Topics in Number Theory - 2nd exercise sheet

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The submission deadlines for the exercise sheet are Oct 15, Oct 22, and Oct 25, always at 3 pm. Each answer will attract full marks if it is perfect, and 0 otherwise. Hopefully, after the third submission everybody will have got full marks for the sheet.

1. Find the smallest positive integer x such that $x \equiv 2 \pmod{15}$ and $x \equiv 1 \pmod 4$. Then characterise the set of all positive integers x satisfying these congruences.

2. (a) Show that if $p \equiv 3 \pmod 4$ is a prime number, then there is no integer a such that $a^2 + 1 \equiv 0 \pmod p$.
   (b) By considering expressions of the form $4(p_1 p_2 \dots p_n)^2 + 1$, show that there are infinitely many primes $p \equiv 1 \pmod 4$.

3. (a) Show that if $2^m + 1$ is an odd prime, then $m = 2^n$ for some $n \in \mathbb{N}$.
   (b) Define $F_n = 2^{2^n} + 1$. By finding a suitable recurrence relation for F_n in terms of $F_1, \dots, F_{n-1}$, show that $\gcd(F_n, F_m) = 1$ for any $n \neq m$.
   (c) Show that if p is a prime divisor of F_n, then $2^{n+1} | (p-1)$.
   (d) Show that for any $n \in \mathbb{N}$, there are infinitely many primes $\equiv 1 \pmod{2^n}$.

4. (a) Show that if $2^p - 1$ is prime, then p must be prime.
   (b) Let p be an odd prime, and let q be a prime divisor of $2^p - 1$. Show that $q \equiv 1 \pmod{2p}$.
   (c) Use this to give an alternative proof of the infinitude of primes.
   (d) A perfect number is a natural number x such that the sum of all its divisors, including 1 and x itself, is equal to $2x$. Show that if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is a perfect number.

5. (a) Show that if p is prime and $m \equiv n \pmod{p-1}$, then for any integer a, $a^m \equiv a^n \pmod p$.
   (b) Show that the above may fail if p is not a prime.