Topics in Number Theory - 7th exercise sheet

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November 18, 2012

Solutions to this exercise sheet should not be submitted.

1. How many distinct zeros does each of the following polynomials have in $\mathbb{Z}_5$?
   (a) $f(x) = x^3 + 5x + 5$;
   (b) $g(x) = x^5 + 2$;
   (c) $h(x, y) = x^2 + y^2$;

2. For each of the following quadratic forms, determine whether the form has a non-trivial zero (you do not need to exhibit it). For this particular question, you may assume any results that were stated in the lectures, even if they were not proved:
   (a) $f(x, y, z) = 2x^2 + 3y^2 - 6z^2$;
   (b) $g(x, y, z) = 2x^2 + 3y^2 - 10z^2$;
   (c) $h(x, y, z) = x^2 + y^2 - 64z^2$.

3. For the following quadratic forms f, show that if $m, n \in \mathbb{Z}$ are represented by f, then so is mn. Hence, classify all positive integers $n \in \mathbb{Z}$ that are represented by f:
   (a) $f(x, y) = x^2 + 2y^2$ (hint: this is the *norm* of a complex number of the form $x + y\sqrt{-2}$, $x, y \in \mathbb{Z}$. Recall that the norm of a complex number of the form $x + iy$ was defined as $x^2 + y^2$, and the duplication formula for numbers that are sums of two squares came just from multiplying numbers of the form $x + iy$, $x, y \in \mathbb{Z}$, and taking the norm of the product.)
   (b) $f(x, y) = x^2 + 3y^2$ (hint: this is the *norm* of a complex number of the form $x + y\sqrt{-3}$, $x, y \in \mathbb{Z}$);
   (c) $f(x, y) = x^2 + xy + y^2$ (hint: this is the *norm* of a complex number of the form $x + ye^{2\pi i/3}$, $x, y \in \mathbb{Z}$; first, show that the product of two numbers of the form $x + ye^{2\pi i/3}$ is again of this form).

4. For the following pairs of binary quadratic forms, determine whether they are equivalent under the action of $\mathrm{SL}_2(\mathbb{Z})$:
   (a) $f_1(x, y) = 5x^2 + xy + y^2$, $f_2(x, y) = 55x^2 + 61xy + 17y^2$;
   (b) $g_1(x, y) = 3x^2 - 4y^2$, $g_2(x, y) = x^2 + 7xy + y^2$;
   (c) $h_1(x, y) = x^2 + 5xy + 10y^2$, $h_1(x, y) = 2x^2 + 5xy + 5y^2$.