How we solve Diophantine equations

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April 13, 2011

Introduction

A **Diophantine problem** is the problem of finding *integer* or *rational* solutions to a given polynomial equation in one or several variables with rational coefficients.

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- Find $(x, y, z) \in \mathbb{Z}^3$ satisfying $x^2 - 5y^2 = 3z^2$. This is a *homogeneous* equation of degree 2.
- Given an integer $n \geq 3$, find all $(x, y, z) \in \mathbb{Z}^2$ satisfying $x^n + y^n = z^n$. This is the famous Fermat equation.

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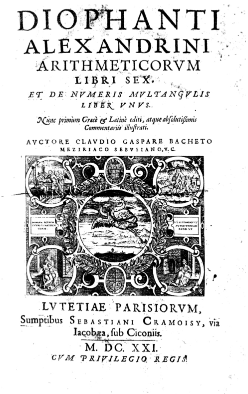
- $x^x y^y = z^z$ is a very interesting equation, but not polynomial in the variables, so not Diophantine.
- $\pi x + ey + \pi^e z = 0$ is not Diophantine, because the coefficients are irrational.

History

A 16th century edition of “Arithmetica” by Diophantus of Alexandria, translated into Latin:



DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO MEZIRIACO SEBVSIANO, V.C.

LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

We want to find rational solutions to $x^2 - 5y^2 = 3$ or, equivalently, integral solutions to $x^2 - 5y^2 = 3z^2$ with $z \neq 0$.

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Since x and y cannot be divisible by arbitrarily large powers of 3, we obtain a contradiction, so there are no integer solutions to $x^2 - 5y^2 = 3z^2$.

This is the **method of infinite descent**, due to Pierre de Fermat.

Moral of the story: for an equation to have integer solutions, it must have solutions modulo p^n for any prime number p and any $n \in \mathbb{N}$. It must also have real solutions.

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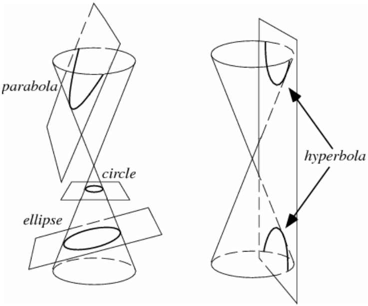
We say that equations of degree 2 satisfy the **Hasse principle**.
This reduces the decision problem to a finite computation, since given an equation, the above condition will be automatically satisfied for almost all primes.

Moreover, a quadratic equation in two variables has either no rational solutions or infinitely many. Once we find one, we find them all:



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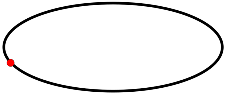

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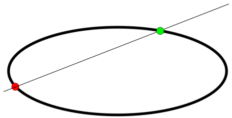

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Famous example, due to Ernst Selmer:

$$3x^3 + 4y^3 + 5z^3 = 0$$

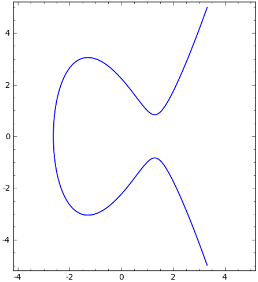
has a non-zero solution in the reals and non-zero solutions modulo all prime powers, but no integral solutions!

Equations of degree 3 differ from those of degree 2 in many other ways. E.g. an equation of the form $y^2 = x^3 + ax + b$, $a, b \in \mathbb{Q}$, can have 0, or finitely many, or infinitely many solutions.

An equation of the form

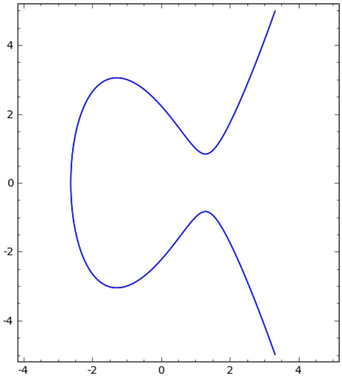
$$E:\ y^2 = x^3 + ax + b,\quad a, b \in \mathbb{Q}$$

describes an elliptic curve.



Addition law on elliptic curves

Given a point on the curve E, we cannot quite repeat the conic trick for finding a new point, but given two points, we can find a third one:



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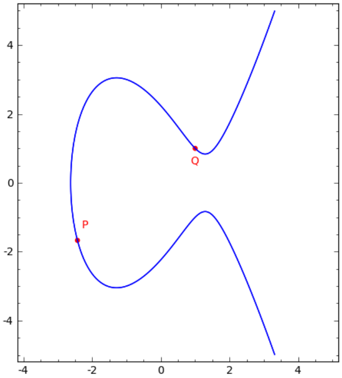

What is a Diophantine equation
The Hasse principle
Elliptic curves
Birch and Swinnerton-Dyer conjecture
Unique factorisation

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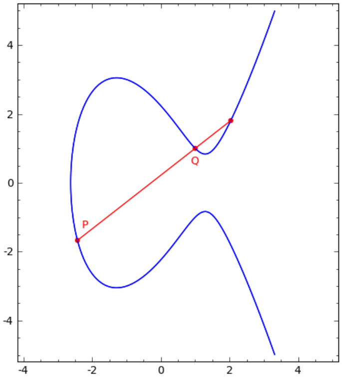

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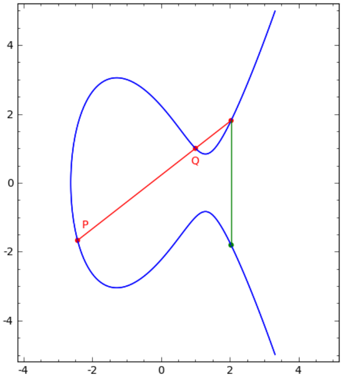

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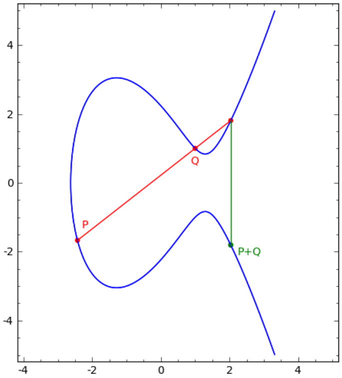

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Theorem (Mordell): Given any elliptic curve E, the group $E(\mathbb{Q})$ is finitely generated. Thus, it is isomorphic to $\Delta \oplus \mathbb{Z}^{r(E)}$, where Δ is a finite abelian group, and $r(E) \geq 0$.

The integer $r(E)$ is called the *rank* of E and is a very mysterious invariant.

One important ingredient in the proof of Mordell's theorem is Fermat's technique of infinite descent. This technique has been vastly generalised.

Even though elliptic curves do not satisfy the Hasse principle, we can still try to count solutions modulo primes. Denote the number of solutions modulo p by $N_E(p)$. It turns out that $N_E(p) = p + 1 - a_p$, where

$$|a_p| \leq 2\sqrt{p}.$$

So, $N_e(p) \sim p$ as $p \to \infty$.

In the 1960s, Bryan Birch and Peter Swinnerton-Dyer computed

$$f_E(X) = \prod_{p \leq X} \frac{N_E(p)}{p}$$

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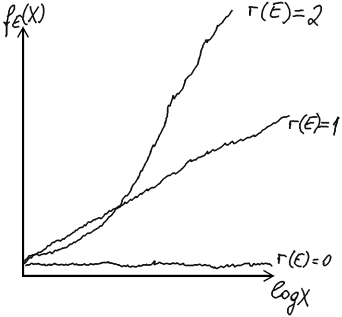

Birch and Swinnerton-Dyer conjecture

This led them to conjecture that

$$f_E(X) \sim c_E(\log X)^{r(E)}.$$

This is the naive form of the famous Birch and Swinnerton-Dyer conjecture. It is a very deep kind of local-global principle, of which the Hasse principle is the simplest example.

Integral points on elliptic curves

Suppose that we want to find integer solutions to

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Idea: Work in the slightly bigger ring $R = \mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$.

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Unique factorisation

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Step 3. Expand and equate coefficients to find the only solutions are $b = 1$, $a = \pm 1$, which correspond to $x = 3$, $y = \pm 5$.

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- We implicitly used in Step 2 that in R, any element can be factorised uniquely into irreducibles, just like in $\mathbb{Z}$.

If we tried to do this for the equation

$$y^2 = x^3 - 1,$$

working in the ring $\mathbb{Z}[\sqrt{-1}]$, then we would have to be careful with the units, since there are the additional units $\pm i$ (they are still all cubes, but in other circumstances they might not be). In fact, if $d > 0$ is square-free and congruent to 3 modulo 4, then $\mathbb{Z}[\sqrt{d}]$ has infinitely many units!

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If we tried to do this for the equation

$$y^2 = x^3 - 6,$$

then things would go completely wrong, since the ring $\mathbb{Z}[\sqrt{-6}]$ does not have unique factorisation into irreducibles.

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Open question: Are there infinitely many real quadratic fields, whose ring of integers has unique factorisation?

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To do that, he introduced a new idea, which was later used by Bryan Birch to produce rational points on elliptic curves. These so-called Heegner points were then used in the 80's in a series of difficult papers by many people to prove a special case of the Birch and Swinnerton-Dyer conjecture in 1990.