Representation theory, exercise sheet 5

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Throughout, G denotes a finite group.

Section A

1. Recall the following presentation of Q_8, and its standard representation:

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle,$$

and

$$\begin{array}{rcl} \rho : Q_8 & \to & \mathrm{GL}_2(\mathbb{Q}(i)) \\ x & \mapsto & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\ y & \mapsto & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{array}$$

By writing $\mathbb{Q}(i)$ as a 2-dimensional vector space over $\mathbb{Q}$, write this map as a 4-dimensional representation over $\mathbb{Q}$. What is the complex representation that you have thus realised over $\mathbb{Q}$?

2. Repeat the above exercise (including the last question) for $G = C_3$, and $\rho : G \to \mathrm{GL}_1(\mathbb{Q}(e^{2\pi i/3}))$ one of the two non-trivial one-dimensional representations.

3. The following are not very well-defined questions, they are just supposed to get you thinking, while standing in the queue:

   (a) Does Artin's induction theorem say anything about Schur indices of irreducible characters?

   (b) Suppose that N is a normal subgroup of a normal subgroup of G (N itself need not be normal in G). If χ is an irreducible character of G, what can you say about $\mathrm{Res}_N \chi$?

Section B

1. Let $H \leq G$, and suppose that there exists a subgroup $U \leq G$ such that $HU = G$ and $U \cap H = \{1\}$. Let $\phi \in \mathrm{Irr}(H)$ and suppose that $\chi = \mathrm{Ind}^H \phi$ is irreducible. Show that $m_{\mathbb{Q}}(\chi)$ divides $\phi(1)$.

2. For a positive integer n, define $r_n : G \to \mathbb{Z}$ by

$$r_n(g) = \#\{h \in G \mid h^n = g\}.$$

(a) Show that r_n is a class function and find the coefficients $a_\chi \in \mathbb{C}$ in the decomposition $r_n = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$.

(b) Show that if $n \geq 3$, then these coefficients can be unbounded, as G and χ vary.
(Hint: for a prime number p, consider the finite group

$$\left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_3(\mathbb{F}_p),$$

which has order p^3 and exponent p.)

3. Let $X \subseteq G$ be a subset with the property that for every $g \in G$, either $gXg^{-1} = X$ or $gXg^{-1} \cap X \subseteq \{1\}$ (note that this is a generalisation of the condition that X is a Frobenius complement). Let N be the normaliser of X in G, and let θ be a class functions on N that vanishes on $N \backslash X$ and on 1 (in particular, θ cannot be a character, but it could be a difference of two characters of the same dimension).

(a) Show that for all $x \in X$, $\mathrm{Ind}^G \theta(x) = \theta(x)$.

(b) Show that if ϕ is any other class function on N that vanishes on $X \backslash N$, then $\langle \mathrm{Ind}^G \theta, \mathrm{Ind}^G \phi \rangle_G = \langle \theta, \phi \rangle_N$.

(c) Let M be a normal subgroup of $H \leq G$, and suppose that $H \cap xHx^{-1} \subseteq M$ for all $x \notin H$. Show that there exists a normal subgroup N of G such that $NH = G$ and $N \cap H = M$.
(Hint: note that the set $X = H \backslash M$ satisfies the above conditions; mimic the proof of Frobenius's theorem on Frobenius complements.)

4. Let F be a subfield of $\mathbb{C}$. Define an F-triple to be a triple (H, N, θ), where

- H is a group, N is a normal subgroup that is equal to its own centraliser in H (in particular N is abelian), and $\theta \in \mathrm{Irr}(H)$ is a faithful character, such that
- the irreducible (1-dimensional) constituents of $\mathrm{Res}_N \theta$ are all Galois conjugate over $F(\theta)$, meaning that for any two such irreducible constituents λ_1 and λ_2, there is a field automorphism of $F(\theta)$ that sends λ_1 to λ_2.

Let (H, N, θ) be such an F-triple, and let λ be an irreducible (1-dimensional) constituent of $\mathrm{Res}_N \theta$.

(a) Show that λ is faithful, and in particular N is automatically cyclic.

(b) Show that $\mathrm{Stab}_H(\lambda) = N$, $\mathrm{Ind}^H \lambda = \theta$, and $F(\theta) \subseteq F(\lambda)$.

(c) θ_N is realisable by a simple $F(\theta)[N]$-module.

(d) (Optional:) H/N is isomorphic to the Galois group of $F(\lambda)/F(\theta)$.

Remark: Such F-triples are used to algorithmically compute Schur indices (see section C).

Section C

1. (Not easy!) Let F be a subfield of $\mathbb{C}$, $\chi \in \mathrm{Irr}(G)$, and suppose that p^a divides $m_F(\chi)$ for some prime p and some integer a. Show that there exists an F-triple (H, N, θ) such that

   (a) N is a subquotient of G, i.e. there are subgroups $M_1 \leq M_2 \leq G$ such that M_1 is normal in M_2 and $M_2/M_1 = H$,

   (b) p^a divides $m_F(\theta)$,

   (c) H/N is a p-group,

   (d) p does not divide the degree $[F(\chi, \theta) : F(\chi)]$.

   (Hints: note that if (H, N, θ) is an E-triple for some field E containing F, then it is also an F-triple, so you can replace F by $F(\chi)$ (why?). Now, argue by induction on $|G|$. You will need Solomon's induction theorem at some point.)

2. (Easier - use questions 1 of sections B and C:) Let $\chi \in \mathrm{Irr}(G)$ and suppose that $p | m_{\mathbb{Q}}(\chi)$ for some prime p. Show that then the Sylow p-subgroups of G are not elementary abelian, and have order at least $p \cdot m_{\mathbb{Q}}(\chi)$.