Topics in Number Theory - 4th exercise sheet

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Please submit solutions to the questions 2,3,4,5 for credit (but do all of them!). The submission deadlines for the exercise sheet are Oct 29, Nov 5, and Nov 8, always at 3 pm. Each answer will attract full marks if it is perfect, and 0 otherwise.

1. Is 1093 a square modulo 2011? Briefly justify every step of your calculation.

2. Find a natural number n such that whether or not 11 is a square modulo a prime p only depends on the congruence class of p modulo n (apart from finitely many exceptions), and find those congruence classes $p \equiv a \pmod{n}$ for which $\left(\frac{11}{p}\right) = 1$.

3. Show that the equation $y^2 = x^3 + 7$ has no integral solutions. (Hint: add 1 to both sides.)

4. (a) Show that the sequence $n^5 - n + 3$, $n \in \mathbb{N}$ does not contain any squares. (Hint: reduce modulo 5.)

   (b) Let $p \equiv 2$ or $3 \pmod{5}$ be an odd prime. Show that the sequence $n! + n^p - n + 5$, $n \in \mathbb{N}$ has at most finitely many squares. How many are there when $p = 3$?

5. Let p, q be odd primes with $p = 2q + 1$.

   (a) Show that 2 is a primitive root modulo p if and only if $q \equiv 1 \pmod{4}$.

   (b) Find a necessary and sufficient condition on q for 3 to be a primitive root modulo p.

6. When thinking about Fermat's theorems on primes of the form $x^2 + ny^2$, Euler originally conjectured the following: let p, q be two distinct odd primes. Then

$$\left(\frac{q}{p}\right) = 1 \Leftrightarrow p \equiv \pm\beta^2 \pmod{4q}$$

   for some odd β. Show that this statement is equivalent to the law of quadratic reciprocity.