UNIVERSITY OF GLASGOW

Department of Mathematics

Mathematics 2Q - Groups, Symmetry and Fractals

Class Test

Candidates should attempt ALL questions.

1. (i) If an isometry $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is either a *reflection* or a *glide reflection*, explain how to determine this from its Seitz symbol $(A \mid \mathbf{t})$. What further information would allow you to decide whether it was a reflection? **2,1**

Given that

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

show that $(A \mid \mathbf{t})$ represents a glide reflection and not a reflection. **5**

(ii) Using the matrix A from (i) and the matrix

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

determine the Seitz symbol of the composition $(A \mid \mathbf{t})(B \mid \mathbf{0})$ and explain why it represents a rotation. Find its centre and angle of rotation. **2,5**

2. (i) Working in the symmetric group S_6,

(a) determine the composition

$$(1\ 4\ 5\ 2\ 6)(3\ 4\ 5)(1\ 5\ 3\ 6),$$

expressing the answer as a product of disjoint cycles; **3**

(b) Express the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix}$$

as a product of disjoint cycles. **3**

(ii) Show that the subset $O(2) \subseteq \mathrm{Euc}(2)$ consisting of all isometries which fix the origin forms a subgroup of the Euclidean group $\mathrm{Euc}(2)$. **4**

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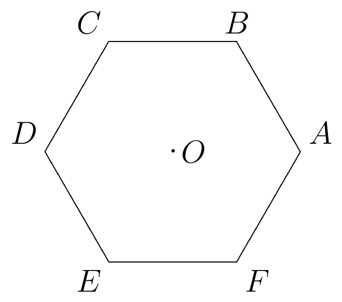
(iii) For the regular hexagon centred at the origin O shown below, write down permutations of the vertices A, B, C, D, E, F corresponding to each of the following symmetries:

$$\alpha = \text{rotation through } \pi/3 \text{ anti-clockwise about } O,$$
$$\beta = \text{reflection in the line } AD.$$

By composing permutations, determine the composition $\alpha \circ \beta$ and describe its geometric effect. **5**



Total Marks 30

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Mathematics 2Q Class Test 2002–3: Solutions

1. (i) $(A \mid \mathbf{t})$ represents a reflection or a glide reflection if $\det A = -1$. It represents a reflection if and only if it has a fixed point, *i.e.*, there is a position vector $\mathbf{x}$ for which $(A \mid \mathbf{t})\mathbf{x} = \mathbf{x}$. **2,1**

As $\det A = -1$, it suffices to show that $(A \mid \mathbf{t})$ has no fixed point by considering the equation

$$A\mathbf{x} + \mathbf{t} = \mathbf{x}, \quad \text{or equivalently} \quad (I_2 - A)\mathbf{x} = \mathbf{t}.$$

Expanding this out we obtain

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

yielding the obviously inconsistent pair of linear equations

$$x + y = 2, \quad x + y = 1.$$

hence no such fixed point can exist and $(A \mid \mathbf{t})$ must represent a glide reflection rather than a reflection.

Alternative approach: Since

$$A = \begin{bmatrix} \cos(-\pi/2) & \sin(-\pi/2) \\ \sin(-\pi/2) & -\cos(-\pi/2) \end{bmatrix},$$

the Seitz symbol $(A \mid \mathbf{0})$ represents reflection in the line $\mathcal{L}$ given by

$$\sin(-\pi/4)x - \cos(-\pi/4)y = 0, \quad i.e., \quad x + y = 0.$$

Then $(A \mid \mathbf{t})$ represents a reflection or glide reflection in a line parallel to $\mathcal{L}$. A unit vector parallel to $\mathcal{L}$ is $\mathbf{w} = (1/\sqrt{2}, -1/\sqrt{2})$, so the component of $\mathbf{t}$ parallel to $\mathcal{L}$ is

$$\begin{aligned} \mathbf{u} &= (\mathbf{w} \cdot \mathbf{t})\,\mathbf{w} = (\sqrt{2} - 1/\sqrt{2})(1/\sqrt{2}, -1/\sqrt{2}) \\ &= (2-1)/\sqrt{2})(1/\sqrt{2}, -1/\sqrt{2}) = 1/\sqrt{2})(1/\sqrt{2}, -1/\sqrt{2}) \\ &= (1/2, -1/2). \end{aligned}$$

Since $\mathbf{u} \neq \mathbf{0}$, $(A \mid \mathbf{t})$ represents a glide reflection in a line parallel to $\mathcal{L}$ with translation by $\mathbf{u}$. **5**

(ii) The Seitz symbol of the composition is

$$(A \mid \mathbf{t})(B \mid \mathbf{0}) = (AB \mid \mathbf{t} + B\mathbf{0}) = (AB \mid \mathbf{t}),$$

where

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since $\det(AB) = 1$ and $AB \neq I_2$, this represents a rotation. **2**

The position vector $\mathbf{c} = (u, v)$ of the centre of rotation satisfies

$$AB\mathbf{c} + \mathbf{t} = \mathbf{c}, \quad i.e., \quad (I_2 - AB)\mathbf{c} = \mathbf{t}.$$

Thus

$$\mathbf{c} = (I_2 - AB)^{-1}\mathbf{t} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}.$$ **3**

Since

$$AB = \begin{bmatrix} \cos 3\pi/2 & \sin 3\pi/2 \\ -\sin 3\pi/2 & \cos 3\pi/2 \end{bmatrix},$$

the angle of rotation is $3\pi/2$ measured anti-clockwise. **2**

2. (i) (a) N.B. The order of composition used is important!

$$(1\ 4\ 5\ 2\ 6)(3\ 4\ 5)(1\ 5\ 3\ 6) = (1\ 3)(2\ 6\ 4)(5) = (1\ 3)(2\ 6\ 4).$$ **3**

(b) The cycles are

$$1 \longrightarrow 5 \longrightarrow 6 \longrightarrow 3 \longrightarrow 1, \quad 2 \longrightarrow 4 \longrightarrow 2,$$

giving

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} = (1\ 5\ 6\ 3)(2\ 4).$$ **3**

(ii) The Euclidean group consists of all isometries $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ under composition of functions, with identity Id and inverses given by inverting functions. **1**

To show that $\mathrm{O}(2) \leqslant \mathrm{Euc}(2)$, first observe that $(A \mid \mathbf{u}) \in \mathrm{O}(2)$ if and only if $\mathbf{u} = \mathbf{0}$ and A is orthogonal ($i.e.$, $A^{-1} = A^T$).

Now for $(A \mid \mathbf{0}), (B \mid \mathbf{0}) \in \mathrm{O}(2)$,

$$(A \mid \mathbf{0})(B \mid \mathbf{0}) = (AB \mid \mathbf{0}), \quad (I_2 \mid \mathbf{0}) = \mathrm{Id}, \quad (A \mid \mathbf{0})^{-1} = (A^{-1} \mid \mathbf{0}) = (A^T \mid \mathbf{0}).$$

So O(2) is closed under composition, contains the identity function and is closed under taking inverses, therefore it is a subgroup of Euc(2). **3**

(iii) $\alpha = (A\ B\ C\ D\ E\ F)$ and $\beta = (B\ F)(C\ E)$. **2**

$\alpha \circ \beta = (A\ B)(C\ F)(D\ E)$ which corresponds to reflection in the line joining the midpoints of the edges AB and DE. **3**

Total marks 30

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