R(G)$ consisting of $\mathbb{Z}$-linear combinations of permutation characters $\mathrm{Ind}_H^G \mathbf{1}$, $H \leq G$. Prove that both $R(G)$ and $P(G)$ are subrings, rather than just subgroups.

Section B

1. A character is called monomial if it is induced from a 1-dimensional character. Let $M(G)$ be the subgroup of $R(G)$ consisting of $\mathbb{Z}$-linear combinations of monomial characters. Show that $M(G) = R(G)$.

2. Let $H_1, \ldots, H_n$ be a set of representatives of conjugacy classes of all subgroups of G (so that G/H_i are precisely all the transitive G-sets up to isomorphism). Form the n-by-n matrix $(\#(G/H_i)^{H_j})$, whose (i,j)-th entry is the number of fixed points under the action of H_j on the set of cosets G/H_i. Show that this matrix is invertible over $\mathbb{Q}$.
   Hint: there is a Mackey theorem for sets: there is an isomorphism of H_j-sets,
$$\mathrm{Res}_{H_j}(G/H_i) = \bigsqcup_{g \in H_i \backslash G / H_j} H_j/(H_j \cap gH_ig^{-1}).$$

3. Let S_n naturally act on the set $X = \{1, \ldots, n\}$. For any $r \leq n/2$, write X_r for the set of all subsets of X of size r. Then, G naturally acts on X_r, write π_r for the corresponding permutation representation.
   (a) Compute $\pi_r(1)$.
   (b) Show that for $1 \leq l \leq k \leq n/2$
$$\langle \pi_k, \pi_l \rangle = l + 1.$$

(c) Set $m = n/2$ if n is even and $m = (n-1)/2$ if n is odd. Show that S_n has distinct irreducible characters

$$\chi^{(n)} = \mathbf{1}_G, \chi^{(n-1,1)}, \chi^{(n-2,2)}, \ldots, \chi^{(n-m,m)}$$

such that for all $r \leq m$,

$$\pi_r = \chi^{(n)} + \chi^{(n-1,1)} + \ldots + \chi^{(n-r,r)}.$$

In particular, the class functions $\pi_r - \pi_{r-1}$ are irreducible characters of S_n for $1 \leq r \leq n/2$.

4. Construct the character table of S_6.

5. In this exercise, we will show that if G is nilpotent[1] then every irreducible character of G is monomial (see question 1).

   Remark: groups for which this is the case are called M-groups. To characterise M-groups purely group theoretically, with no reference to characters, is still an open problem.

   (a) Call a character primitive if it is not induced from a character of a proper subgroup. Show that if χ is a primitive irreducible character of G and N is a normal subgroup, then $\mathrm{Res}_N \chi$ is isotypical, i.e. is a multiple of an irreducible character of N.

   (b) Show that if G has a primitive faithful irreducible character, then any normal abelian subgroup of G is contained in the centre $Z(G)$.

   (c) (Harder:) Using the fact that subgroups and quotient groups of nilpotent groups are also nilpotent, and that any nilpotent group has a normal abelian self-centralising subgroup, show that nilpotent groups are M-groups.

   Hint: given an arbitrary irreducible character χ of G, take a minimal subgroup $H \leq G$ such that there exists $\psi \in H$ with $\chi = \mathrm{Ind}^G \psi$. Note that sich a ψ must be primitive.

Section C

1. Let $G = C_3 \rtimes C_8$, where C_8 acts on C_3 non-trivially, through the quotient C_8/C_4. So a presentation of G is

$$G = \langle a, b \mid a^3 = b^8 = id, bab^{-1} = a^{-1} \rangle.$$

   (a) Classify all the conjugacy classes of G.

   (b) Let χ be a faithful irreducible character of $A = C_3$. What is its stabiliser in C_8?

[1] There are lots of equivalent definitions of "nilpotent". For example a finite group is nilpotent if it is a direct product of its p-Sylow subgroups as p runs over the distinct prime divisors of $|G|$. Or if its lower central series, defined by $G = G_1$, $G_{i+1} = [G_i, G] = \langle hgh^{-1}g^{-1} \mid h \in G_i, g \in G \rangle$ terminates in the trivial group.

(c) Extend χ trivially to its stabiliser $S = \mathrm{Stab}_{C_8}(\chi)$, let τ be a faithful irreducible character of $S \cong AS/A$, thought of as a character of AS. Note that by sheet 2, Section B, qn. 4, $\phi = \mathrm{Ind}^G(\chi \otimes \tau)$ is irreducible. Compute this character on all the conjugacy classes.

(d) The absolute Galois group of $\mathbb{Q}$ acts on the character values of ϕ. Let $\overline{\phi}$ be the sum over all the Galois conjugates of ϕ. So $\overline{\phi}$ is a $\mathbb{Q}$-valued character. Show that $\overline{\phi}$ is not an integral linear combination of permutation characters.

(e) (Harder:) Show that $\overline{\phi}$ corresponds to a representation that can be defined over $\mathbb{Q}$.

2. Repeat the last two parts of the previous question for the following situation: $G = Q_8 \times C_3$, $\phi = \tau \otimes (\chi + \bar{\chi})$, where τ is the standard character of Q_8, and χ is a faithful character of C_3.