Topics in Number Theory - 5th exercise sheet

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These questions are not for credit, but nevertheless important. Please do them!

1. Show that the field of p-adic numbers is complete, i.e. that a sequence of p-adic numbers converges if and only if it is Cauchy.

2. (a) For which primes p does the series $\sum_{i=0}^{\infty}\left(\frac{10}{11}\right)^i$ converge p-adically and, when it does, to what limit?

   (b) Compute the p-adic order of $(p^n)! = p^n \cdot (p^n-1)\cdot(p^n-2)\cdot\ldots\cdot 2\cdot 1$.

   (c) For what primes p does the series $1! + 2! + 3! + 4! + \ldots$ converge p-adically?

3. This question shows that a sequence of rationals may converge to rational numbers both with respect to the Euclidean and to the p-adic absolute value, but to different numbers! So let us fix a prime number p.

   (a) Verify that $\frac{p^n}{p^n-1}$ converges p-adically to 0, while $\frac{1}{p^n-1}$ converges p-adically to 1.

   (b) Hence, given two rational numbers r, s, write down a sequence of rationals that converges to r with respect to the Archimedean absolute value, and to s with respect to the p-adic absolute value.

4. (a) Show that a sequence a_n in $\mathbb{Q}_p$ is Cauchy if and only if

   $$|a_n - a_{n-1}|_p \to 0$$

   as $n \to \infty$. (Note that this is far from true for the reals, as the example $s_n = \sum_{i=0}^{n} 1/i$ shows!)

   (b) Consider the sequence of rational numbers

   $$a_1 = 1+3,\ a_2 = 1+\frac{3}{1+3},\ a_3 = 1+\frac{3}{1+\frac{3}{1+3}},\ a_4 = 1+\frac{3}{1+\frac{3}{1+\frac{3}{1+3}}},\ldots$$

   By establishing a recurrence relation and using induction, or otherwise, show that this sequence is 3-adically Cauchy.

   (c) Let $\alpha \in \mathbb{Q}_3$ be the element represented by the above Cauchy sequence. Find a polynomial with integral coefficients that has α as a root. Hence show that $\alpha \notin \mathbb{Q}$.