# Topics in Number Theory - 8th exercise sheet

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Solutions to this exercise sheet should not be submitted.

1. For each of the following positive definite quadratic forms, find the unique reduced form representing the same equivalence class:
   (a) $17x^2 - 11xy + 2y^2$;
   (b) $3x^2 + 8xy + 7y^2$.

2. Prove that $h(-19) = 1$ and write down the unique reduced positive definite form of discriminant -19. Find a congruence condition on a positive integer $n$ to be properly representable by this form. Hence, classify all positive integers that are representable (properly or improperly) by this form.

3. Recall the definition of the Jacobi symbol: given an integer $a$ and a positive odd integer $n = \prod_i p_i^{e_i}$, written as a product of odd primes, define

$$\left(\frac{a}{n}\right) = \prod_i \left(\frac{a}{p_i}\right)^{e_i},$$

   where $\left(\frac{a}{p_i}\right)$ is the usual Legendre symbol. Prove the following properties of the Jacobi symbol:

   (a) If $a, b$ are integers, and $n, m$ are odd positive integers, then

$$\left(\frac{ab}{n}\right) = \left(\frac{a}{n}\right)\left(\frac{b}{n}\right),$$

   and

$$\left(\frac{a}{nm}\right) = \left(\frac{a}{n}\right)\left(\frac{a}{m}\right).$$

   (b) If $n$ is an odd positive integer, then

$$\left(\frac{-1}{n}\right) = (-1)^{(m-1)/2},$$

   and

$$\left(\frac{2}{n}\right) = (-1)^{(m^2-1)/8}.$$

   (c) If $a$ and $n$ are both positive and odd, then

$$\left(\frac{a}{n}\right) = (-1)^{\frac{a-1}{2}\frac{n-1}{2}}\left(\frac{n}{a}\right).$$

(d) If $d \equiv 1 \pmod 4$ and $m, n$ are odd positive with $m \equiv n \pmod d$, then

$$\left(\frac{d}{m}\right) = \left(\frac{d}{n}\right).$$

4. (a) Let $d = -4n$, where $n$ has at least 2 distinct prime divisors. Prove that $h(d) > 1$, by exhibiting at least two positive definite reduced forms.

   (b) Let $d = -4n$, where $n + 1 = ac$ with $2 \leq a < c$, $\gcd(a, c) = 1$. Show that $h(d) > 1$.